Gap Labelling for Periodic Jacobi Matrices on Trees

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After discussing periodic Jacobi matrices on trees and this theorem, I'll discuss the more general context of gap labelling and the historical model of Floquet theory for Hill's equation. Then after describing the tools we'll need, I'll focus on a miraculous formula that will prove Sunada's theorem.

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After discussing periodic Jacobi matrices on trees and this theorem, I'll discuss the more general context of gap labelling and the historical model of Floquet theory for Hill's equation. Then after describing the tools we'll need, I'll focus on a miraculous formula that will prove Sunada's theorem. If there is time I'll discuss another result for which our new formula provides a new proof.

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We will assign an orientation for each edge, e, using $\check{e}$ for the oppositely directed edge. $\sigma(e)$ is the initial vertex and $\tau(e)$ the final of the directed edge e, so for example, $\sigma(\check{e}) = \tau(e)$. We let $\tilde{\mathbb{E}}$ denote the set of all edges with arbitrary assigned orientation so that $\#(\tilde{\mathbb{E}}) = 2\#(\mathbb{E})$.

Jacobi Matrices on Graphs

A *Jacobi matrix* on Γ is defined by *Jacobi parameters*, i.e. a potential, $b(v) \in \mathbb{R}$, to each vertex and coupling, $a(e) = a(\check{e}) > 0$, to each edge.

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The Jacobi matrix is indexed by pairs of vertices and defines an operator on $\ell^2(\mathbb{V}(\Gamma))$ by taking

$$H_{vw} = \begin{cases} b(v), & \text{if } v = w \\ a(e), & \text{if } (vw) = e \text{ an edge in } \tilde{\mathbb{E}}(\Gamma) \\ 0, & \text{otherwise} \end{cases}$$

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If the graph has n-vertices, $\{1, \ldots, n\}$ with edges between j and $j+1$, this is a classical tridiagonal Jacobi matrix. If the graph has vertex set $\mathbb{Z}$ with neighboring edges, we get a classical (doubly) infinite Jacobi matrix.

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Let $\mathcal{T}$ be the universal cover of Γ. It is a tree. There is a cover map $\pi : \mathcal{T} \to \Gamma$ and a family of deck transformations isomorphic to $\mathbb{F}_\ell$ which acts transitively on each $\pi^{-1}(v)$ for each $v \in \mathbb{V}(\Gamma)$. Given a Jabobi matrix, J_Γ, on Γ, with Jacobi parameters, b and a, there is a unique lift to Jacobi parameters on $\mathcal{T}$ given by $b(\tilde{v}) = b(\pi(\tilde{v})), a(\tilde{e}) = a(\pi(\tilde{e}))$.

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Two simple and canonical example are where first Γ is a single cycle with p vertices and second where Γ has a two vertices with d edges connecting them. In the first case $H_{\mathcal{T}}$ is a conventional periodic Jacobi matrix of period p (which is where our notion of period comes from) - a subject on which there is truly enormous history and literature which we will discuss in part below.

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The model of the homogeneous tree and these models more generally are connected to modular forms and so this subject is to interest to mathematical physicists, spectral theorists and number theorists.

The DOS and Gap Labelling

Deck transformations induce unitary maps on $\ell^2(\mathbb{V}(\mathcal{T}))$ which commute with $H_{\mathcal{T}}$. In particular, for every $v \in V(\Gamma)$, the spectral measure, $d\mu_{\tilde{v}}$, for $H_{\mathcal{T}}$ and $\tilde{v}$ are the same for all $\tilde{v} \in \mathbb{V}(\mathcal{T})$ with $\pi(\tilde{v}) = v$.

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One defines the *density of states* measure, $dk(E)$ (and *integrated density of states*, aka IDS, $k(E) = dk((-\infty, E))$), by

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The big theorem of Sunada (1992), called *gap labelling*, says the following

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Sunada first noted that in the general context of operators on trees of the type we looked at, the sum of diagonal matrix elements, one from each equivalence class of vertices, for operators commuting with our action of $\mathbb{F}_\ell$ (a natural von Neumann algebra) defines a natural normalized trace.

The Sunada Proof

$k(E)$ is just this normalized trace applied to the spectral projection $P_{(-\infty,E)}(H_{\mathcal{T}})$. This projection lies in the C^*-algebra generated by $H_{\mathcal{T}}$ if the projection is of the form $f(H_{\mathcal{T}})$ for a continuous function f and this is true if E is in a gap.

