GNR Approach

Szegő Coefficient Side

Szegő Measure Side

Killip Simon via LDP

Further Developements

Large Deviations and Sum Rules for Orthogonal Polynomials

CLAPEM XIV

Universidad de Costa Rica, December, 2016

Barry Simon

IBM Professor of Mathematics and Theoretical Physics, Emeritus
California Institute of Technology
Pasadena, CA, U.S.A.

Lecture 4: GNR Proof of Sum Rules

GNR Proof of Sum Rules

- Lecture 1: OPRL, OPUC and Sum Rules
- Lecture 2: Meromorphic Herglotz Functions and Proof of KS Sum Rule
- Lecture 3: The Theory of Large Deviations
- Lecture 4: GNR Proof of Sum Rules

References

[GNR1] F. Gamboa, J. Nagel, and A. Rouault, *Sum rules via large deviations*, J. Funct. Anal. **270**, (2016), 509–559.

[BSZ1] J. Breuer, B. Simon and O. Zeitouni *Large Deviations and Sum Rules for Spectral Theory – A Pedagogical Approach*, J. Spec. Th, to appear

[AGZ] G. Anderson, A. Guionnet and O. Zeitouni, *An Introduction to Random Matrics*, Cambridge University Press, 2010

[BAG] G. Ben Arous and A. Guionnet, *Large deviations for Wigner's law and Voiculescu's non-commutative entropy*, Probab. Theory Rel. Fields, **108** (1997), 517–542.

[DE] I. Dumitriu, and A. Edelman, A. (2002). *Matrix models for beta ensembles*, J. Math. Phys. **43** (2002), 5830–5847.

LDP and Sum Rules

Gamboa, Nagel and Rouault had the following lovely idea. Let X be the set of probability measures on $\partial\mathbb{D}$ or on $\mathbb{R}$ (with some song and dance to handle measures which don't have compact support) and suppose we have a sequence of probability measures on X with an LDP.

LDP and Sum Rules

Gamboa, Nagel and Rouault had the following lovely idea. Let X be the set of probability measures on $\partial\mathbb{D}$ or on $\mathbb{R}$ (with some song and dance to handle measures which don't have compact support) and suppose we have a sequence of probability measures on X with an LDP. The Verblunsky and Jacobi maps are continuous to sequences of Verblunsky coefficients or Jacobi parameters and so one has an LDP on sequence space.

LDP and Sum Rules

Gamboa, Nagel and Rouault had the following lovely idea. Let X be the set of probability measures on $\partial\mathbb{D}$ or on $\mathbb{R}$ (with some song and dance to handle measures which don't have compact support) and suppose we have a sequence of probability measures on X with an LDP. The Verblunsky and Jacobi maps are continuous to sequences of Verblunsky coefficients or Jacobi parameters and so one has an LDP on sequence space. But the rate functions are clearly the same, so we have the equality of a function of the spectral measures and of a function of the parameters

LDP and Sum Rules

Gamboa, Nagel and Rouault had the following lovely idea. Let X be the set of probability measures on $\partial\mathbb{D}$ or on $\mathbb{R}$ (with some song and dance to handle measures which don't have compact support) and suppose we have a sequence of probability measures on X with an LDP. The Verblunsky and Jacobi maps are continuous to sequences of Verblunsky coefficients or Jacobi parameters and so one has an LDP on sequence space. But the rate functions are clearly the same, so we have the equality of a function of the spectral measures and of a function of the parameters *and as rate functions, these functions are automatically non-negative*!!!!!

LDP and Sum Rules

Gamboa, Nagel and Rouault had the following lovely idea. Let X be the set of probability measures on $\partial\mathbb{D}$ or on $\mathbb{R}$ (with some song and dance to handle measures which don't have compact support) and suppose we have a sequence of probability measures on X with an LDP. The Verblunsky and Jacobi maps are continuous to sequences of Verblunsky coefficients or Jacobi parameters and so one has an LDP on sequence space. But the rate functions are clearly the same, so we have the equality of a function of the spectral measures and of a function of the parameters *and as rate functions, these functions are automatically non-negative*!!!!! We thus have a way to generate positive sum rules and demanding they be finite gives us a gem.

LDP and Sum Rules

GNR had the further idea that the measures on the spectral measures should come from random matrix measures with a cyclic vector in the limit as the matrix dimension goes to infinity.

LDP and Sum Rules

GNR had the further idea that the measures on the spectral measures should come from random matrix measures with a cyclic vector in the limit as the matrix dimension goes to infinity.

Of course, the issue becomes to effectively compute the rate function on both sides and alas, we haven't yet found a magic way to do these calculations in a general context.

LDP and Sum Rules

GNR had the further idea that the measures on the spectral measures should come from random matrix measures with a cyclic vector in the limit as the matrix dimension goes to infinity.

Of course, the issue becomes to effectively compute the rate function on both sides and alas, we haven't yet found a magic way to do these calculations in a general context.

The reception of the GNR paper illustrates the dangers of working in between two disparate areas.

LDP and Sum Rules

GNR had the further idea that the measures on the spectral measures should come from random matrix measures with a cyclic vector in the limit as the matrix dimension goes to infinity.

Of course, the issue becomes to effectively compute the rate function on both sides and alas, we haven't yet found a magic way to do these calculations in a general context.

The reception of the GNR paper illustrates the dangers of working in between two disparate areas. They wrote the paper in a way that only experts on large deviations could understand it,

LDP and Sum Rules

GNR had the further idea that the measures on the spectral measures should come from random matrix measures with a cyclic vector in the limit as the matrix dimension goes to infinity.

Of course, the issue becomes to effectively compute the rate function on both sides and alas, we haven't yet found a magic way to do these calculations in a general context.

The reception of the GNR paper illustrates the dangers of working in between two disparate areas. They wrote the paper in a way that only experts on large deviations could understand it, but such experts didn't understand the spectral theory context.

LDP and Sum Rules

Jonathan Breuer and I couldn't understand the paper, so we consulted Ofer Zeitouni, who said he'd looked quickly at the paper and there didn't seem to be much new there!

LDP and Sum Rules

Jonathan Breuer and I couldn't understand the paper, so we consulted Ofer Zeitouni, who said he'd looked quickly at the paper and there didn't seem to be much new there! In fact, the calculations of rate functions on the two sides wasn't so far from prior calculations of rate functions.

LDP and Sum Rules

Jonathan Breuer and I couldn't understand the paper, so we consulted Ofer Zeitouni, who said he'd looked quickly at the paper and there didn't seem to be much new there! In fact, the calculations of rate functions on the two sides wasn't so far from prior calculations of rate functions. What was new was the realization that because a rate function could be computed in two ways, one is able to prove interesting equalities.

LDP and Sum Rules

Jonathan Breuer and I couldn't understand the paper, so we consulted Ofer Zeitouni, who said he'd looked quickly at the paper and there didn't seem to be much new there! In fact, the calculations of rate functions on the two sides wasn't so far from prior calculations of rate functions. What was new was the realization that because a rate function could be computed in two ways, one is able to prove interesting equalities. So they had some troubles getting published what I regard as one of the more interesting recent papers in spectral theory.

LDP and Sum Rules

Jonathan Breuer and I couldn't understand the paper, so we consulted Ofer Zeitouni, who said he'd looked quickly at the paper and there didn't seem to be much new there! In fact, the calculations of rate functions on the two sides wasn't so far from prior calculations of rate functions. What was new was the realization that because a rate function could be computed in two ways, one is able to prove interesting equalities. So they had some troubles getting published what I regard as one of the more interesting recent papers in spectral theory. In the end, Jonathan, Ofer and I used their methods to study higher order sum rules

LDP and Sum Rules

Jonathan Breuer and I couldn't understand the paper, so we consulted Ofer Zeitouni, who said he'd looked quickly at the paper and there didn't seem to be much new there! In fact, the calculations of rate functions on the two sides wasn't so far from prior calculations of rate functions. What was new was the realization that because a rate function could be computed in two ways, one is able to prove interesting equalities. So they had some troubles getting published what I regard as one of the more interesting recent papers in spectral theory. In the end, Jonathan, Ofer and I used their methods to study higher order sum rules and we also wrote a pedagogic translation of their paper accessible to spectral theorists.

CUE and GUE

To be explicit about the random matrix models:

CUE and GUE

To be explicit about the random matrix models:

- the Szegő–Verblunsky sum rule comes from CUE, aka Circular Unitary Ensemble, the family on the spectral measures induced by Haar measure on $\mathbb{U}(n)$.

CUE and GUE

To be explicit about the random matrix models:

- the Szegő–Verblunsky sum rule comes from CUE, aka Circular Unitary Ensemble, the family on the spectral measures induced by Haar measure on $\mathbb{U}(n)$.
- the Killip–Simon sum rules comes from GUE, aka Gaussian Unitary Ensemble, the measure on random $n \times n$ self–adjoint matrices has $\{\mathrm{Re} M_{ij}^{(n)}\}_{1 \le i \le j \le n}$ and $\{\mathrm{Im} M_{ij}^{(n)}\}_{1 \le i < j \le n}$ Gaussian iid with mean zero and $\mathbb{E}([M_{ii}^{(n)}]^2) = n^{-1}$.

CUE and GUE

To be explicit about the random matrix models:

- the Szegő–Verblunsky sum rule comes from CUE, aka Circular Unitary Ensemble, the family on the spectral measures induced by Haar measure on $\mathbb{U}(n)$.
- the Killip–Simon sum rules comes from GUE, aka Gaussian Unitary Ensemble, the measure on random $n \times n$ self–adjoint matrices has $\{\mathrm{Re} M_{ij}^{(n)}\}_{1 \le i \le j \le n}$ and $\{\mathrm{Im} M_{ij}^{(n)}\}_{1 \le i < j \le n}$ Gaussian iid with mean zero and $\mathbb{E}([M_{ii}^{(n)}]^2) = n^{-1}$.

(GNR use GOE rather than GUE but that only means our sum rules are twice theirs).

CUE and GUE

To be explicit about the random matrix models:

- the Szegő–Verblunsky sum rule comes from CUE, aka Circular Unitary Ensemble, the family on the spectral measures induced by Haar measure on $\mathbb{U}(n)$.
- the Killip–Simon sum rules comes from GUE, aka Gaussian Unitary Ensemble, the measure on random $n \times n$ self–adjoint matrices has $\{\mathrm{Re} M_{ij}^{(n)}\}_{1 \le i \le j \le n}$ and $\{\mathrm{Im} M_{ij}^{(n)}\}_{1 \le i < j \le n}$ Gaussian iid with mean zero and $\mathbb{E}([M_{ii}^{(n)}]^2) = n^{-1}$.

(GNR use GOE rather than GUE but that only means our sum rules are twice theirs). Note the curious fact that on the support of the measures $\mathbb{P}_n$ (which is easily seen to be the measures with at most n pure points (only)), we have that $I = \infty$ because there is no a.c. part.

Topology of VCs

In the rest of the lectures, we'll describe the CUE proof in some detail and then sketch the GUE proof.

Topology of VCs

In the rest of the lectures, we'll describe the CUE proof in some detail and then sketch the GUE proof. We begin by describing the set of Verblunsky coefficients and the topology on it. Let

Topology of VCs

In the rest of the lectures, we'll describe the CUE proof in some detail and then sketch the GUE proof. We begin by describing the set of Verblunsky coefficients and the topology on it. Let

$$Y_\infty = \mathbb{D}^\infty \qquad Y_n = \left(\prod_{j=0}^{n-2} \mathbb{D} \right) \times \partial\mathbb{D} \qquad Y = Y_\infty \cup \bigcup_{n=1}^{\infty} Y_n$$

Topology of VCs

In the rest of the lectures, we'll describe the CUE proof in some detail and then sketch the GUE proof. We begin by describing the set of Verblunsky coefficients and the topology on it. Let

$$Y_\infty = \mathbb{D}^\infty \qquad Y_n = \left(\prod_{j=0}^{n-2} \mathbb{D}\right) \times \partial\mathbb{D} \qquad Y = Y_\infty \cup \bigcup_{n=1}^{\infty} Y_n$$

The topology is metrizable with convergence given by $\alpha^{(n)} \to \alpha^{(\infty)}$ with $\alpha^{(\infty)} \in Y_\infty \iff \alpha_j^{(n)} \to \alpha_j^{(\infty)}$ for all j

Topology of VCs

In the rest of the lectures, we'll describe the CUE proof in some detail and then sketch the GUE proof. We begin by describing the set of Verblunsky coefficients and the topology on it. Let

$$Y_\infty = \mathbb{D}^\infty \qquad Y_n = \left(\prod_{j=0}^{n-2} \mathbb{D}\right) \times \partial\mathbb{D} \qquad Y = Y_\infty \cup \bigcup_{n=1}^{\infty} Y_n$$

The topology is metrizable with convergence given by $\alpha^{(n)} \to \alpha^{(\infty)}$ with $\alpha^{(\infty)} \in Y_\infty \iff \alpha_j^{(n)} \to \alpha_j^{(\infty)}$ for all j and if $\alpha^{(\infty)} \in Y_m$, then for eventually, $\alpha^{(n)}$ $\in Y_\infty \cup (\cup_{n=m}^{\infty} Y_n)$ and $\alpha_j^{(n)} \to \alpha_j^{(\infty)}$, $j = 0, \dots, m-1$.

