# The Tale of a Wrong Conjecture: Borg's Theorem for Periodic Jacobi Matrices on Trees

Barry Simon
IBM Professor of Mathematics and Theoretical Physics, Emeritus
California Institute of Technology
Pasadena, CA, U.S.A.

# The Tale of a Wrong Conjecture: Borg's Theorem for Periodic Jacobi Matrices on Trees

Barry Simon
IBM Professor of Mathematics and Theoretical Physics, Emeritus
California Institute of Technology
Pasadena, CA, U.S.A.

Joint Work with Nir Avni (Northwestern) and Jonathan Breuer (HUJI)

# Parameter Counting

It is always interesting to figure out how rare a rare thing is.

# Parameter Counting



It is always interesting to figure out how rare a rare thing is. If we look at points in $\mathbb{R}^n$, most will have unequal coordinates.

# Parameter Counting

It is always interesting to figure out how rare a rare thing is. If we look at points in $\mathbb{R}^n$, most will have unequal coordinates. We can ask the codimension of the set of points, $\{\mathbf{x} \mid x_i = x_j \text{ for some } i \neq j\}$, with not all distinct coordinates,

# Parameter Counting

It is always interesting to figure out how rare a rare thing is. If we look at points in $\mathbb{R}^n$, most will have unequal coordinates. We can ask the codimension of the set of points, $\{\mathbf{x} \mid x_i = x_j \text{ for some } i \neq j\}$, with not all distinct coordinates, clearly codimension $1$ as a finite union of hyperplanes.

# Parameter Counting



It is always interesting to figure out how rare a rare thing is. If we look at points in $\mathbb{R}^n$, most will have unequal coordinates. We can ask the codimension of the set of points, $\{\mathbf{x} \mid x_i = x_j \text{ for some } i \neq j\}$, with not all distinct coordinates, clearly codimension $1$ as a finite union of hyperplanes. We could quickly figure out what the answer is by looking at the simplest case, $n = 2$ where it is a single line

# Parameter Counting



It is always interesting to figure out how rare a rare thing is. If we look at points in $\mathbb{R}^n$, most will have unequal coordinates. We can ask the codimension of the set of points, $\{\mathbf{x} \mid x_i = x_j \text{ for some } i \neq j\}$, with not all distinct coordinates, clearly codimension $1$ as a finite union of hyperplanes. We could quickly figure out what the answer is by looking at the simplest case, $n = 2$ where it is a single line and arguing that coincidence happens a pair at time so that should be the general case.

# Parameter Counting

It is always interesting to figure out how rare a rare thing is. If we look at points in $\mathbb{R}^n$, most will have unequal coordinates. We can ask the codimension of the set of points, $\{\mathbf{x} \mid x_i = x_j \text{ for some } i \neq j\}$, with not all distinct coordinates, clearly codimension $1$ as a finite union of hyperplanes. We could quickly figure out what the answer is by looking at the simplest case, $n = 2$ where it is a single line and arguing that coincidence happens a pair at time so that should be the general case.

Harder, but not a lot more, is looking at self-adjoint matrices and asking for the codimension of those with a degenerate eigenvalue.

# Parameter Counting

It is always interesting to figure out how rare a rare thing is. If we look at points in $\mathbb{R}^n$, most will have unequal coordinates. We can ask the codimension of the set of points, $\{\mathbf{x} \mid x_i = x_j \text{ for some } i \neq j\}$, with not all distinct coordinates, clearly codimension $1$ as a finite union of hyperplanes. We could quickly figure out what the answer is by looking at the simplest case, $n = 2$ where it is a single line and arguing that coincidence happens a pair at time so that should be the general case.

Harder, but not a lot more, is looking at self-adjoint matrices and asking for the codimension of those with a degenerate eigenvalue. Again we start with the simplest case, $2 \times 2$ matrices.

# Parameter Counting



If we write the general $2 \times 2$ self-adjoint matrix as

$$\begin{pmatrix} a+b & c \\ \bar{c} & a-b \end{pmatrix}$$

with $a, b \in \mathbb{R}, c \in \mathbb{C}$,

# Parameter Counting

If we write the general $2 \times 2$ self-adjoint matrix as

$$\begin{pmatrix} a+b & c \\ \bar{c} & a-b \end{pmatrix}$$

with $a, b \in \mathbb{R}, c \in \mathbb{C}$, we see the set of such matrices has real dimension $4$ and those with only one eigenvalue (so $b = c = 0$) dimension $1$ so codimension $3$.