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The Pimsner-Voiculescu theorem is proven by them by using an exact sequence of K-theory groups. While Effros and others have a simpler proof of their theorem, there is no elementary proof. This ends the introduction. My goal in the rest of the talk is our new proof of Sunada's gap labelling theorem which is so elementary we think of it as "the proof from the book"

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$$-u''(x) + V(x)u(x) = \lambda u(x)$$

where λ is a (usually) real parameter and V is a real periodic function, i.e. $V(x+L) = V(x)$ for all real x and some $L \geq 0$.

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This is, of course, the $1D$ Schrödinger equation but Hill (the rare great 19^{th} century American scientist (he worked at Rutgers, the NJ state university, whose math building is named after him) was 40 years before Schrödinger and was studying perturbations of the moon orbit.

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$$-a_{n+1}u(n+1) + b_n u(n) + a_n u(n-1) = \lambda u(n)$$

which is of course a periodic Jacobi matrix on a tree where the tree is $\mathbb{Z}$ (and if the period is p, the finite graph is a cyclic of length p.)

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We remark there is a version of Floquet theory and gap labelling for such periodic operators on $\mathbb{Z}^\nu$ - an abelian extension of the $1D$ theory as opposed to the tree theory which is a non-Abelian extension.

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where $T(kp;\lambda) = T(p;\lambda)^k$ because of periodicity. By a simple calculation, $\det(T(p;\lambda)) = 1$, so the two eigenvalues of $T(p;\lambda)$,

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$$T(m; \lambda) \begin{pmatrix} u(1) \\ a(0)u(0) \end{pmatrix} = \begin{pmatrix} u(m+1) \\ a(m)u(m) \end{pmatrix}$$

where $T(kp; \lambda) = T(p; \lambda)^k$ because of periodicity. By a simple calculation, $\det(T(p; \lambda)) = 1$, so the two eigenvalues of $T(p; \lambda)$, called *Floquet eigenvalues* denoted $\alpha_\pm(\lambda)$, obey $\alpha_-(\lambda) = \alpha_+(\lambda)^{-1}$.

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The solutions of the difference equation on $\mathbb{Z}$, $u_\pm$, with initial conditions the eigenfunctions of $T(p; \lambda)$ are called *Floquet solutions*.

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Almost Periodic Gap Labelling

Before leaving the $1D$ case, we should mention a result going back to the 1980's that popularized the name "gap labelling" in a related but distinct context, namely for almost periodic classical Jacobi matrices, where a_n and b_n are almost periodic rather than periodic.

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The point is that the set of possible values is dense in $[0, 1]$ so that if all (or many) values occur, the spectrum is a Cantor set. The famous *ten martini problem* (which is a theorem of Avila-Jitomirskaya with important partial results by others, especially Puig) is that for all $\beta \neq 0$ and all irrational α, the almost Mathieu spectrum is a Cantor set.

m-functions

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Dropping the edge $\tilde{e}$ from $\mathcal{T}$ breaks $\ell^2(\mathcal{T}) = \ell^2(\mathcal{T}_{\tilde{e}-}) \oplus \ell^2(\mathcal{T}_{\tilde{e}+})$ where $\ell^2(\mathcal{T}_{\tilde{e}+})$ is the subspace with $\tau(\tilde{e})$ and $\ell^2(\mathcal{T}_{\tilde{e}-})$ is the subspace with $\sigma(\tilde{e})$.

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$$m_e(z) = \langle \delta_{\tau(\tilde{e})}, (H_{\tilde{e}}^{+} - z)^{-1} \delta_{\tau(\tilde{e})} \rangle$$

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Schur Complements

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$$\frac{1}{G_u(z)} = -z + b_u - \sum_{f\in\tilde{E}\colon\, \sigma(f)=u} a_f^2 m_f(z)$$

$$\frac{1}{m_f(z)} = -z + b_u - \sum_{\substack{f'\in\tilde{E}, f'\neq\check{f} \\ \sigma(f')=\tau(f)}} a_{f'}^2 m_{f'}(z)$$

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which implies for any $e \in \tilde{E}$ that

$$G_{\sigma(e)} = \frac{1}{m_{\check{e}}^{-1} - a_e^2 m_e} = \frac{m_{\check{e}}}{1 - a_e^2 m_e m_{\check{e}}}$$

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These equations imply the important result (of Chomsky-Schützenberger (1963) - that Chomsky!!), that G and m are algebraic functions , which implies (Avni-Breuer-Simon) that there is no signular continuous spectrum.