Topology of VCs

In the rest of the lectures, we'll describe the CUE proof in some detail and then sketch the GUE proof. We begin by describing the set of Verblunsky coefficients and the topology on it. Let

$$Y_\infty = \mathbb{D}^\infty \qquad Y_n = \left(\prod_{j=0}^{n-2} \mathbb{D}\right) \times \partial\mathbb{D} \qquad Y = Y_\infty \cup \bigcup_{n=1}^{\infty} Y_n$$

The topology is metrizable with convergence given by $\alpha^{(n)} \to \alpha^{(\infty)}$ with $\alpha^{(\infty)} \in Y_\infty \iff \alpha_j^{(n)} \to \alpha_j^{(\infty)}$ for all j and if $\alpha^{(\infty)} \in Y_m$, then for eventually, $\alpha^{(n)}$ $\in Y_\infty \cup (\cup_{n=m}^{\infty} Y_n)$ and $\alpha_j^{(n)} \to \alpha_j^{(\infty)}$, $j = 0, \ldots, m-1$. This topology is such that the map from probability measures to Y is a homeomorphism.

Topology of VCs

In the rest of the lectures, we'll describe the CUE proof in some detail and then sketch the GUE proof. We begin by describing the set of Verblunsky coefficients and the topology on it. Let

$$Y_\infty = \mathbb{D}^\infty \qquad Y_n = \left(\prod_{j=0}^{n-2} \mathbb{D}\right) \times \partial\mathbb{D} \qquad Y = Y_\infty \cup \bigcup_{n=1}^{\infty} Y_n$$

The topology is metrizable with convergence given by $\alpha^{(n)} \to \alpha^{(\infty)}$ with $\alpha^{(\infty)} \in Y_\infty \iff \alpha_j^{(n)} \to \alpha_j^{(\infty)}$ for all j and if $\alpha^{(\infty)} \in Y_m$, then for eventually, $\alpha^{(n)}$ $\in Y_\infty \cup (\cup_{n=m}^{\infty} Y_n)$ and $\alpha_j^{(n)} \to \alpha_j^{(\infty)}$, $j = 0, \dots, m-1$. This topology is such that the map from probability measures to Y is a homeomorphism.

Let $X = \overline{\mathbb{D}}^\infty$.

Topology of VCs

In the rest of the lectures, we'll describe the CUE proof in some detail and then sketch the GUE proof. We begin by describing the set of Verblunsky coefficients and the topology on it. Let

$$Y_\infty = \mathbb{D}^\infty \qquad Y_n = \left(\prod_{j=0}^{n-2} \mathbb{D}\right) \times \partial\mathbb{D} \qquad Y = Y_\infty \cup \bigcup_{n=1}^{\infty} Y_n$$

The topology is metrizable with convergence given by $\alpha^{(n)} \to \alpha^{(\infty)}$ with $\alpha^{(\infty)} \in Y_\infty \iff \alpha_j^{(n)} \to \alpha_j^{(\infty)}$ for all j and if $\alpha^{(\infty)} \in Y_m$, then for eventually, $\alpha^{(n)}$ $\in Y_\infty \cup (\cup_{n=m}^{\infty} Y_n)$ and $\alpha_j^{(n)} \to \alpha_j^{(\infty)}$, $j = 0, \dots, m-1$. This topology is such that the map from probability measures to Y is a homeomorphism.

Let $X = \overline{\mathbb{D}}^\infty$. Then the map $H : X \to Y$ by dropping all α_j after the first one in $\partial\mathbb{D}$ is continuous.

Computation of I on Y

Let $\mathbb{P}_N$ by the measure on X given by the Killip–Nenciu formula on the first N factors and a point mass at 0 on the remaining coordinates.

Computation of I on Y

Let $\mathbb{P}_N$ by the measure on X given by the Killip–Nenciu formula on the first N factors and a point mass at 0 on the remaining coordinates. Let X_j be $\overline{\mathbb{D}}^j$ and $\pi_j : X \to X_j$ projection onto the first j coordinates.

Computation of I on Y

Let $\mathbb{P}_N$ by the measure on X given by the Killip–Nenciu formula on the first N factors and a point mass at 0 on the remaining coordinates. Let X_j be $\overline{\mathbb{D}}^j$ and $\pi_j : X \to X_j$ projection onto the first j coordinates. By our result on and LDP for measures of the form $F(x)e^{-NG(x)}d^\nu x$, we see that $\pi_j^*(\mathbb{P}_N)$ obeys and LDP with speed N and rate $I_j(\{\alpha_k\}_{k=0}^{j-1}) = -\sum_{k=0}^{j-1} \log(1-|\alpha_k|^2)$.

Computation of I on Y

Let $\mathbb{P}_N$ by the measure on X given by the Killip–Nenciu formula on the first N factors and a point mass at 0 on the remaining coordinates. Let X_j be $\overline{\mathbb{D}}^j$ and $\pi_j : X \to X_j$ projection onto the first j coordinates. By our result on and LDP for measures of the form $F(x)e^{-NG(x)}d^\nu x$, we see that $\pi_j^*(\mathbb{P}_N)$ obeys and LDP with speed N and rate $I_j(\{\alpha_k\}_{k=0}^{j-1}) = -\sum_{k=0}^{j-1} \log(1-|\alpha_k|^2)$. It follows by the projective limit theorem that $\mathbb{P}_N$ has an LDP with speed N and rate function $I(\{\alpha_k\}_{k=0}^{\infty}) = -\sum_{k=0}^{\infty} \log(1-|\alpha_k|^2)$.

Computation of I on Y

Let $\mathbb{P}_N$ by the measure on X given by the Killip–Nenciu formula on the first N factors and a point mass at 0 on the remaining coordinates. Let X_j be $\overline{\mathbb{D}}^j$ and $\pi_j : X \to X_j$ projection onto the first j coordinates. By our result on and LDP for measures of the form $F(x)e^{-NG(x)}d^\nu x$, we see that $\pi_j^*(\mathbb{P}_N)$ obeys and LDP with speed N and rate $I_j(\{\alpha_k\}_{k=0}^{j-1}) = -\sum_{k=0}^{j-1} \log(1-|\alpha_k|^2)$. It follows by the projective limit theorem that $\mathbb{P}_N$ has an LDP with speed N and rate function $I(\{\alpha_k\}_{k=0}^{\infty}) = -\sum_{k=0}^{\infty} \log(1-|\alpha_k|^2)$.

Given the map H from the set of allowed Verblunsky coefficients and X, one notes that the Killip–Nenciu Theorem says that $\mathbb{P}_N^{(H)}$ is precisely the measure on VCs induced by Haar measure on $\mathbb{U}(n)$.

Computation of I on Y

Let $\mathbb{P}_N$ by the measure on X given by the Killip–Nenciu formula on the first N factors and a point mass at 0 on the remaining coordinates. Let X_j be $\overline{\mathbb{D}}^j$ and $\pi_j : X \to X_j$ projection onto the first j coordinates. By our result on and LDP for measures of the form $F(x)e^{-NG(x)}d^{\nu}x$, we see that $\pi_j^*(\mathbb{P}_N)$ obeys and LDP with speed N and rate $I_j(\{\alpha_k\}_{k=0}^{j-1}) = -\sum_{k=0}^{j-1}\log(1-|\alpha_k|^2)$. It follows by the projective limit theorem that $\mathbb{P}_N$ has an LDP with speed N and rate function $I(\{\alpha_k\}_{k=0}^{\infty}) = -\sum_{k=0}^{\infty}\log(1-|\alpha_k|^2)$.

Given the map H from the set of allowed Verblunsky coefficients and X, one notes that the Killip–Nenciu Theorem says that $\mathbb{P}_N^{(H)}$ is precisely the measure on VCs induced by Haar measure on $\mathbb{U}(n)$. Applying the contraction principle, we see these measures obey an LDP with rate I as above, one side of the Szegő–Verblunsky sum rule.

Distribution of Haar distributed spectral measures

We begin our presentation of the calculation of the rate function on the measure side by specifying the distribution of spectral measures induced by CUE(n) which we'll also call CUE(n).

Distribution of Haar distributed spectral measures

We begin our presentation of the calculation of the rate function on the measure side by specifying the distribution of spectral measures induced by CUE(n) which we'll also call CUE(n). Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{C}^n$. It is easy to see that for a.e. U, e_1 is a cyclic vector for U so that U and e_1 define a spectral measure

Distribution of Haar distributed spectral measures

We begin our presentation of the calculation of the rate function on the measure side by specifying the distribution of spectral measures induced by CUE(n) which we'll also call CUE(n). Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{C}^n$. It is easy to see that for a.e. U, e_1 is a cyclic vector for U so that U and e_1 define a spectral measure

$$d\mu(\theta) = \sum_{j=1}^{n} w_j \delta_{\lambda_j}$$

Distribution of Haar distributed spectral measures

We begin our presentation of the calculation of the rate function on the measure side by specifying the distribution of spectral measures induced by CUE(n) which we'll also call CUE(n). Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{C}^n$. It is easy to see that for a.e. U, e_1 is a cyclic vector for U so that U and e_1 define a spectral measure

$$d\mu(\theta) = \sum_{j=1}^{n} w_j \delta_{\lambda_j}$$

on $\partial\mathbb{D}$, with precisely n pure points (aka atoms) $\lambda_j = e^{i\theta_j}, j = 1, \ldots, n$.

Distribution of Haar distributed spectral measures

We begin our presentation of the calculation of the rate function on the measure side by specifying the distribution of spectral measures induced by CUE(n) which we'll also call CUE(n). Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{C}^n$. It is easy to see that for a.e. U, e_1 is a cyclic vector for U so that U and e_1 define a spectral measure

$$d\mu(\theta) = \sum_{j=1}^{n} w_j \delta_{\lambda_j}$$

on $\partial\mathbb{D}$, with precisely n pure points (aka atoms) $\lambda_j = e^{i\theta_j}, j = 1, \ldots, n$. Letting $\{\varphi_j\}_{j=1}^n$ be the orthonormal basis of eigenvectors of U, so that $U\varphi_j = \lambda_j \varphi_j$, we have $w_j = |\langle \varphi_j, e_1 \rangle|^2$. Of course, since $\|e_1\| = 1$,

Distribution of Haar distributed spectral measures

We begin our presentation of the calculation of the rate function on the measure side by specifying the distribution of spectral measures induced by CUE(n) which we'll also call CUE(n). Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{C}^n$. It is easy to see that for a.e. U, e_1 is a cyclic vector for U so that U and e_1 define a spectral measure

$$d\mu(\theta) = \sum_{j=1}^{n} w_j \delta_{\lambda_j}$$

on $\partial\mathbb{D}$, with precisely n pure points (aka atoms) $\lambda_j = e^{i\theta_j}, j = 1, \ldots, n$. Letting $\{\varphi_j\}_{j=1}^n$ be the orthonormal basis of eigenvectors of U, so that $U\varphi_j = \lambda_j \varphi_j$, we have $w_j = |\langle \varphi_j, e_1 \rangle|^2$. Of course, since $\|e_1\| = 1$,

$$\sum_{j=1}^{n} w_j = 1$$

Distribution of Haar distributed spectral measures

For $\tilde{U}$ an arbitrary unitary, $\tilde{U}U\tilde{U}^{-1}$ has the same eigenvalues as U and $\langle \varphi_j(\tilde{U}U\tilde{U}^{-1}), e_1 \rangle = \langle \tilde{U}\varphi_j(U), e_1 \rangle$.