# Parameter Counting

If we write the general $2 \times 2$ self-adjoint matrix as

$$\begin{pmatrix} a+b & c \\ \bar{c} & a-b \end{pmatrix}$$

with $a, b \in \mathbb{R}, c \in \mathbb{C}$, we see the set of such matrices has real dimension $4$ and those with only one eigenvalue (so $b = c = 0$) dimension $1$ so codimension $3$. If we only look at real matrices, $c$ is real so the real codimension is $2$.

# Parameter Counting

If we write the general $2 \times 2$ self-adjoint matrix as

$$\begin{pmatrix} a+b & c \\ \bar{c} & a-b \end{pmatrix}$$

with $a, b \in \mathbb{R}, c \in \mathbb{C}$, we see the set of such matrices has real dimension $4$ and those with only one eigenvalue (so $b = c = 0$) dimension $1$ so codimension $3$. If we only look at real matrices, $c$ is real so the real codimension is $2$. Again, eigenvalue coincidences occur in pairs, so we expect those are the right codimensions in general.

# Wigner-von Neumann and Weyl



That this expectation is correct is a famous theorem of Wigner and von-Neumann published in 1929.

# Wigner-von Neumann and Weyl

That this expectation is correct is a famous theorem of Wigner and von-Neumann published in 1929. They went to the same high school and were lifelong friends and, while postdocs, published two back-to-back papers in Physikalische Zeitschrift

# Wigner-von Neumann and Weyl



That this expectation is correct is a famous theorem of Wigner and von-Neumann published in 1929. They went to the same high school and were lifelong friends and, while postdocs, published two back-to-back papers in Physikalische Zeitschrift (the first on embedded eigenvalues is perhaps more famous among spectral theorists;

# Wigner-von Neumann and Weyl



That this expectation is correct is a famous theorem of Wigner and von-Neumann published in 1929. They went to the same high school and were lifelong friends and, while postdocs, published two back-to-back papers in Physikalische Zeitschrift (the first on embedded eigenvalues is perhaps more famous among spectral theorists; the second with the theorem below is much more quoted in the physics community at large).

# Wigner-von Neumann and Weyl

That this expectation is correct is a famous theorem of Wigner and von-Neumann published in 1929. They went to the same high school and were lifelong friends and, while postdocs, published two back-to-back papers in Physikalische Zeitschrift (the first on embedded eigenvalues is perhaps more famous among spectral theorists; the second with the theorem below is much more quoted in the physics community at large).

**Theorem** *In the* $\frac{n(n+1)}{2}$ *dimensional space of self-adjoint real* $n \times n$ *matrices, those with a degenerate eigenvalue are a variety of dimension* $\frac{n(n+1)}{2} - 2$.

# Wigner-von Neumann and Weyl

That this expectation is correct is a famous theorem of Wigner and von-Neumann published in 1929. They went to the same high school and were lifelong friends and, while postdocs, published two back-to-back papers in Physikalische Zeitschrift (the first on embedded eigenvalues is perhaps more famous among spectral theorists; the second with the theorem below is much more quoted in the physics community at large).

**Theorem** *In the* $\frac{n(n+1)}{2}$ *dimensional space of self-adjoint real* $n \times n$ *matrices, those with a degenerate eigenvalue are a variety of dimension* $\frac{n(n+1)}{2} - 2$*. In the* $n^2$ *dimensional space of self-adjoint complex* $n \times n$ *matrices, those with a degenerate eigenvalue are a variety of dimension* $n^2 - 3$.

# Wigner-von Neumann and Weyl

The argument that WvN use is simple. They counted dimension by looking at the eigenvalues and at the fact that given the eigenvalues, you have to pick frames of eigenvectors (i.e. orthonormal eigenvectors up to phase).