The Floquet Function

The key to the new proof is a remarkable equality involving a new function we introduced and called the *Floquet function* defined by

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Careful analysis of the imaginary part of the $\log$ in the gap, shows that the imaginary part of this last integral if $-p\pi k(E_0)$ is E_0 is a point in a gap.

A Remarkable Formula

The key to our proof is the following formula involving Φ, G and m (or and Q)

$$\Phi(z) = \frac{\prod_{e \in E(\mathcal{G})} Q_e(z)}{\prod_{u \in V(\mathcal{G})} G_u(z)}$$

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Using the Schur complement formula for G_u^{-1}, one gets a formula for $(\log(G_u))'$ and from the definition of Q_e a formula for $(\log(Q_e))'$ from which one sees that $\sum_{e\in E}(\log(Q_e))' = \sum_{u\in V}[-G_u + (\log(G_u))']$ so that

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The left side is just $[\log(\Psi)]'$ and the right is $[\log(\Phi)]'$ proving the magic formula.

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Statement

Given an eigenvalue, λ, define $X_1(\lambda)$ to be the set of vertices, $v \in V$, so that for some $\tilde{v}$ with $\pi(\tilde{v}) = v$ there is some eigenfunction ψ associated to λ, with $\psi(\tilde{v}) \neq 0$. Define $\partial X_1(\lambda)$ to be those $v \in V$ not in $X_1(\lambda)$ but neighbors of points in $X_1(\lambda)$, and we let $E(\lambda)$ be the set of edges with both endpoints in $X_1(\lambda)$.

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A consequence of this theorem is that if Γ has a fixed degree (equivalently, $\mathcal{T}$ does), then $H_{\mathcal{T}}$ has no point spectrum.

Remarks

The original proof of Aomoto is opaque and our new proof is simpler but there is a 2022 paper of Banks, Garza-Vargas and Mukhejee with a particularly elegant way to understand point spectrum including a lovely proof of the index theorem.

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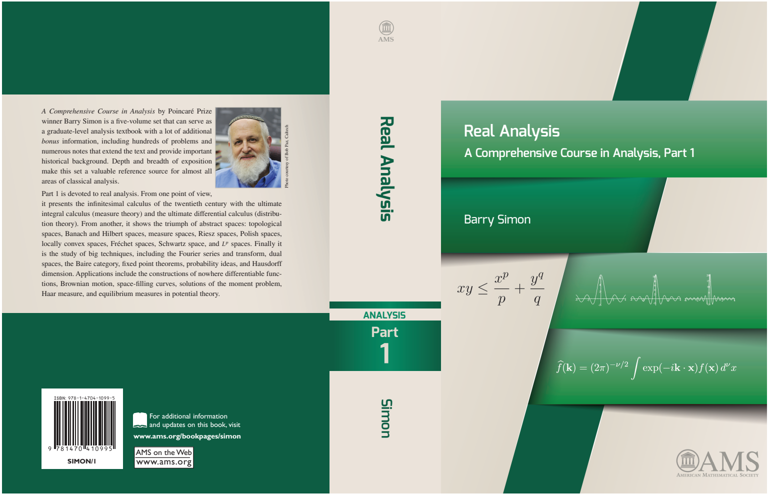

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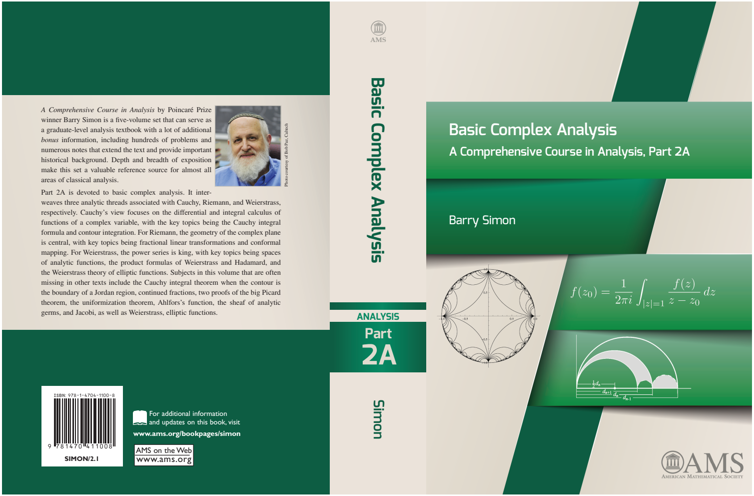