Distribution of Haar distributed spectral measures

For $\tilde{U}$ an arbitrary unitary, $\tilde{U}U\tilde{U}^{-1}$ has the same eigenvalues as U and $\langle \varphi_j(\tilde{U}U\tilde{U}^{-1}), e_1\rangle = \langle \tilde{U}\varphi_j(U), e_1\rangle$. Since $U \mapsto \tilde{U}U\tilde{U}^{-1}$ leaves Haar measure invariant, we see that the distribution of the unit vector $(\langle \varphi_1(U), e_1\rangle, \langle \varphi_2(U), e_1\rangle, \ldots, \langle \varphi_n(U), e_1\rangle) \in \mathbb{C}^n$ is invariant under unitary transformations,

Distribution of Haar distributed spectral measures

For $\tilde{U}$ an arbitrary unitary, $\tilde{U}U\tilde{U}^{-1}$ has the same eigenvalues as U and $\langle \varphi_j(\tilde{U}U\tilde{U}^{-1}), e_1\rangle = \langle \tilde{U}\varphi_j(U), e_1\rangle$. Since $U \mapsto \tilde{U}U\tilde{U}^{-1}$ leaves Haar measure invariant, we see that the distribution of the unit vector $(\langle \varphi_1(U), e_1\rangle, \langle \varphi_2(U), e_1\rangle, \ldots, \langle \varphi_n(U), e_1\rangle) \in \mathbb{C}^n$ is invariant under unitary transformations, which implies it is the Euclidean measure restricted to the sphere.

Distribution of Haar distributed spectral measures

For $\tilde{U}$ an arbitrary unitary, $\tilde{U}U\tilde{U}^{-1}$ has the same eigenvalues as U and $\langle \varphi_j(\tilde{U}U\tilde{U}^{-1}), e_1 \rangle = \langle \tilde{U}\varphi_j(U), e_1 \rangle$. Since $U \mapsto \tilde{U}U\tilde{U}^{-1}$ leaves Haar measure invariant, we see that the distribution of the unit vector $(\langle \varphi_1(U), e_1 \rangle, \langle \varphi_2(U), e_1 \rangle, \dots, \langle \varphi_n(U), e_1 \rangle) \in \mathbb{C}^n$ is invariant under unitary transformations, which implies it is the Euclidean measure restricted to the sphere. By using the fact that that $d^2z = \frac{1}{2} d\theta d(|z|^2)$ (which shows it is essential we work in $\mathbb{C}$), it is not hard to show that the squares of the components of a complex n–vector uniformly distributed on the sphere are uniformly distributed on the simplex.

Distribution of Haar distributed spectral measures

For $\tilde{U}$ an arbitrary unitary, $\tilde{U}U\tilde{U}^{-1}$ has the same eigenvalues as U and $\langle\varphi_j(\tilde{U}U\tilde{U}^{-1}), e_1\rangle = \langle\tilde{U}\varphi_j(U), e_1\rangle$. Since $U \mapsto \tilde{U}U\tilde{U}^{-1}$ leaves Haar measure invariant, we see that the distribution of the unit vector $(\langle\varphi_1(U), e_1\rangle, \langle\varphi_2(U), e_1\rangle, \ldots, \langle\varphi_n(U), e_1\rangle) \in \mathbb{C}^n$ is invariant under unitary transformations, which implies it is the Euclidean measure restricted to the sphere. By using the fact that that $d^2z = \frac{1}{2}d\theta d(|z|^2)$ (which shows it is essential we work in $\mathbb{C}$), it is not hard to show that the squares of the components of a complex n–vector uniformly distributed on the sphere are uniformly distributed on the simplex. Thus we get that the $\{w_j\}_{j=1}^n$ are independent of the eigenvalues and have $\mathbb{P}_n$-distribution.

Distribution of Haar distributed spectral measures

For $\tilde{U}$ an arbitrary unitary, $\tilde{U}U\tilde{U}^{-1}$ has the same eigenvalues as U and $\langle\varphi_j(\tilde{U}U\tilde{U}^{-1}), e_1\rangle = \langle\tilde{U}\varphi_j(U), e_1\rangle$. Since $U \mapsto \tilde{U}U\tilde{U}^{-1}$ leaves Haar measure invariant, we see that the distribution of the unit vector $(\langle\varphi_1(U), e_1\rangle, \langle\varphi_2(U), e_1\rangle, \ldots, \langle\varphi_n(U), e_1\rangle) \in \mathbb{C}^n$ is invariant under unitary transformations, which implies it is the Euclidean measure restricted to the sphere. By using the fact that that $d^2z = \frac{1}{2}d\theta d(|z|^2)$ (which shows it is essential we work in $\mathbb{C}$), it is not hard to show that the squares of the components of a complex n–vector uniformly distributed on the sphere are uniformly distributed on the simplex. Thus we get that the $\{w_j\}_{j=1}^n$ are independent of the eigenvalues and have $\mathbb{P}_n$-distribution.

$$(n-1)!\chi_{\{\sum_{j=1}^{n-1} w_j \ \leq 1\,;\, w_j \geq 0\}}(w)dw_1 \ldots dw_{n-1}$$

Weyl Integration Formula

The distribution of the eigenvalues is given by the celebrated Weyl integration formula which says that the distribution of the eigenvalues under Haar measure is

Weyl Integration Formula

The distribution of the eigenvalues is given by the celebrated Weyl integration formula which says that the distribution of the eigenvalues under Haar measure is

$$\frac{1}{n!}|\Delta(e^{i\theta_1}, \ldots, e^{i\theta_n})|^2 \prod_{j=1}^{n} \frac{d\theta_j}{2\pi}$$

$$\Delta(\lambda_1, \ldots, \lambda_n) \equiv \prod_{i<j}(\lambda_i - \lambda_j)$$

Weyl Integration Formula

The distribution of the eigenvalues is given by the celebrated Weyl integration formula which says that the distribution of the eigenvalues under Haar measure is

$$\frac{1}{n!}|\Delta(e^{i\theta_1}, \ldots, e^{i\theta_n})|^2 \prod_{j=1}^{n} \frac{d\theta_j}{2\pi}$$

$$\Delta(\lambda_1, \ldots, \lambda_n) \equiv \prod_{i<j}(\lambda_i - \lambda_j)$$

For proofs of this formula from two different points of view, see Anderson et al Random Matrices book or my group representation book.

Weyl Integration Formula

The distribution of the eigenvalues is given by the celebrated Weyl integration formula which says that the distribution of the eigenvalues under Haar measure is

$$\frac{1}{n!}|\Delta(e^{i\theta_1}, \dots, e^{i\theta_n})|^2 \prod_{j=1}^{n} \frac{d\theta_j}{2\pi}$$

$$\Delta(\lambda_1, \dots, \lambda_n) \equiv \prod_{i<j} (\lambda_i - \lambda_j)$$

For proofs of this formula from two different points of view, see Anderson et al Random Matrices book or my group representation book. Summarizing

Weyl Integration Formula

The distribution of the eigenvalues is given by the celebrated Weyl integration formula which says that the distribution of the eigenvalues under Haar measure is

$$\frac{1}{n!}|\Delta(e^{i\theta_1}, \dots, e^{i\theta_n})|^2 \prod_{j=1}^{n} \frac{d\theta_j}{2\pi}$$

$$\Delta(\lambda_1, \dots, \lambda_n) \equiv \prod_{i<j} (\lambda_i - \lambda_j)$$

For proofs of this formula from two different points of view, see Anderson et al Random Matrices book or my group representation book. Summarizing

$$d\mathbb{P}_n(\theta_1, \dots, \theta_n, w_1, \dots, w_n) = \frac{1}{n(2\pi)^n} \chi_{\{\sum_{j=1}^{n-1} w_j \ \leq 1\, ;\, w_j \geq 0\}}(w)$$

$$|\Delta(e^{i\theta_1}, \dots, e^{i\theta_n}|^2 d\theta_1 \dots d\theta_n \, dw_1 \dots dw_{n-1}$$

LDP for the Empirical Measure

As a preliminary to computing the measure side rate, one needs to look at what spectral theorists call the density of states, OP workers the density of zeroes and probabilists the empirical measure, namely

LDP for the Empirical Measure

As a preliminary to computing the measure side rate, one needs to look at what spectral theorists call the density of states, OP workers the density of zeroes and probabilists the empirical measure, namely

$$\mu^{(E)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}$$

LDP for the Empirical Measure

As a preliminary to computing the measure side rate, one needs to look at what spectral theorists call the density of states, OP workers the density of zeroes and probabilists the empirical measure, namely

$$\mu^{(E)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}$$

where λ_j are the atoms of μ.

LDP for the Empirical Measure

As a preliminary to computing the measure side rate, one needs to look at what spectral theorists call the density of states, OP workers the density of zeroes and probabilists the empirical measure, namely

$$\mu^{(E)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}$$

where λ_j are the atoms of μ. That is, we drop the weights from the spectral measure.

LDP for the Empirical Measure

As a preliminary to computing the measure side rate, one needs to look at what spectral theorists call the density of states, OP workers the density of zeroes and probabilists the empirical measure, namely

$$\mu^{(E)} = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j}$$

where λ_j are the atoms of μ. That is, we drop the weights from the spectral measure.

$\mathbb{P}_n$ induces a distribution $\mathbb{P}_n^{(E)}$ on point measures of the above form, essentially given by the Weyl Integration Formula.

LDP for the Empirical Measure

One has the following result of Ben Arous and Guionnet – their results discuss GUE, not CUE – the analog for CUE uses the same ideas and is even simpler:

LDP for the Empirical Measure

One has the following result of Ben Arous and Guionnet – their results discuss GUE, not CUE – the analog for CUE uses the same ideas and is even simpler:

BAG Theorem $\mathbb{P}_n^{(E)}$ obeys a LDP with speed n^2 and good rate function

LDP for the Empirical Measure

One has the following result of Ben Arous and Guionnet – their results discuss GUE, not CUE – the analog for CUE uses the same ideas and is even simpler:

BAG Theorem $\mathbb{P}_n^{(E)}$ obeys a LDP with speed n^2 and good rate function

$$I(\mu) = -\int \log(|z-w|)\, d\mu(z)\, d\mu(w)$$

LDP for the Empirical Measure

One has the following result of Ben Arous and Guionnet – their results discuss GUE, not CUE – the analog for CUE uses the same ideas and is even simpler:

BAG Theorem $\mathbb{P}_n^{(E)}$ obeys a LDP with speed n^2 and good rate function

$$I(\mu) = -\int \log(|z-w|)\, d\mu(z)\, d\mu(w)$$

Remark. In the formula for I, z and w lie in the unit circle and $|z-w|$ is a two dimensional distance. This is a $2D$ Coulomb energy. There is a close connection between this result and Johansson's proof of the Strong Szegő Theorem.

LDP for the Empirical Measure

We will not give a formal proof of the BAG Theorem but instead indicate the basic intuition.

LDP for the Empirical Measure

We will not give a formal proof of the BAG Theorem but instead indicate the basic intuition. For distinct λ_is,

LDP for the Empirical Measure

We will not give a formal proof of the BAG Theorem but instead indicate the basic intuition. For distinct λ_is,

$$\prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|^2 = \exp\left(-n^2 J_n(\lambda_1, \dots, \lambda_n)\right)$$

$$\begin{aligned} J_n(\lambda_1, \dots, \lambda_n) &= -\frac{2}{n^2} \sum_{i<j} \log(|\lambda_i - \lambda_j|) \\ &= -\frac{1}{n^2} \sum_{i \neq j} \log(|\lambda_i - \lambda_j|) \end{aligned}$$

LDP for the Empirical Measure

We will not give a formal proof of the BAG Theorem but instead indicate the basic intuition. For distinct λ_is,

$$\prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|^2 = \exp\left(-n^2 J_n(\lambda_1, \dots, \lambda_n)\right)$$

$$J_n(\lambda_1, \dots, \lambda_n) = -\frac{2}{n^2} \sum_{i<j} \log(|\lambda_i - \lambda_j|)$$
$$= -\frac{1}{n^2} \sum_{i\neq j} \log(|\lambda_i - \lambda_j|)$$

If $\mu^{(E)}$ is an n–point measure near μ and the λ have reasonable local spacing,

LDP for the Empirical Measure

We will not give a formal proof of the BAG Theorem but instead indicate the basic intuition. For distinct λ_is,

$$\prod_{i<j} |e^{i\theta_i} - e^{i\theta_j}|^2 = \exp\left(-n^2 J_n(\lambda_1, \dots, \lambda_n)\right)$$

$$\begin{aligned} J_n(\lambda_1, \dots, \lambda_n) &= -\frac{2}{n^2} \sum_{i<j} \log(|\lambda_i - \lambda_j|) \\ &= -\frac{1}{n^2} \sum_{i\neq j} \log(|\lambda_i - \lambda_j|) \end{aligned}$$

If $\mu^{(E)}$ is an n–point measure near μ and the λ have reasonable local spacing, the final sum, which is a discrete Coulomb energy should be near the integral which gives a continuum Coulomb energy.