# Wigner-von Neumann and Weyl



The argument that WvN use is simple. They counted dimension by looking at the eigenvalues and at the fact that given the eigenvalues, you have to pick frames of eigenvectors (i.e. orthonormal eigenvectors up to phase). For example, one dimension of the lost two in the real case comes from the lower dimension of the set of distinct eigenvalues

# Wigner-von Neumann and Weyl



The argument that WvN use is simple. They counted dimension by looking at the eigenvalues and at the fact that given the eigenvalues, you have to pick frames of eigenvectors (i.e. orthonormal eigenvectors up to phase). For example, one dimension of the lost two in the real case comes from the lower dimension of the set of distinct eigenvalues and the other one comes from the fact that if the last two eigenvalues are the equal ones, their eigenspace is determined as the space orthogonal to the first $n-2$

# Wigner-von Neumann and Weyl

The argument that WvN use is simple. They counted dimension by looking at the eigenvalues and at the fact that given the eigenvalues, you have to pick frames of eigenvectors (i.e. orthonormal eigenvectors up to phase). For example, one dimension of the lost two in the real case comes from the lower dimension of the set of distinct eigenvalues and the other one comes from the fact that if the last two eigenvalues are the equal ones, their eigenspace is determined as the space orthogonal to the first $n-2$ whereas if those last two are unequal, one has to choose a unit vector in a two dimensional space, an extra real parameter.

# Wigner-von Neumann and Weyl

One can argue that while this result is attributed to Wigner-von Neumann in 1929, it is in essence in 1926 work of Weyl (or even earlier work of Szegő).

# Wigner-von Neumann and Weyl



One can argue that while this result is attributed to Wigner-von Neumann in 1929, it is in essence in 1926 work of Weyl (or even earlier work of Szegő). For it is easy to argue we must get the same answer for $U(n)$ as for $n \times n$ complex matrices and for $O(n)$ and real matrices.

# Wigner-von Neumann and Weyl

One can argue that while this result is attributed to Wigner-von Neumann in 1929, it is in essence in 1926 work of Weyl (or even earlier work of Szegő). For it is easy to argue we must get the same answer for $U(n)$ as for $n \times n$ complex matrices and for $O(n)$ and real matrices. That the Weyl formula for the Haar measure projected onto the sets of eigenvectors has factors of $|\lambda_i - \lambda_j|^m$ with $m = 2$ for $U(n)$ and $m = 1$ for $O(n)$

# Wigner-von Neumann and Weyl



One can argue that while this result is attributed to Wigner-von Neumann in 1929, it is in essence in 1926 work of Weyl (or even earlier work of Szegő). For it is easy to argue we must get the same answer for $U(n)$ as for $n \times n$ complex matrices and for $O(n)$ and real matrices. That the Weyl formula for the Haar measure projected onto the sets of eigenvectors has factors of $|\lambda_i - \lambda_j|^m$ with $m = 2$ for $U(n)$ and $m = 1$ for $O(n)$ (as made famous by random matrix theory) is precisely an expression of the extra lower codimensions.

# Perturbation Theory

Rather than the $WvN$ picture of direct dimension counting, one can use eigenvalue perturbation theory to understand where codimension $2$ comes from.

# Perturbation Theory

Rather than the $WvN$ picture of direct dimension counting, one can use eigenvalue perturbation theory to understand where codimension $2$ comes from. To see if a degenerate eigenvalue splits to first order, one looks at the projection, $P$, onto the unperturbed eigenspace and then at $PVP$ where $V$ is the perturbation.

# Perturbation Theory

Rather than the $WvN$ picture of direct dimension counting, one can use eigenvalue perturbation theory to understand where codimension $2$ comes from. To see if a degenerate eigenvalue splits to first order, one looks at the projection, $P$, onto the unperturbed eigenspace and then at $PVP$ where $V$ is the perturbation. If this has distinct eigenvalues, there is splitting to first order. We saw above in this effective $2D$ case the codimensions were $2$ and $3$.

# Perturbation Theory

Rather than the $WvN$ picture of direct dimension counting, one can use eigenvalue perturbation theory to understand where codimension $2$ comes from. To see if a degenerate eigenvalue splits to first order, one looks at the projection, $P$, onto the unperturbed eigenspace and then at $PVP$ where $V$ is the perturbation. If this has distinct eigenvalues, there is splitting to first order. We saw above in this effective $2D$ case the codimensions were $2$ and $3$.