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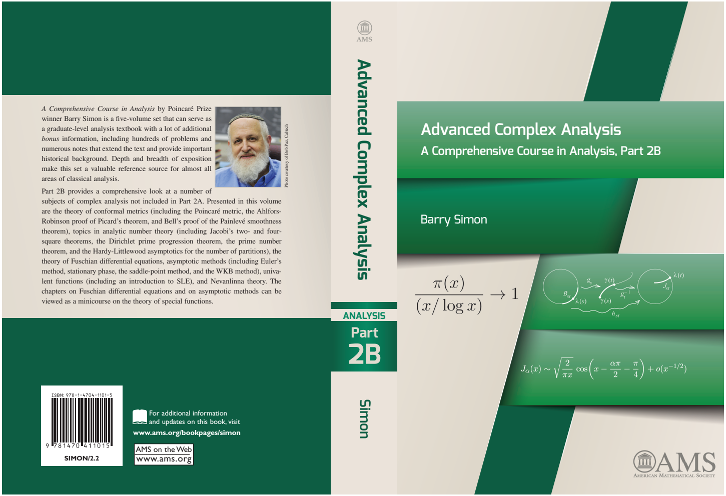

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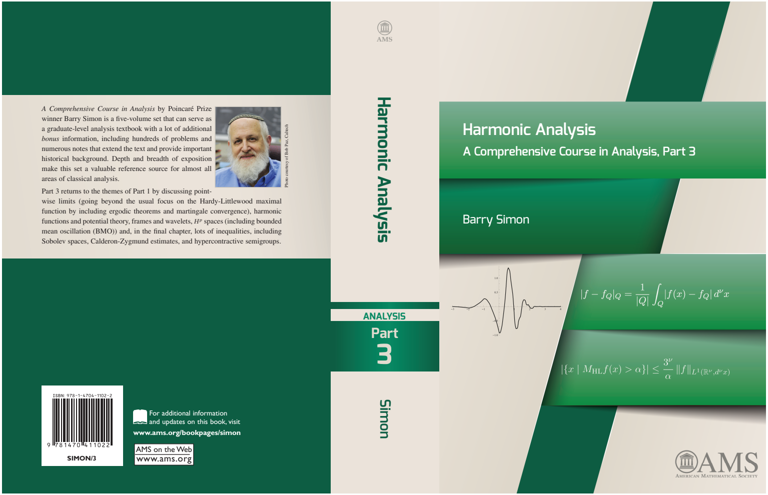

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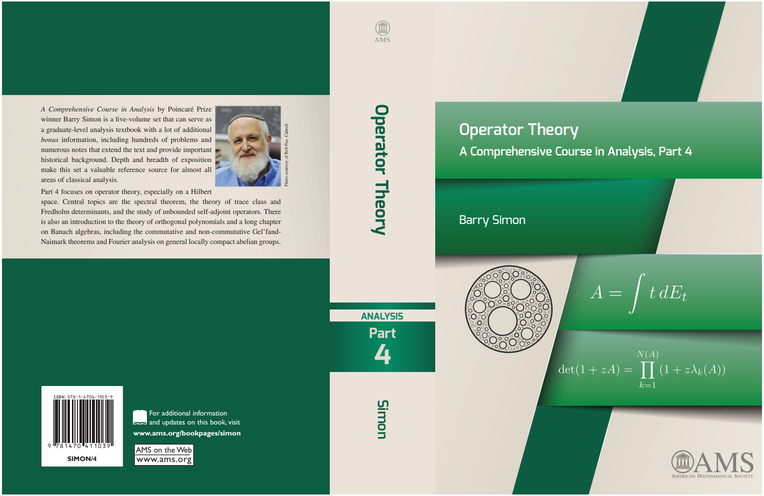

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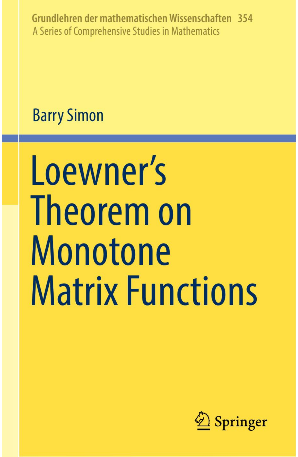

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