Slightly Simplified Problem

The weights and eigenvalues are independent. We'll consider a **fixed** triangular array of eigenvalues $\{\lambda_\ell^{(n)}\}_{1\le\ell\le n;\, n=1,\ldots}$ where we suppose that

Slightly Simplified Problem

The weights and eigenvalues are independent. We'll consider a **fixed** triangular array of eigenvalues $\{\lambda_\ell^{(n)}\}_{1\le\ell\le n;\, n=1,\ldots}$ where we suppose that

$$\frac{1}{n}\sum_{\ell=1}^{n}\delta_{\lambda_\ell^{(n)}} \to \frac{d\theta}{2\pi}$$

weakly.

Slightly Simplified Problem

The weights and eigenvalues are independent. We'll consider a **fixed** triangular array of eigenvalues $\{\lambda_\ell^{(n)}\}_{1\leq\ell\leq n;\, n=1,\ldots}$ where we suppose that

$$\frac{1}{n}\sum_{\ell=1}^{n}\delta_{\lambda_\ell^{(n)}} \to \frac{d\theta}{2\pi}$$

weakly. We distribute weights uniformly on the simplex and look at

Slightly Simplified Problem

The weights and eigenvalues are independent. We'll consider a **fixed** triangular array of eigenvalues $\{\lambda_\ell^{(n)}\}_{1\le\ell\le n;\, n=1,\ldots}$ where we suppose that

$$\frac{1}{n}\sum_{\ell=1}^{n}\delta_{\lambda_\ell^{(n)}} \to \frac{d\theta}{2\pi}$$

weakly. We distribute weights uniformly on the simplex and look at

$$\{w_\ell\}_{\ell=1}^{n} \mapsto \sum_{\ell=1}^{n} w_\ell \delta_{\lambda_\ell^{(n)}} \equiv \mu_n(w_\ell)$$

Slightly Simplified Problem

The weights and eigenvalues are independent. We'll consider a **fixed** triangular array of eigenvalues $\{\lambda_\ell^{(n)}\}_{1\leq \ell\leq n;\, n=1,\ldots}$ where we suppose that

$$\frac{1}{n}\sum_{\ell=1}^{n}\delta_{\lambda_\ell^{(n)}} \to \frac{d\theta}{2\pi}$$

weakly. We distribute weights uniformly on the simplex and look at

$$\{w_\ell\}_{\ell=1}^{n} \mapsto \sum_{\ell=1}^{n} w_\ell \delta_{\lambda_\ell^{(n)}} \equiv \mu_n(w_\ell)$$

This gives a distribution, $\mathbb{P}_n^{(\lambda)}$, on measures

Slightly Simplified Problem

The weights and eigenvalues are independent. We'll consider a **fixed** triangular array of eigenvalues $\{\lambda_\ell^{(n)}\}_{1\leq\ell\leq n;\, n=1,\ldots}$ where we suppose that

$$\frac{1}{n}\sum_{\ell=1}^{n}\delta_{\lambda_\ell^{(n)}} \to \frac{d\theta}{2\pi}$$

weakly. We distribute weights uniformly on the simplex and look at

$$\{w_\ell\}_{\ell=1}^{n} \mapsto \sum_{\ell=1}^{n} w_\ell \delta_{\lambda_\ell^{(n)}} \equiv \mu_n(w_\ell)$$

This gives a distribution, $\mathbb{P}_n^{(\lambda)}$, on measures and we'll prove these measures obey a LDP with speed n and rate function $H(\frac{d\theta}{2\pi}, \mu)$, the KL divergence.

Slightly Simplified Problem

The weights and eigenvalues are independent. We'll consider a **fixed** triangular array of eigenvalues $\{\lambda_\ell^{(n)}\}_{1\le\ell\le n;\, n=1,\dots}$ where we suppose that

$$\frac{1}{n}\sum_{\ell=1}^{n}\delta_{\lambda_\ell^{(n)}}\to\frac{d\theta}{2\pi}$$

weakly. We distribute weights uniformly on the simplex and look at

$$\{w_\ell\}_{\ell=1}^{n}\mapsto\sum_{\ell=1}^{n}w_\ell\delta_{\lambda_\ell^{(n)}}\equiv\mu_n(w_\ell)$$

This gives a distribution, $\mathbb{P}_n^{(\lambda)}$, on measures and we'll prove these measures obey a LDP with speed n and rate function $H(\frac{d\theta}{2\pi},\mu)$, the KL divergence. A full analysis depends on proving for each $\epsilon>0$, j and $k=1,\dots,2^j$, the probability that $\left|\frac{2^j}{n}\#(\ell\,|\,\lambda_\ell^{(n)}\in I_k^{(j)})-1\right|\ge\epsilon$ (with $I_k^{(j)}\equiv\{e^{2\pi i\theta}\,|\,\frac{k-1}{2^j}\le\theta<\frac{k}{2^j}\}$) goes to zero faster than exponentially in n.

Slightly Simplified Problem

The weights and eigenvalues are independent. We'll consider a **fixed** triangular array of eigenvalues $\{\lambda_\ell^{(n)}\}_{1\le\ell\le n;\, n=1,\dots}$ where we suppose that

$$\frac{1}{n}\sum_{\ell=1}^{n}\delta_{\lambda_\ell^{(n)}} \to \frac{d\theta}{2\pi}$$

weakly. We distribute weights uniformly on the simplex and look at

$$\{w_\ell\}_{\ell=1}^{n} \mapsto \sum_{\ell=1}^{n} w_\ell \delta_{\lambda_\ell^{(n)}} \equiv \mu_n(w_\ell)$$

This gives a distribution, $\mathbb{P}_n^{(\lambda)}$, on measures and we'll prove these measures obey a LDP with speed n and rate function $H(\frac{d\theta}{2\pi},\mu)$, the KL divergence. A full analysis depends on proving for each $\epsilon > 0$, j and $k = 1,\dots,2^j$, the probability that $\left|\frac{2^j}{n}\#(\ell \mid \lambda_\ell^{(n)} \in I_k^{(j)}) - 1\right| \ge \epsilon$ (with $I_k^{(j)} \equiv \{e^{2\pi i\theta} \mid \frac{k-1}{2^j} \le \theta < \frac{k}{2^j}\}$) goes to zero faster than exponentially in n. This depends on the BAG Theorem.

LDP for Projected Haar Distribution

The proof will be to use projective limits with the maps $\pi_j : \mathcal{M}_{+,1}(\partial\mathbb{D}) \to \mathbb{R}^{2^j}$ given by $\mu \mapsto \mu(I_k^{(j)})$.

LDP for Projected Haar Distribution

The proof will be to use projective limits with the maps $\pi_j : \mathcal{M}_{+,1}(\partial\mathbb{D}) \to \mathbb{R}^{2^j}$ given by $\mu \mapsto \mu(I_k^{(j)})$. We'll get a LDP for the projections using our LDP for sums of exponential random variables and control the $\sup$ of the projected rate functions by a general continuity result.

LDP for Projected Haar Distribution

The proof will be to use projective limits with the maps $\pi_j : \mathcal{M}_{+,1}(\partial\mathbb{D}) \to \mathbb{R}^{2^j}$ given by $\mu \mapsto \mu(I_k^{(j)})$. We'll get a LDP for the projections using our LDP for sums of exponential random variables and control the $\sup$ of the projected rate functions by a general continuity result. It is this last fact that will show singular parts of the measure only change the rate by their impact on the total weight of the a.c. part.

LDP for Projected Haar Distribution

The proof will be to use projective limits with the maps $\pi_j : \mathcal{M}_{+,1}(\partial\mathbb{D}) \to \mathbb{R}^{2^j}$ given by $\mu \mapsto \mu(I_k^{(j)})$. We'll get a LDP for the projections using our LDP for sums of exponential random variables and control the $\sup$ of the projected rate functions by a general continuity result. It is this last fact that will show singular parts of the measure only change the rate by their impact on the total weight of the a.c. part.

For each $j = 1, \ldots$ and $k = 1, \ldots, 2^j$, let $I_k^{(j)}$ be given as above and $\pi_j(\mu)$ the measure with constant a.c. weight on each $I_k^{(j)}$ which gives the same weight to each $I_k^{(j)}$ as μ.

LDP for Projected Haar Distribution

The proof will be to use projective limits with the maps $\pi_j : \mathcal{M}_{+,1}(\partial\mathbb{D}) \to \mathbb{R}^{2^j}$ given by $\mu \mapsto \mu(I_k^{(j)})$. We'll get a LDP for the projections using our LDP for sums of exponential random variables and control the $\sup$ of the projected rate functions by a general continuity result. It is this last fact that will show singular parts of the measure only change the rate by their impact on the total weight of the a.c. part.

For each $j = 1, \dots$ and $k = 1, \dots, 2^j$, let $I_k^{(j)}$ be given as above and $\pi_j(\mu)$ the measure with constant a.c. weight on each $I_k^{(j)}$ which gives the same weight to each $I_k^{(j)}$ as μ. This is exactly the setup we described in Lecture 3 for an example of projective limits.

LDP for Projected Haar Distribution

Given $\{w_\ell\}_{\ell=1}^n$, let $\tilde{\mu}_n^j(w_\ell)$ be the measure on $\partial\mathbb{D}$ with constant a.c. weight on each $I_k^{(j)}$ so that

LDP for Projected Haar Distribution

Given $\{w_\ell\}_{\ell=1}^n$, let $\tilde{\mu}_n^j(w_\ell)$ be the measure on $\partial\mathbb{D}$ with constant a.c. weight on each $I_k^{(j)}$ so that

$$\tilde{\mu}_n^j(I_k^{(j)}) = \sum_{\lambda_\ell^{(n)} \in I_k^{(j)}} w_\ell$$

LDP for Projected Haar Distribution

Given $\{w_\ell\}_{\ell=1}^n$, let $\tilde{\mu}_n^j(w_\ell)$ be the measure on $\partial\mathbb{D}$ with constant a.c. weight on each $I_k^{(j)}$ so that

$$\tilde{\mu}_n^j(I_k^{(j)}) = \sum_{\lambda_\ell^{(n)} \in I_k^{(j)}} w_\ell$$

Thus we have that $\pi_j(\mu_n(w_\ell)) = \tilde{\mu}_n^j(w_\ell)$.

LDP for Projected Haar Distribution

Given $\{w_\ell\}_{\ell=1}^n$, let $\tilde{\mu}_n^j(w_\ell)$ be the measure on $\partial\mathbb{D}$ with constant a.c. weight on each $I_k^{(j)}$ so that

$$\tilde{\mu}_n^j(I_k^{(j)}) = \sum_{\lambda_\ell^{(n)} \in I_k^{(j)}} w_\ell$$

Thus we have that $\pi_j(\mu_n(w_\ell)) = \tilde{\mu}_n^j(w_\ell)$. The w_j are almost independent except for the bothersome normalization condition.

LDP for Projected Haar Distribution

Given $\{w_\ell\}_{\ell=1}^n$, let $\tilde{\mu}_n^j(w_\ell)$ be the measure on $\partial\mathbb{D}$ with constant a.c. weight on each $I_k^{(j)}$ so that

$$\tilde{\mu}_n^j(I_k^{(j)}) = \sum_{\lambda_\ell^{(n)} \in I_k^{(j)}} w_\ell$$

Thus we have that $\pi_j(\mu_n(w_\ell)) = \tilde{\mu}_n^j(w_\ell)$. The w_j are almost independent except for the bothersome normalization condition. We will deal this by noting that if $\{W_j\}_{j=1}^n$ are iidrv with exponential distribution, then $w_j = W_j / \sum_{k=1}^n W_k$ are distributed uniformly on a simplex.

LDP for Projected Haar Distribution

We will be able to prove a LDP for subsums of W's and then use the contraction principle to pass to w's.

LDP for Projected Haar Distribution

We will be able to prove a LDP for subsums of W's and then use the contraction principle to pass to w's.