A final remark before leaving this subject. In quantum mechanics without magnetic fields, Hamiltonians commute with a complex conjugation (essentially by time reversal invariance)

# Perturbation Theory

Rather than the $WvN$ picture of direct dimension counting, one can use eigenvalue perturbation theory to understand where codimension $2$ comes from. To see if a degenerate eigenvalue splits to first order, one looks at the projection, $P$, onto the unperturbed eigenspace and then at $PVP$ where $V$ is the perturbation. If this has distinct eigenvalues, there is splitting to first order. We saw above in this effective $2D$ case the codimensions were $2$ and $3$.

A final remark before leaving this subject. In quantum mechanics without magnetic fields, Hamiltonians commute with a complex conjugation (essentially by time reversal invariance) so the relevant codimension is $2$. Once there is a magnetic field, things are effectively complex, so codimension $3$.

# Generic Continuum Schrödinger

Related to this theme is the following theorem that I proved in 1976:

## Generic Continuum Schrödinger

Related to this theme is the following theorem that I proved in 1976:

**Theorem** Let $Y$ be the Frèchet space of $C^\infty$ period $1$ functions on $\mathbb{R}$ with the seminorms $\|V\|_n \equiv \sup |V^{(n)}(x)|$.

# Generic Continuum Schrödinger



Related to this theme is the following theorem that I proved in 1976:

**Theorem** Let $Y$ be the Frèchet space of $C^\infty$ period $1$ functions on $\mathbb{R}$ with the seminorms $\|V\|_n \equiv \sup |V^{(n)}(x)|$. Then the set of $V$'s so that $h = -\frac{d^2}{dx^2} + V(x)$ has all gaps open is a dense open set.

# Generic Continuum Schrödinger

Related to this theme is the following theorem that I proved in 1976:

**Theorem** Let $Y$ be the Frèchet space of $C^\infty$ period $1$ functions on $\mathbb{R}$ with the seminorms $\|V\|_n \equiv \sup|V^{(n)}(x)|$. Then the set of $V$'s so that $h = -\frac{d^2}{dx^2} + V(x)$ has all gaps open is a dense open set.

We recall that these periodic Hamiltonians have an integrated density of states, $k(E)$

# Generic Continuum Schrödinger



Related to this theme is the following theorem that I proved in 1976:

**Theorem** Let $Y$ be the Frèchet space of $C^\infty$ period $1$ functions on $\mathbb{R}$ with the seminorms $\|V\|_n \equiv \sup|V^{(n)}(x)|$. Then the set of $V$'s so that $h = -\frac{d^2}{dx^2} + V(x)$ has all gaps open is a dense open set.

We recall that these periodic Hamiltonians have an integrated density of states, $k(E)$ (one definition is that $k(E)$ is the limit as $m \to \infty$ of $m^{-1}$ times the number of eigenvalues less than $E$ of $h$ restricted to $[0, m]$ with periodic boundary conditions).

# Generic Continuum Schrödinger



Related to this theme is the following theorem that I proved in 1976:

**Theorem** Let $Y$ be the Frèchet space of $C^\infty$ period $1$ functions on $\mathbb{R}$ with the seminorms $\|V\|_n \equiv \sup|V^{(n)}(x)|$. Then the set of $V$'s so that $h = -\frac{d^2}{dx^2} + V(x)$ has all gaps open is a dense open set.

We recall that these periodic Hamiltonians have an integrated density of states, $k(E)$ (one definition is that $k(E)$ is the limit as $m \to \infty$ of $m^{-1}$ times the number of eigenvalues less than $E$ of $h$ restricted to $[0, m]$ with periodic boundary conditions). In the periodic case, $k$ is strictly monotone precisely on the spectrum of $h$ with gaps in the spectrum where $k$ is constant and that there is a potential gap at the energies where $k(E) = n$ for $n = 1, 2, \ldots$.

# Generic Continuum Schrödinger



The proof is easy if one uses band theory.

# Generic Continuum Schrödinger



The proof is easy if one uses band theory. A closed gap corresponds to a degenerate periodic or antiperiodic eigenvalue and an explicit calculation shows such a degeneracy is removed in perturbation theory for some perturbations,

# Generic Continuum Schrödinger



The proof is easy if one uses band theory. A closed gap corresponds to a degenerate periodic or antiperiodic eigenvalue and an explicit calculation shows such a degeneracy is removed in perturbation theory for some perturbations, so the set where a given gap is open is a dense open set.