So let $\widetilde{\mathbb{P}}_n^{(j)}$ be the measure on $\mathbb{R}^{2^j}$ but where now the w_ℓ are replaced by iid exponential random variables, W_ℓ. Thus, $\widetilde{\mathbb{P}}_n^{(j)}$ is the probability measure for the $\mathbb{R}^{2^j}$-valued random variable given by

LDP for Projected Haar Distribution

We will be able to prove a LDP for subsums of W's and then use the contraction principle to pass to w's.

So let $\widetilde{\mathbb{P}}_n^{(j)}$ be the measure on $\mathbb{R}^{2^j}$ but where now the w_ℓ are replaced by iid exponential random variables, W_ℓ. Thus, $\widetilde{\mathbb{P}}_n^{(j)}$ is the probability measure for the $\mathbb{R}^{2^j}$-valued random variable given by

$$\beta_k^n = \sum_{\lambda_\ell^{(n)} \in I_k^{(j)}} W_\ell$$

LDP for Projected Haar Distribution

We will be able to prove a LDP for subsums of W's and then use the contraction principle to pass to w's.

So let $\widetilde{\mathbb{P}}_n^{(j)}$ be the measure on $\mathbb{R}^{2^j}$ but where now the w_ℓ are replaced by iid exponential random variables, W_ℓ. Thus, $\widetilde{\mathbb{P}}_n^{(j)}$ is the probability measure for the $\mathbb{R}^{2^j}$-valued random variable given by

$$\beta_k^n = \sum_{\lambda_\ell^{(n)} \in I_k^{(j)}} W_\ell$$

Fix j and take $n \to \infty$. By our analysis of sums of exponential iidrvs, $\widetilde{\mathbb{P}}_n^{(j)}$ obeys a LDP with speed n and rate function at the point $\vec{\beta} \equiv \{\beta_\ell\}_{\ell=1}^{2^j} \in \mathbb{R}^{2^j}$

LDP for Projected Haar Distribution

We will be able to prove a LDP for subsums of W's and then use the contraction principle to pass to w's.

So let $\widetilde{\mathbb{P}}_n^{(j)}$ be the measure on $\mathbb{R}^{2^j}$ but where now the w_ℓ are replaced by iid exponential random variables, W_ℓ. Thus, $\widetilde{\mathbb{P}}_n^{(j)}$ is the probability measure for the $\mathbb{R}^{2^j}$-valued random variable given by

$$\beta_k^n = \sum_{\lambda_\ell^{(n)} \in I_k^{(j)}} W_\ell$$

Fix j and take $n \to \infty$. By our analysis of sums of exponential iidrvs, $\widetilde{\mathbb{P}}_n^{(j)}$ obeys a LDP with speed n and rate function at the point $\vec{\beta} \equiv \{\beta_\ell\}_{\ell=1}^{2^j} \in \mathbb{R}^{2^j}$

$$\varphi(\vec{\beta}) = \sum_{\ell=1}^{2^j} \left[(\beta_\ell - 2^{-j}) - 2^{-j} \log(2^j \beta_\ell) \right]$$

LDP for Projected Haar Distribution

Recall that given two probability measures μ and ν on the same space, their KL divergence, $H(\mu|\nu)$, is given by the negative of a $\log$ integral.

LDP for Projected Haar Distribution

Recall that given two probability measures μ and ν on the same space, their KL divergence, $H(\mu|\nu)$, is given by the negative of a $\log$ integral. Write $\beta_\ell = \beta s_\ell$ with $\beta = \sum_{q=1}^{2^j} \beta_q$ so that $\vec{s}$ lies in a 2^j-simplex. Write $\mu_{\vec{s}}$ for the probability measure giving uniform weight s_k to $I_k^{(j)}$ and let ν be normalized Lebesgue measure on the circle (i.e. $\mu_{\vec{s}}$ for the $\vec{s}$ with equal components, 2^{-j}). Then φ can be rewritten:

LDP for Projected Haar Distribution

Recall that given two probability measures μ and ν on the same space, their KL divergence, $H(\mu|\nu)$, is given by the negative of a $\log$ integral. Write $\beta_\ell = \beta s_\ell$ with $\beta = \sum_{q=1}^{2^j} \beta_q$ so that $\vec{s}$ lies in a 2^j-simplex. Write $\mu_{\vec{s}}$ for the probability measure giving uniform weight s_k to $I_k^{(j)}$ and let ν be normalized Lebesgue measure on the circle (i.e. $\mu_{\vec{s}}$ for the $\vec{s}$ with equal components, 2^{-j}). Then φ can be rewritten:

$$\varphi(\vec{\beta}) = \beta - 1 - \log(\beta) + H(\nu|\mu_{\vec{s}})$$

LDP for Projected Haar Distribution

Recall that given two probability measures μ and ν on the same space, their KL divergence, $H(\mu|\nu)$, is given by the negative of a $\log$ integral. Write $\beta_\ell = \beta s_\ell$ with $\beta = \sum_{q=1}^{2^j} \beta_q$ so that $\vec{s}$ lies in a 2^j-simplex. Write $\mu_{\vec{s}}$ for the probability measure giving uniform weight s_k to $I_k^{(j)}$ and let ν be normalized Lebesgue measure on the circle (i.e. $\mu_{\vec{s}}$ for the $\vec{s}$ with equal components, 2^{-j}). Then φ can be rewritten:

$$\varphi(\vec{\beta}) = \beta - 1 - \log(\beta) + H(\nu|\mu_{\vec{s}})$$

Note this is the sum of a function of β only and a function of the s's only.

LDP for Projected Haar Distribution

Recall that given two probability measures μ and ν on the same space, their KL divergence, $H(\mu|\nu)$, is given by the negative of a $\log$ integral. Write $\beta_\ell = \beta s_\ell$ with $\beta = \sum_{q=1}^{2^j} \beta_q$ so that $\vec{s}$ lies in a 2^j-simplex. Write $\mu_{\vec{s}}$ for the probability measure giving uniform weight s_k to $I_k^{(j)}$ and let ν be normalized Lebesgue measure on the circle (i.e. $\mu_{\vec{s}}$ for the $\vec{s}$ with equal components, 2^{-j}). Then φ can be rewritten:

$$\varphi(\vec{\beta}) = \beta - 1 - \log(\beta) + H(\nu|\mu_{\vec{s}})$$

Note this is the sum of a function of β only and a function of the s's only. This is a consequence of the fact that for independent exponential random variables, $\sum_{k=1}^N X_k$ is independent of $\{X_j / \sum_{k=1}^N X_k\}_{j=1}^N$. It makes the use of the contraction principle (which, in general, is already simple), extremely simple.

LDP for Projected Haar Distribution

For fixed λ's, let $\mathbb{P}_n^{(j)} = \pi_j^* \left(\mathbb{P}_n^{(\lambda)}\right)$. This is just the contraction of $\widetilde{\mathbb{P}}_n^{(j)}$ under the map $G(\vec{\beta}) \equiv \vec{\beta}/\beta$ from $\mathbb{R}^{2^j}$ to the 2^j–simplex. By the contraction principle and

LDP for Projected Haar Distribution

For fixed λ's, let $\mathbb{P}_n^{(j)} = \pi_j^* \left(\mathbb{P}_n^{(\lambda)}\right)$. This is just the contraction of $\widetilde{\mathbb{P}}_n^{(j)}$ under the map $G(\vec{\beta}) \equiv \vec{\beta}/\beta$ from $\mathbb{R}^{2^j}$ to the 2^j–simplex. By the contraction principle and

$$\inf_{\beta > 0} [\beta - 1 - \log(\beta)] = 0$$

LDP for Projected Haar Distribution

For fixed λ's, let $\mathbb{P}_n^{(j)} = \pi_j^* \left(\mathbb{P}_n^{(\lambda)}\right)$. This is just the contraction of $\widetilde{\mathbb{P}}_n^{(j)}$ under the map $G(\vec{\beta}) \equiv \vec{\beta}/\beta$ from $\mathbb{R}^{2^j}$ to the 2^j–simplex. By the contraction principle and

$$\inf_{\beta>0} [\beta - 1 - \log(\beta)] = 0$$

(as it must as the rate function, for averages of exponentials),

LDP for Projected Haar Distribution

For fixed λ's, let $\mathbb{P}_n^{(j)} = \pi_j^* \left(\mathbb{P}_n^{(\lambda)}\right)$. This is just the contraction of $\widetilde{\mathbb{P}}_n^{(j)}$ under the map $G(\vec{\beta}) \equiv \vec{\beta}/\beta$ from $\mathbb{R}^{2^j}$ to the 2^j–simplex. By the contraction principle and

$$\inf_{\beta>0} [\beta - 1 - \log(\beta)] = 0$$

(as it must as the rate function, for averages of exponentials), we see that for each fixed j, $\mathbb{P}_n^{(j)}$ obeys a LDP with speed n and rate function $H(\nu|\mu_{\vec{s}})$.
Given the projection theorem, the following completes the proof that the measure theory rate function is $H(\nu|\mu)$.

LDP for Projected Haar Distribution

For fixed λ's, let $\mathbb{P}_n^{(j)} = \pi_j^* \left(\mathbb{P}_n^{(\lambda)} \right)$. This is just the contraction of $\widetilde{\mathbb{P}}_n^{(j)}$ under the map $G(\vec{\beta}) \equiv \vec{\beta}/\beta$ from $\mathbb{R}^{2^j}$ to the 2^j–simplex. By the contraction principle and

$$\inf_{\beta > 0} [\beta - 1 - \log(\beta)] = 0$$

(as it must as the rate function, for averages of exponentials), we see that for each fixed j, $\mathbb{P}_n^{(j)}$ obeys a LDP with speed n and rate function $H(\nu|\mu_{\vec{s}})$.
Given the projection theorem, the following completes the proof that the measure theory rate function is $H(\nu|\mu)$.

Key Fact. Let μ be an arbitrary probability measure on $\partial\mathbb{D}$ and $\nu = \frac{d\theta}{2\pi}$. Then

LDP for Projected Haar Distribution

For fixed λ's, let $\mathbb{P}_n^{(j)} = \pi_j^*\left(\mathbb{P}_n^{(\lambda)}\right)$. This is just the contraction of $\widetilde{\mathbb{P}}_n^{(j)}$ under the map $G(\vec{\beta}) \equiv \vec{\beta}/\beta$ from $\mathbb{R}^{2^j}$ to the 2^j–simplex. By the contraction principle and

$$\inf_{\beta>0}[\beta - 1 - \log(\beta)] = 0$$

(as it must as the rate function, for averages of exponentials), we see that for each fixed j, $\mathbb{P}_n^{(j)}$ obeys a LDP with speed n and rate function $H(\nu|\mu_{\vec{s}})$.
Given the projection theorem, the following completes the proof that the measure theory rate function is $H(\nu|\mu)$.

Key Fact. Let μ be an arbitrary probability measure on $\partial\mathbb{D}$ and $\nu = \frac{d\theta}{2\pi}$. Then

$$\lim_{k\to\infty} H(\pi_j(\nu)|\pi_j(\mu)) = H(\nu|\mu)$$

Limit Theorem for KL Divergences

Before turning to the proof of the Key Fact, a quick remark: $\pi_j(\nu) = \nu$ for this ν.

Limit Theorem for KL Divergences

Before turning to the proof of the Key Fact, a quick remark: $\pi_j(\nu) = \nu$ for this ν. We write it this way because with a slight change in the proof, it holds for any ν (and μ).

Limit Theorem for KL Divergences

Before turning to the proof of the Key Fact, a quick remark: $\pi_j(\nu) = \nu$ for this ν. We write it this way because with a slight change in the proof, it holds for any ν (and μ). This extended version is needed for the Killip–Simon theorem and other cases where the limiting empirical measure is not unweighted Lebesgue measure.

Limit Theorem for KL Divergences

Before turning to the proof of the Key Fact, a quick remark: $\pi_j(\nu) = \nu$ for this ν. We write it this way because with a slight change in the proof, it holds for any ν (and μ). This extended version is needed for the Killip–Simon theorem and other cases where the limiting empirical measure is not unweighted Lebesgue measure.

We'll prove the limit result in two parts.

Limit Theorem for KL Divergences

Before turning to the proof of the Key Fact, a quick remark: $\pi_j(\nu) = \nu$ for this ν. We write it this way because with a slight change in the proof, it holds for any ν (and μ). This extended version is needed for the Killip–Simon theorem and other cases where the limiting empirical measure is not unweighted Lebesgue measure.

We'll prove the limit result in two parts. We'll prove a general upper bound: $H(\pi_j(\nu)|\pi_j(\mu)) \leq H(\nu|\mu)$.

Limit Theorem for KL Divergences

Before turning to the proof of the Key Fact, a quick remark: $\pi_j(\nu) = \nu$ for this ν. We write it this way because with a slight change in the proof, it holds for any ν (and μ). This extended version is needed for the Killip–Simon theorem and other cases where the limiting empirical measure is not unweighted Lebesgue measure.

We'll prove the limit result in two parts. We'll prove a general upper bound: $H(\pi_j(\nu)|\pi_j(\mu)) \leq H(\nu|\mu)$. (By slightly expanding the argument, one sees that $H(\pi_j(\nu)|\pi_j(\mu))$ is monotone increasing in j.)

Limit Theorem for KL Divergences

Before turning to the proof of the Key Fact, a quick remark: $\pi_j(\nu) = \nu$ for this ν. We write it this way because with a slight change in the proof, it holds for any ν (and μ). This extended version is needed for the Killip–Simon theorem and other cases where the limiting empirical measure is not unweighted Lebesgue measure.

We'll prove the limit result in two parts. We'll prove a general upper bound: $H(\pi_j(\nu)|\pi_j(\mu)) \leq H(\nu|\mu)$. (By slightly expanding the argument, one sees that $H(\pi_j(\nu)|\pi_j(\mu))$ is monotone increasing in j.)

The other direction – that $H(\nu|\mu) \leq \liminf H(\pi_j(\nu)|\pi_j(\mu))$ comes from weak convergence, $\lim \pi_j(\eta) = \eta$ (for any probability measure η) and the lower semi–continuity.

Limit Theorem for KL Divergences

To get the upper bound, note that by convexity of $y \mapsto -\log y$ and Jensen's inequality, for any positive function h and probability measure $d\eta(y)$, we have that

Limit Theorem for KL Divergences

To get the upper bound, note that by convexity of $y \mapsto -\log y$ and Jensen's inequality, for any positive function h and probability measure $d\eta(y)$, we have that

$$-\int \log h(y)\, d\eta(y) \geq -\log \left(\int h(y)\, d\eta(y) \right)$$

Limit Theorem for KL Divergences

To get the upper bound, note that by convexity of $y \mapsto -\log y$ and Jensen's inequality, for any positive function h and probability measure $d\eta(y)$, we have that

$$-\int \log h(y)\, d\eta(y) \geq -\log\left(\int h(y)\, d\eta(y)\right)$$

In just the same way that this implies that $H(\nu|\mu) \geq 0$, it implies that

Limit Theorem for KL Divergences

To get the upper bound, note that by convexity of $y \mapsto -\log y$ and Jensen's inequality, for any positive function h and probability measure $d\eta(y)$, we have that

$$-\int \log h(y)\, d\eta(y) \geq -\log\left(\int h(y)\, d\eta(y)\right)$$

In just the same way that this implies that $H(\nu|\mu) \geq 0$, it implies that

$$-\int_{I_k^{(j)}} \log(w(\theta))\, 2^j \frac{d\theta}{2\pi} \geq -\log\left(2^j \mu(I_k^{(j)})\right)$$

Limit Theorem for KL Divergences

To get the upper bound, note that by convexity of $y \mapsto -\log y$ and Jensen's inequality, for any positive function h and probability measure $d\eta(y)$, we have that

$$-\int \log h(y)\, d\eta(y) \geq -\log\left(\int h(y)\, d\eta(y)\right)$$

In just the same way that this implies that $H(\nu|\mu) \geq 0$, it implies that

$$-\int_{I_k^{(j)}} \log(w(\theta))\, 2^j \frac{d\theta}{2\pi} \geq -\log\left(2^j \mu(I_k^{(j)})\right)$$

Summing this yields the upper bound.

Killip Simon via LDP

The large deviation proof of the Killip–Simon sum rule is similar to the one I just presented for Szegő–Verblunsky sum rule with some changes and additions which we briefly describe.

Killip Simon via LDP

The large deviation proof of the Killip–Simon sum rule is similar to the one I just presented for Szegő–Verblunsky sum rule with some changes and additions which we briefly describe.

1. One uses GUE instead of CUE. Thus the measure on random $n \times n$ self–adjoint matrices has $\{\mathrm{Re} M_{ij}^{(n)}\}_{1 \leq i \leq j \leq n}$ and $\{\mathrm{Im} M_{ij}^{(n)}\}_{1 \leq i < j \leq n}$ Gaussian iid with mean zero and $\mathbb{E}([M_{ii}^{(n)}]^2) = n^{-1}$.

Killip Simon via LDP

2. The eigenvalue distribution has $\lambda_j \in \mathbb{R}$ with distribution

$$\left[\prod_{i<j} |\lambda_i - \lambda_j|^2\right] e^{-n \sum_{j=1}^{n} \lambda_j^2} \tag{4.1}$$

Killip Simon via LDP

2. The eigenvalue distribution has $\lambda_j \in \mathbb{R}$ with distribution

$$\left[\prod_{i<j} |\lambda_i - \lambda_j|^2\right] e^{-n \sum_{j=1}^{n} \lambda_j^2} \qquad (4.1)$$

so the empirical measure converges to the equilibrium measure in a quadratic external field, i.e. the minimizer for $-\int \log|x-y|\, d\mu(x)\, d\mu(y) + 2\int x^2\, d\mu(x)$.

Killip Simon via LDP

2 The eigenvalue distribution has $\lambda_j \in \mathbb{R}$ with distribution

$$\left[\prod_{i<j} |\lambda_i - \lambda_j|^2\right] e^{-n\sum_{j=1}^n \lambda_j^2} \qquad (4.1)$$

so the empirical measure converges to the equilibrium measure in a quadratic external field, i.e. the minimizer for $-\int \log|x-y|\,d\mu(x)\,d\mu(y) + 2\int x^2\,d\mu(x)$. It is well–known that this minimizer is the semicircle law $d\nu_0(x) \equiv \pi^{-1}(1-x^2)^{1/2}\chi_{[-1,1]}(x)dx$.

Killip Simon via LDP

2. The eigenvalue distribution has $\lambda_j \in \mathbb{R}$ with distribution

$$\left[\prod_{i<j} |\lambda_i - \lambda_j|^2\right] e^{-n \sum_{j=1}^n \lambda_j^2} \tag{4.1}$$

so the empirical measure converges to the equilibrium measure in a quadratic external field, i.e. the minimizer for $-\int \log|x-y|\,d\mu(x)\,d\mu(y) + 2\int x^2\,d\mu(x)$. It is well–known that this minimizer is the semicircle law $d\nu_0(x) \equiv \pi^{-1}(1-x^2)^{1/2}\chi_{[-1,1]}(x)dx$. To agree with the Killip–Simon notation, one rescales the matrix so the support is $[-2,2]$.

Killip Simon via LDP

3. The empirical measure converges to ν_0.

Killip Simon via LDP

3. The empirical measure converges to ν_0. By mimicking the argument above, the contribution of the part of the spectral measure on $[-2, 2]$ is just $H(\nu_0|\mu)$. Thus the weight in the Killip–Simon quasi–Szegő integral is exactly the Wigner semicircle weight.

Killip Simon via LDP

4. As we've seen, a single point in the measure, if the point is in the bulk, involves the increase of $H(\nu|\mu)$ due to the weight having a smaller integral.

Killip Simon via LDP

4. As we've seen, a single point in the measure, if the point is in the bulk, involves the increase of $H(\nu|\mu)$ due to the weight having a smaller integral. But if the point is outside $[-2, 2]$, there is a contribution due to the location, λ_0, of the eigenvalue.

Killip Simon via LDP

4. As we've seen, a single point in the measure, if the point is in the bulk, involves the increase of $H(\nu|\mu)$ due to the weight having a smaller integral. But if the point is outside $[-2, 2]$, there is a contribution due to the location, λ_0, of the eigenvalue. By looking at the $\log$ of the part of the weight depending on λ_0, one sees that the decrease in the eigenvalue density involves λ_0 interacting with n eigenvalues.

Killip Simon via LDP

4. As we've seen, a single point in the measure, if the point is in the bulk, involves the increase of $H(\nu|\mu)$ due to the weight having a smaller integral. But if the point is outside $[-2,2]$, there is a contribution due to the location, λ_0, of the eigenvalue. By looking at the $\log$ of the part of the weight depending on λ_0, one sees that the decrease in the eigenvalue density involves λ_0 interacting with n eigenvalues. The decrease is approximately $\exp(-nF(\lambda_0))$ where F is the potential in the quadratic external field in the equilibrium measure (this idea is due to Ben Arous, Dembo and Guionnet).

Killip Simon via LDP

4. As we've seen, a single point in the measure, if the point is in the bulk, involves the increase of $H(\nu|\mu)$ due to the weight having a smaller integral. But if the point is outside $[-2, 2]$, there is a contribution due to the location, λ_0, of the eigenvalue. By looking at the $\log$ of the part of the weight depending on λ_0, one sees that the decrease in the eigenvalue density involves λ_0 interacting with n eigenvalues. The decrease is approximately $\exp(-nF(\lambda_0))$ where F is the potential in the quadratic external field in the equilibrium measure (this idea is due to Ben Arous, Dembo and Guionnet). It is known that this function is the same as the Killip–Simon F.

Killip Simon via LDP

5. For finitely many eigenvalues outside $[-2, 2]$ you just get the sums of single costs since the interaction between eigenvalues is $\mathrm{O}(1)$, not $\mathrm{O}(n)$.

Killip Simon via LDP

5. For finitely many eigenvalues outside $[-2, 2]$ you just get the sums of single costs since the interaction between eigenvalues is $\mathrm{O}(1)$, not $\mathrm{O}(n)$. Handling infinitely many eigenvalues converging to ± 2 requires a careful use of projective limits.

Killip Simon via LDP

5. For finitely many eigenvalues outside $[-2, 2]$ you just get the sums of single costs since the interaction between eigenvalues is $\mathrm{O}(1)$, not $\mathrm{O}(n)$. Handling infinitely many eigenvalues converging to ± 2 requires a careful use of projective limits.
6. For the coefficient side, Killip–Nenciu is replaced by earlier results of Dumitriu–Edelman (whose work motivated Killip and Nenciu) who found the distribution of Jacobi parameters for GUE and GOE.

Killip Simon via LDP

5. For finitely many eigenvalues outside $[-2, 2]$ you just get the sums of single costs since the interaction between eigenvalues is $\mathrm{O}(1)$, not $\mathrm{O}(n)$. Handling infinitely many eigenvalues converging to ± 2 requires a careful use of projective limits.
6. For the coefficient side, Killip–Nenciu is replaced by earlier results of Dumitriu–Edelman (whose work motivated Killip and Nenciu) who found the distribution of Jacobi parameters for GUE and GOE. The $\{b_j\}_{j=1}^{n}$ are Gaussian (with $\mathrm{O}(n)$ widths leading to the b_j^2 term in the Killip–Simon sum rule).

Killip Simon via LDP

5. For finitely many eigenvalues outside $[-2, 2]$ you just get the sums of single costs since the interaction between eigenvalues is $\mathrm{O}(1)$, not $\mathrm{O}(n)$. Handling infinitely many eigenvalues converging to ± 2 requires a careful use of projective limits.
6. For the coefficient side, Killip–Nenciu is replaced by earlier results of Dumitriu–Edelman (whose work motivated Killip and Nenciu) who found the distribution of Jacobi parameters for GUE and GOE. The $\{b_j\}_{j=1}^{n}$ are Gaussian (with $\mathrm{O}(n)$ widths leading to the b_j^2 term in the Killip–Simon sum rule). The $\{a_j^2\}_{j=1}^{n-1}$ are gamma distributed, essentially behaving like sums of exponential random variables and so we get the $G(a_j)$ terms.

Killip Simon via LDP

5. For finitely many eigenvalues outside $[-2, 2]$ you just get the sums of single costs since the interaction between eigenvalues is $\mathrm{O}(1)$, not $\mathrm{O}(n)$. Handling infinitely many eigenvalues converging to ± 2 requires a careful use of projective limits.
6. For the coefficient side, Killip–Nenciu is replaced by earlier results of Dumitriu–Edelman (whose work motivated Killip and Nenciu) who found the distribution of Jacobi parameters for GUE and GOE. The $\{b_j\}_{j=1}^{n}$ are Gaussian (with $\mathrm{O}(n)$ widths leading to the b_j^2 term in the Killip–Simon sum rule). The $\{a_j^2\}_{j=1}^{n-1}$ are gamma distributed, essentially behaving like sums of exponential random variables and so we get the $G(a_j)$ terms. Thus G occurs in the sum rule as the rate function for suitable gamma distributions.