# Generic Continuum Schrödinger

The proof is easy if one uses band theory. A closed gap corresponds to a degenerate periodic or antiperiodic eigenvalue and an explicit calculation shows such a degeneracy is removed in perturbation theory for some perturbations, so the set where a given gap is open is a dense open set. The magic of the Baire category theorem then completes the proof.

# Generic Discrete Jacobi

Much more can be understood easily in the period $p$ Jacobi case.

# Generic Discrete Jacobi



Much more can be understood easily in the period $p$ Jacobi case. In that case, the space of $2p$ Jacobi parameters supports the Toda dynamical system which is completely integrable.

# Generic Discrete Jacobi

Much more can be understood easily in the period $p$ Jacobi case. In that case, the space of $2p$ Jacobi parameters supports the Toda dynamical system which is completely integrable. The parameter space foliates into isospectral tori of dimension $2\ell$, where $\ell$ is the number of gaps.

# Generic Discrete Jacobi

Much more can be understood easily in the period $p$ Jacobi case. In that case, the space of $2p$ Jacobi parameters supports the Toda dynamical system which is completely integrable. The parameter space foliates into isospectral tori of dimension $2\ell$, where $\ell$ is the number of gaps. This is a precise expression that one looses two dimensions each time a gap closes.

# Generic Discrete Jacobi



Much more can be understood easily in the period $p$ Jacobi case. In that case, the space of $2p$ Jacobi parameters supports the Toda dynamical system which is completely integrable. The parameter space foliates into isospectral tori of dimension $2\ell$, where $\ell$ is the number of gaps. This is a precise expression that one looses two dimensions each time a gap closes. In the Schrödinger case, there also the KdV dynamical systems but since all dimensions are infinite, it is more complicated to discuss codimensions.

# The Three Martini Problem



The same theme occurs in a more famous story, what is called the *Ten Martini Problem*. This concerns one of the most famous examples in mathematical physics which I named the almost Mathieu operator (acting on $\ell^2(\mathbb{Z})$)

# The Three Martini Problem



The same theme occurs in a more famous story, what is called the *Ten Martini Problem*. This concerns one of the most famous examples in mathematical physics which I named the almost Mathieu operator (acting on $\ell^2(\mathbb{Z})$)

$$H_{\lambda,\alpha,\theta}u(n) = u(n+1) + u(n-1) + 2\lambda\cos(\pi\alpha n + \theta)u(n)$$

# The Three Martini Problem

The same theme occurs in a more famous story, what is called the *Ten Martini Problem*. This concerns one of the most famous examples in mathematical physics which I named the almost Mathieu operator (acting on $\ell^2(\mathbb{Z})$)

$$H_{\lambda,\alpha,\theta}u(n) = u(n+1) + u(n-1) + 2\lambda\cos(\pi\alpha n + \theta)u(n)$$

This is periodic if $\alpha$ is rational but only almost periodic if $\alpha$ is irrational.

# The Three Martini Problem



The same theme occurs in a more famous story, what is called the *Ten Martini Problem*. This concerns one of the most famous examples in mathematical physics which I named the almost Mathieu operator (acting on $\ell^2(\mathbb{Z})$)

$$H_{\lambda,\alpha,\theta}u(n) = u(n+1) + u(n-1) + 2\lambda\cos(\pi\alpha n + \theta)u(n)$$

This is periodic if $\alpha$ is rational but only almost periodic if $\alpha$ is irrational. If $\alpha = p/q$, then there is a possible gap when $k(E) = j/q$; $j = 1, \dots, q-1$ and so the spectrum has $q$ (or fewer) bands.

# The Three Martini Problem

The same theme occurs in a more famous story, what is called the *Ten Martini Problem*. This concerns one of the most famous examples in mathematical physics which I named the almost Mathieu operator (acting on $\ell^2(\mathbb{Z})$)

$$H_{\lambda,\alpha,\theta}u(n) = u(n+1) + u(n-1) + 2\lambda\cos(\pi\alpha n + \theta)u(n)$$

This is periodic if $\alpha$ is rational but only almost periodic if $\alpha$ is irrational. If $\alpha = p/q$, then there is a possible gap when $k(E) = j/q$; $j = 1, \dots, q-1$ and so the spectrum has $q$ (or fewer) bands. If $\alpha$ is irrational, one can prove (Johnson-Moser & Bellisard) that on any potential gap $k(E) = [m\alpha]$, the fractional part of $m\alpha$.