Killip Simon via LDP

7. There is a technical issue involving the equality of the two sides of the sum rule that we want to discuss, addressed in related ways by Gamboa-Rouault and by BSZ.

Killip Simon via LDP

7. There is a technical issue involving the equality of the two sides of the sum rule that we want to discuss, addressed in related ways by Gamboa-Rouault and by BSZ. The natural setting for the LDP for measures is the space, X', of all probability measures on $\mathbb{R}$, and for Jacobi parameters the Polish space $Y' \equiv [\mathbb{R} \times (0, \infty)]^\infty$ with finite sequences added to it.

Killip Simon via LDP

7 There is a technical issue involving the equality of the two sides of the sum rule that we want to discuss, addressed in related ways by Gamboa-Rouault and by BSZ. The natural setting for the LDP for measures is the space, X', of all probability measures on $\mathbb{R}$, and for Jacobi parameters the Polish space $Y' \equiv [\mathbb{R} \times (0, \infty)]^\infty$ with finite sequences added to it. The issue is that the inverse Jacobi map isn't defined for all measures but only those with all moments finite and, in general, this inverse map is many–to–one in certain cases where the measure has unbounded support.

Killip Simon via LDP

7. There is a technical issue involving the equality of the two sides of the sum rule that we want to discuss, addressed in related ways by Gamboa-Rouault and by BSZ. The natural setting for the LDP for measures is the space, X', of all probability measures on $\mathbb{R}$, and for Jacobi parameters the Polish space $Y' \equiv [\mathbb{R} \times (0, \infty)]^\infty$ with finite sequences added to it. The issue is that the inverse Jacobi map isn't defined for all measures but only those with all moments finite and, in general, this inverse map is many–to–one in certain cases where the measure has unbounded support. BSZ handle this by restricting to measures supported in $[-k-2, k+2]$

Killip Simon via LDP

7 There is a technical issue involving the equality of the two sides of the sum rule that we want to discuss, addressed in related ways by Gamboa-Rouault and by BSZ. The natural setting for the LDP for measures is the space, X', of all probability measures on $\mathbb{R}$, and for Jacobi parameters the Polish space $Y' \equiv [\mathbb{R} \times (0, \infty)]^\infty$ with finite sequences added to it. The issue is that the inverse Jacobi map isn't defined for all measures but only those with all moments finite and, in general, this inverse map is many–to–one in certain cases where the measure has unbounded support. BSZ handle this by restricting to measures supported in $[-k-2, k+2]$ and its image under the Jacobi map.

Mysteries Solved

We can now solve the mysteries:

1. *Why are there any positive combinations?*

Mysteries Solved

We can now solve the mysteries:

1. *Why are there any positive combinations?* This is the basic GNR theory of positive sum rules.

Mysteries Solved

We can now solve the mysteries:

1. *Why are there any positive combinations?* This is the basic GNR theory of positive sum rules.
2. *It is easy to understand the* $(4 - x^2)^{-1/2}\, dx$ *of the Szegő condition but where the heck does the* $(4 - x^2)^{1/2}\, dx$ *come from?*

Mysteries Solved

We can now solve the mysteries:

1. *Why are there any positive combinations?* This is the basic GNR theory of positive sum rules.
2. *It is easy to understand the* $(4 - x^2)^{-1/2}\,dx$ *of the Szegő condition but where the heck does the* $(4 - x^2)^{1/2}\,dx$ *come from?* This is the Wigner semi–circle law;

Mysteries Solved

We can now solve the mysteries:

1. *Why are there any positive combinations?* This is the basic GNR theory of positive sum rules.
2. *It is easy to understand the* $(4 - x^2)^{-1/2}\, dx$ *of the Szegő condition but where the heck does the* $(4 - x^2)^{1/2}\, dx$ *come from?* This is the Wigner semi–circle law; essentially the measure is the potential theory equilibrium measure in quadratic external field.

Mysteries Solved

We can now solve the mysteries:

1. *Why are there any positive combinations?* This is the basic GNR theory of positive sum rules.
2. *It is easy to understand the* $(4-x^2)^{-1/2}\,dx$ *of the Szegő condition but where the heck does the* $(4-x^2)^{1/2}\,dx$ *come from?* This is the Wigner semi–circle law; essentially the measure is the potential theory equilibrium measure in quadratic external field.
3. *What does the function*
$$G(a) = a^2 - 1 - \log(a^2)$$
mean?

Mysteries Solved

We can now solve the mysteries:

1. *Why are there any positive combinations?* This is the basic GNR theory of positive sum rules.
2. *It is easy to understand the* $(4 - x^2)^{-1/2}\,dx$ *of the Szegő condition but where the heck does the* $(4 - x^2)^{1/2}\,dx$ *come from?* This is the Wigner semi–circle law; essentially the measure is the potential theory equilibrium measure in quadratic external field.
3. *What does the function*
$$G(a) = a^2 - 1 - \log(a^2)$$
mean? As we've seen, this is the rate function for square roots of sums of exponential RVs.

Mysteries Solved

We can now solve the mysteries:

1. *Why are there any positive combinations?* This is the basic GNR theory of positive sum rules.
2. *It is easy to understand the* $(4-x^2)^{-1/2}\,dx$ *of the Szegő condition but where the heck does the* $(4-x^2)^{1/2}\,dx$ *come from?* This is the Wigner semi–circle law; essentially the measure is the potential theory equilibrium measure in quadratic external field.
3. *What does the function*
$$G(a) = a^2 - 1 - \log(a^2)$$
mean? As we've seen, this is the rate function for square roots of sums of exponential RVs.
4. *What does the function*
$$F(E) = \tfrac{1}{4}[\beta^2 - \beta^{-2} - \log\beta^4]; \quad E = \beta + \beta^{-1}$$
mean?

Mysteries Solved

We can now solve the mysteries:

1. *Why are there any positive combinations?* This is the basic GNR theory of positive sum rules.
2. *It is easy to understand the* $(4-x^2)^{-1/2}\,dx$ *of the Szegő condition but where the heck does the* $(4-x^2)^{1/2}\,dx$ *come from?* This is the Wigner semi–circle law; essentially the measure is the potential theory equilibrium measure in quadratic external field.
3. *What does the function*
$$G(a) = a^2 - 1 - \log(a^2)$$
mean? As we've seen, this is the rate function for square roots of sums of exponential RVs.
4. *What does the function*
$$F(E) = \tfrac{1}{4}[\beta^2 - \beta^{-2} - \log\beta^4]; \quad E = \beta + \beta^{-1}$$
mean? This is the Coulomb potential of the Wigner semi–circle distribution plus a quadratic external field.

Higher Order Sum Rules

In OPUC1, I found a sum rule involving $-\int (1-\cos(\theta)) \log(w(\theta)) \frac{d\theta}{2\pi}$ on the measure side and made a conjecture concerning

Higher Order Sum Rules

In OPUC1, I found a sum rule involving $-\int (1-\cos(\theta))\log(w(\theta))\,\frac{d\theta}{2\pi}$ on the measure side and made a conjecture concerning

$$-\int \log(w(\theta))\,d\eta(\theta)$$

Higher Order Sum Rules

In OPUC1, I found a sum rule involving $-\int (1-\cos(\theta)) \log(w(\theta)) \frac{d\theta}{2\pi}$ on the measure side and made a conjecture concerning

$$-\int \log(w(\theta))\, d\eta(\theta)$$

where

$$d\eta(\theta) = Z^{-1} \prod_{j=1}^{k} (1-\cos(\theta-\theta_j))^{m_j}\, d\theta$$

Higher Order Sum Rules

In OPUC1, I found a sum rule involving $-\int(1-\cos(\theta))\log(w(\theta))\,\frac{d\theta}{2\pi}$ on the measure side and made a conjecture concerning

$$-\int \log(w(\theta))\,d\eta(\theta)$$

where

$$d\eta(\theta) = Z^{-1}\prod_{j=1}^{k}(1-\cos(\theta-\theta_j))^{m_j}\,d\theta$$

where Z is a normalization factor to make $d\eta$ into a probability measure.

Higher Order Sum Rules

In OPUC1, I found a sum rule involving $-\int(1-\cos(\theta))\log(w(\theta))\,\frac{d\theta}{2\pi}$ on the measure side and made a conjecture concerning

$$-\int \log(w(\theta))\,d\eta(\theta)$$

where

$$d\eta(\theta) = Z^{-1}\prod_{j=1}^{k}(1-\cos(\theta-\theta_j))^{m_j}\,d\theta$$

where Z is a normalization factor to make $d\eta$ into a probability measure. There developed a huge literature on these so called higher order sum rules for OPUC and OPRL including papers by Denissov, Golinskii, Kupin, Laptev et al, Lukic and Nazarov et al.

Higher Order Sum Rules

The key to understanding such sum rules (for OPUC) in the context of large deviations is to replace Haar measure, $d\mathbb{P}_N$, by

Higher Order Sum Rules

The key to understanding such sum rules (for OPUC) in the context of large deviations is to replace Haar measure, $d\mathbb{P}_N$, by

$$Z_N^{-1} \exp\left[-N \sum_{j=1}^{N} V(\lambda_j)\right] d\mathbb{P}_N$$

Higher Order Sum Rules

The key to understanding such sum rules (for OPUC) in the context of large deviations is to replace Haar measure, $d\mathbb{P}_N$, by

$$Z_N^{-1} \exp\left[-N\sum_{j=1}^{N} V(\lambda_j)\right] d\mathbb{P}_N$$

where V is a function on $\partial\mathbb{D}$ and $\{\lambda_j\}_{j=1}^{N}$ are the eigenvalues.

Higher Order Sum Rules

The key to understanding such sum rules (for OPUC) in the context of large deviations is to replace Haar measure, $d\mathbb{P}_N$, by

$$Z_N^{-1} \exp\left[-N \sum_{j=1}^{N} V(\lambda_j)\right] d\mathbb{P}_N$$

where V is a function on $\partial\mathbb{D}$ and $\{\lambda_j\}_{j=1}^{N}$ are the eigenvalues. It is well known in the random matrix literature that when V is nice enough, we will get $d\eta$ as the empirical measure if

Higher Order Sum Rules

The key to understanding such sum rules (for OPUC) in the context of large deviations is to replace Haar measure, $d\mathbb{P}_N$, by

$$Z_N^{-1} \exp\left[-N \sum_{j=1}^{N} V(\lambda_j)\right] d\mathbb{P}_N$$

where V is a function on $\partial\mathbb{D}$ and $\{\lambda_j\}_{j=1}^N$ are the eigenvalues. It is well known in the random matrix literature that when V is nice enough, we will get $d\eta$ as the empirical measure if

$$V(e^{i\theta}) = 2 \int \log |e^{i\theta} - e^{i\psi}| \, d\eta(\psi)$$

Higher Order Sum Rules

In a forthcoming paper BSZ study this when $d\eta$ is given as above.

Higher Order Sum Rules

In a forthcoming paper BSZ study this when $d\eta$ is given as above. In the cases we study, $V(e^{i\theta})$ is a finite linear combination of $\cos(m\theta)$.

Higher Order Sum Rules

In a forthcoming paper BSZ study this when $d\eta$ is given as above. In the cases we study, $V(e^{i\theta})$ is a finite linear combination of $\cos(m\theta)$. In terms of U, if $e^{i\theta_j}$ are the eigenvalues, $\sum_{j=1}^{n} \cos(m\theta_j) = \mathrm{Re}(\mathrm{Tr}(U^m))$ which one can write in terms of Verblunsky coefficients using the CMV (or the GGT) representation of U.

Higher Order Sum Rules

In a forthcoming paper BSZ study this when $d\eta$ is given as above. In the cases we study, $V(e^{i\theta})$ is a finite linear combination of $\cos(m\theta)$. In terms of U, if $e^{i\theta_j}$ are the eigenvalues, $\sum_{j=1}^{n} \cos(m\theta_j) = \mathrm{Re}(\mathrm{Tr}(U^m))$ which one can write in terms of Verblunsky coefficients using the CMV (or the GGT) representation of U. We obtain a large deviations proof of the $(1 - \cos(\theta))$ sum rule of Simon and the gems of Simon–Zlatoš.