# The Three Martini Problem



The same theme occurs in a more famous story, what is called the *Ten Martini Problem*. This concerns one of the most famous examples in mathematical physics which I named the almost Mathieu operator (acting on $\ell^2(\mathbb{Z})$)

$$H_{\lambda,\alpha,\theta}u(n) = u(n+1) + u(n-1) + 2\lambda\cos(\pi\alpha n + \theta)u(n)$$

This is periodic if $\alpha$ is rational but only almost periodic if $\alpha$ is irrational. If $\alpha = p/q$, then there is a possible gap when $k(E) = j/q$; $j = 1, \dots, q-1$ and so the spectrum has $q$ (or fewer) bands. If $\alpha$ is irrational, one can prove (Johnson-Moser & Bellisard) that on any potential gap $k(E) = [m\alpha]$, the fractional part of $m\alpha$. If all gaps are open, the spectrum is a Cantor set (i.e. closed and nowhere dense).

# The Three Martini Problem

Mark Kac and I discussed this situation at lunch one day in 1981 and agreed that it was an interesting conjecture to prove that $H_{\lambda,\alpha,\theta}$ had a Cantor spectrum for all irrational $\alpha$ and $\lambda \neq 0$

# The Three Martini Problem

Mark Kac and I discussed this situation at lunch one day in 1981 and agreed that it was an interesting conjecture to prove that $H_{\lambda,\alpha,\theta}$ had a Cantor spectrum for all irrational $\alpha$ and $\lambda \neq 0$ (if $\alpha$ is irrational, it is known (Avron-Simon) that the spectrum is $\theta$ independent).

# The Three Martini Problem



Mark Kac and I discussed this situation at lunch one day in 1981 and agreed that it was an interesting conjecture to prove that $H_{\lambda,\alpha,\theta}$ had a Cantor spectrum for all irrational $\alpha$ and $\lambda \neq 0$ (if $\alpha$ is irrational, it is known (Avron-Simon) that the spectrum is $\theta$ independent). "That's a grand conjecture", said Mark, "I'll offer ten Martini's for its solution."

# The Three Martini Problem

Mark Kac and I discussed this situation at lunch one day in 1981 and agreed that it was an interesting conjecture to prove that $H_{\lambda,\alpha,\theta}$ had a Cantor spectrum for all irrational $\alpha$ and $\lambda \neq 0$ (if $\alpha$ is irrational, it is known (Avron-Simon) that the spectrum is $\theta$ independent). "That's a grand conjecture", said Mark, "I'll offer ten Martini's for its solution." He later repeated this offer at an AMS meeting and I popularized it as the ten Martini problem.

# The Three Martini Problem



Mark Kac and I discussed this situation at lunch one day in 1981 and agreed that it was an interesting conjecture to prove that $H_{\lambda,\alpha,\theta}$ had a Cantor spectrum for all irrational $\alpha$ and $\lambda \neq 0$ (if $\alpha$ is irrational, it is known (Avron-Simon) that the spectrum is $\theta$ independent). "That's a grand conjecture", said Mark, "I'll offer ten Martini's for its solution." He later repeated this offer at an AMS meeting and I popularized it as the ten Martini problem. Solved in full in 2004 by Avila-Jitomirskaya after an important partial result by Puig.

# The Three Martini Problem



Mark Kac and I discussed this situation at lunch one day in 1981 and agreed that it was an interesting conjecture to prove that $H_{\lambda,\alpha,\theta}$ had a Cantor spectrum for all irrational $\alpha$ and $\lambda \neq 0$ (if $\alpha$ is irrational, it is known (Avron-Simon) that the spectrum is $\theta$ independent). "That's a grand conjecture", said Mark, "I'll offer ten Martini's for its solution." He later repeated this offer at an AMS meeting and I popularized it as the ten Martini problem. Solved in full in 2004 by Avila-Jitomirskaya after an important partial result by Puig. This is weaker than the result that all gaps are open, something known as the dry form of the ten Martini problem.

# The Three Martini Problem

A year after my lunch with Kac, Bellisard and I used the strategy of my periodic result.