Higher Order Sum Rules

In a forthcoming paper BSZ study this when $d\eta$ is given as above. In the cases we study, $V(e^{i\theta})$ is a finite linear combination of $\cos(m\theta)$. In terms of U, if $e^{i\theta_j}$ are the eigenvalues, $\sum_{j=1}^{n} \cos(m\theta_j) = \mathrm{Re}(\mathrm{Tr}(U^m))$ which one can write in terms of Verblunsky coefficients using the CMV (or the GGT) representation of U. We obtain a large deviations proof of the $(1 - \cos(\theta))$ sum rule of Simon and the gems of Simon–Zlatoš. In addition, we prove a partial special case of a conjecture of Lukic that replaces a wrong conjecture of Simon, providing evidence for Lukic's conjecture.

Higher Order Sum Rules

GNR have a paper that discusses in some detail the case $V(\theta) = \cos(\theta)$ where the random matrix model has been studied by Gross–Witten whose names GNR apply to the model.

Higher Order Sum Rules

GNR have a paper that discusses in some detail the case $V(\theta) = \cos(\theta)$ where the random matrix model has been studied by Gross–Witten whose names GNR apply to the model. They note that formally the large deviations argument leads to a sum rule but for technical reasons, they aren't able to provide a proof.

Higher Order Sum Rules

GNR have a paper that discusses in some detail the case $V(\theta) = \cos(\theta)$ where the random matrix model has been studied by Gross–Witten whose names GNR apply to the model. They note that formally the large deviations argument leads to a sum rule but for technical reasons, they aren't able to provide a proof. By using some results from the theory of OPUC, we do prove sum rules in this and the other cases.

Finite Gap OPUC

There has been very little work on Killip–Simon type theorems for finite gap sets in OPUC.

Finite Gap OPUC

There has been very little work on Killip–Simon type theorems for finite gap sets in OPUC. In a recent preprint, GNR obtain a sum rule and gem for $\mathfrak{e} = \{e^{i\theta} \,|\, \alpha \leq \theta \leq 2\pi - \alpha\}$ for $0 < \alpha < \pi$.

Finite Gap OPUC

There has been very little work on Killip–Simon type theorems for finite gap sets in OPUC. In a recent preprint, GNR obtain a sum rule and gem for $\mathfrak{e} = \{e^{i\theta} \,|\, \alpha \leq \theta \leq 2\pi - \alpha\}$ for $0 < \alpha < \pi$. For real α, the Verblunsky side has the expected $\sum |\alpha_j - a|^2$ form but for general α, it has the form $\sum |\gamma_j - a|^2$ where γ_j is a non–local function of the α's.

Finite Gap OPUC

There has been very little work on Killip–Simon type theorems for finite gap sets in OPUC. In a recent preprint, GNR obtain a sum rule and gem for $\mathfrak{e} = \{e^{i\theta} \,|\, \alpha \leq \theta \leq 2\pi - \alpha\}$ for $0 < \alpha < \pi$. For real α, the Verblunsky side has the expected $\sum |\alpha_j - a|^2$ form but for general α, it has the form $\sum |\gamma_j - a|^2$ where γ_j is a non–local function of the α's. In particular, it is not clear if the finiteness of their Verblunsky side only depends on the behavior near $j = \infty$.

Finite Gap OPUC

There has been very little work on Killip–Simon type theorems for finite gap sets in OPUC. In a recent preprint, GNR obtain a sum rule and gem for $\mathfrak{e} = \{e^{i\theta} \mid \alpha \leq \theta \leq 2\pi - \alpha\}$ for $0 < \alpha < \pi$. For real α, the Verblunsky side has the expected $\sum |\alpha_j - a|^2$ form but for general α, it has the form $\sum |\gamma_j - a|^2$ where γ_j is a non–local function of the α's. In particular, it is not clear if the finiteness of their Verblunsky side only depends on the behavior near $j = \infty$. At least for the real case, it would be interesting to get the sum rule via the Poisson–Jensen methods used in the original Killip–Simon proof.

Finite Gap OPUC

There has been very little work on Killip–Simon type theorems for finite gap sets in OPUC. In a recent preprint, GNR obtain a sum rule and gem for $\mathfrak{e} = \{e^{i\theta} \,|\, \alpha \leq \theta \leq 2\pi - \alpha\}$ for $0 < \alpha < \pi$. For real α, the Verblunsky side has the expected $\sum |\alpha_j - a|^2$ form but for general α, it has the form $\sum |\gamma_j - a|^2$ where γ_j is a non–local function of the α's. In particular, it is not clear if the finiteness of their Verblunsky side only depends on the behavior near $j = \infty$. At least for the real case, it would be interesting to get the sum rule via the Poisson–Jensen methods used in the original Killip–Simon proof. It would also be interesting to understand the γ_j's in a more conventional setting.

Finite Gap OPUC

There has been very little work on Killip–Simon type theorems for finite gap sets in OPUC. In a recent preprint, GNR obtain a sum rule and gem for $\mathfrak{e} = \{e^{i\theta} \,|\, \alpha \leq \theta \leq 2\pi - \alpha\}$ for $0 < \alpha < \pi$. For real α, the Verblunsky side has the expected $\sum |\alpha_j - a|^2$ form but for general α, it has the form $\sum |\gamma_j - a|^2$ where γ_j is a non–local function of the α's. In particular, it is not clear if the finiteness of their Verblunsky side only depends on the behavior near $j = \infty$. At least for the real case, it would be interesting to get the sum rule via the Poisson–Jensen methods used in the original Killip–Simon proof. It would also be interesting to understand the γ_j's in a more conventional setting.

Understanding perturbations of periodic and the more general finite gap OPUC remains open.

Half Line Schródinger Operators

Finally, we note that Killip–Simon have proven a sum rule and gem for half–line Schrödinger operators when $V \in L^2((0,\infty);dx)$.

Half Line Schródinger Operators

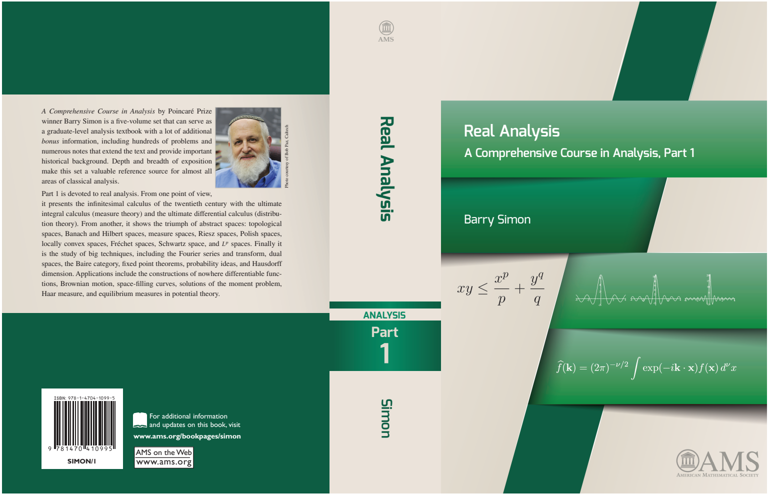
Finally, we note that Killip–Simon have proven a sum rule and gem for half–line Schrödinger operators when $V \in L^2((0,\infty); dx)$. It would be very interesting to find a large deviation proof of this result.

Half Line Schródinger Operators

Finally, we note that Killip–Simon have proven a sum rule and gem for half–line Schrödinger operators when $V \in L^2((0,\infty);dx)$. It would be very interesting to find a large deviation proof of this result. In particular, what is the analog of random matrix models for the study of Schrödinger operators?

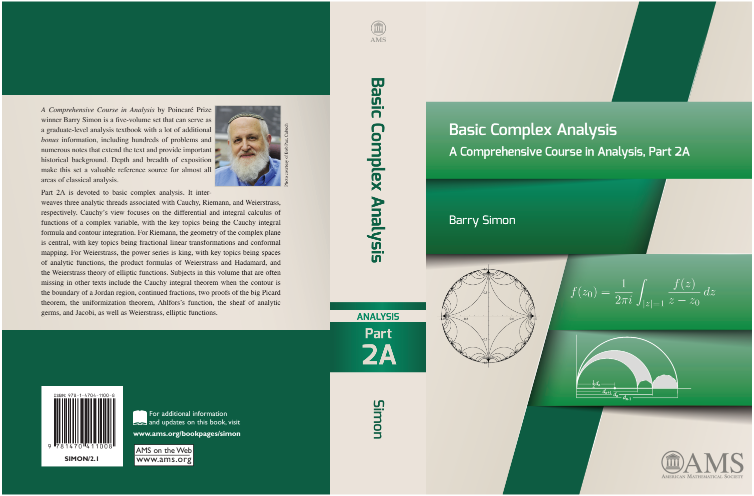
And Now a Word from Our Sponsor

And Now a Word from Our Sponsor



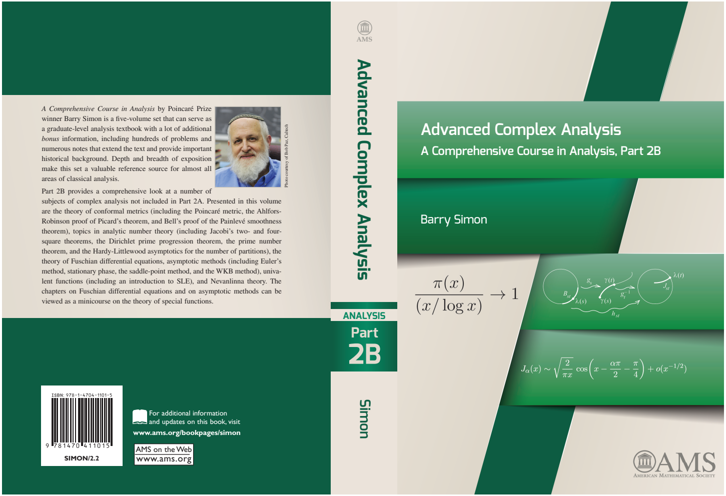
Google *simon comprehensive course*

And Now a Word from Our Sponsor



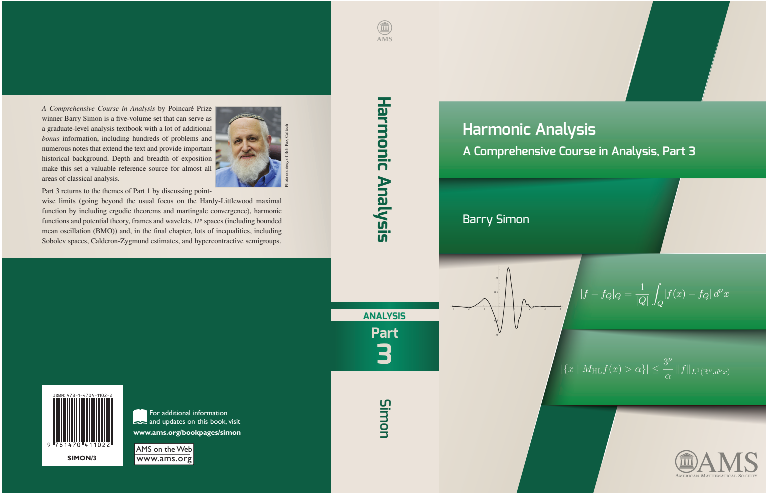
Google *simon comprehensive course*

And Now a Word from Our Sponsor



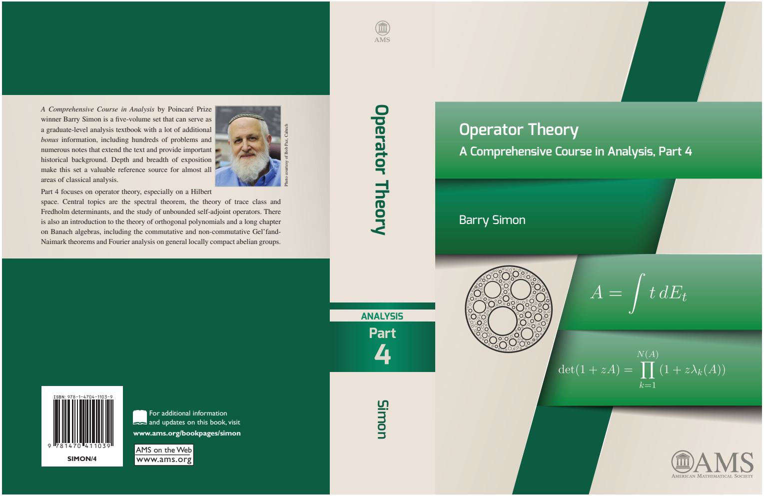
Google *simon comprehensive course*

And Now a Word from Our Sponsor



Google *simon comprehensive course*

And Now a Word from Our Sponsor



Google *simon comprehensive course*