# The Three Martini Problem

A year after my lunch with Kac, Bellisard and I used the strategy of my periodic result. We first proved that if $\alpha = p/q$ is rational and $q\theta$ is not a multiple of $\pi$, then all gaps were open (i.e. the spectrum had $q-1$ gaps).

# The Three Martini Problem

A year after my lunch with Kac, Bellisard and I used the strategy of my periodic result. We first proved that if $\alpha = p/q$ is rational and $q\theta$ is not a multiple of $\pi$, then all gaps were open (i.e. the spectrum had $q-1$ gaps). This non-trivial analytic fact was proven using ideas motivated by the classical result of Ince that the continuum Mathieu operator $(-\frac{d^2}{dx^2} + \lambda\cos(x))$ has all gaps open.

# The Three Martini Problem

A year after my lunch with Kac, Bellisard and I used the strategy of my periodic result. We first proved that if $\alpha = p/q$ is rational and $q\theta$ is not a multiple of $\pi$, then all gaps were open (i.e. the spectrum had $q-1$ gaps). This non-trivial analytic fact was proven using ideas motivated by the classical result of Ince that the continuum Mathieu operator $(-\frac{d^2}{dx^2} + \lambda\cos(x))$ has all gaps open. But once we knew that and had some continuity results on $k(E)$ of Avron-Simon, the magic of the Baire category theorem showed that for a Baire generic set of $(\alpha, \lambda)$, the spectrum is a Cantor set!

# The Three Martini Problem

A year after my lunch with Kac, Bellisard and I used the strategy of my periodic result. We first proved that if $\alpha = p/q$ is rational and $q\theta$ is not a multiple of $\pi$, then all gaps were open (i.e. the spectrum had $q-1$ gaps). This non-trivial analytic fact was proven using ideas motivated by the classical result of Ince that the continuum Mathieu operator $(-\frac{d^2}{dx^2} + \lambda\cos(x))$ has all gaps open. But once we knew that and had some continuity results on $k(E)$ of Avron-Simon, the magic of the Baire category theorem showed that for a Baire generic set of $(\alpha, \lambda)$, the spectrum is a Cantor set! It is remarkable that with one's Baire hands one can learn something about the irrational case (Cantor spectrum) by knowing something about the rational case even though, of course, in the rational case the spectrum is never Cantor.

# The Three Martini Problem

When I told Mark about this on the phone admitting it wasn't the full result, he remarked "But it is still interesting! I'll give you three martini's for it."

# The Three Martini Problem



When I told Mark about this on the phone admitting it wasn't the full result, he remarked "But it is still interesting! I'll give you three martini's for it." So I always think of this as the three Martini result.

# The Three Martini Problem

When I told Mark about this on the phone admitting it wasn't the full result, he remarked "But it is still interesting! I'll give you three martini's for it." So I always think of this as the three Martini result. Alas, before we met again, Mark was dead of pancreatic cancer (the same disease that felled the other half of the Feynman-Kac formula).

# Continuum Schrödinger

Related to these themes is the following 1946 theorem of Borg:

**Theorem** *Let $V$ be a periodic function on $\mathbb{R}$ so that $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R}, dx)$ has spectrum $[\Sigma, \infty)$. Then $V$ is constant.*

# Continuum Schrödinger

Related to these themes is the following 1946 theorem of Borg:

**Theorem** *Let $V$ be a periodic function on $\mathbb{R}$ so that $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R}, dx)$ has spectrum $[\Sigma, \infty)$. Then $V$ is constant.*

In other words, if $V$ is not constant, at least one gap is open.

# Jacobi Matrices



In 1975, Hochstadt proved the analog for Jacobi matrices

# Jacobi Matrices



In 1975, Hochstadt proved the analog for Jacobi matrices

**Theorem** *Let* $\{a_n, b_n\}_{n\in\mathbb{Z}}$ *be a periodic in* $n$ *so that the corresponding two sided Jacobi matrix on* $\ell^2(\mathbb{Z})$ *has spectrum* $[a, b]$*. Then* $a$ *and* $b$ *are each constant.*

# Hochstadt's Theorem



We recall, that if $V$ is a function on $\mathbb{R}$ with period $L$, then gaps occur at energies where $k(E) = n/L$.

# Hochstadt's Theorem

We recall, that if $V$ is a function on $\mathbb{R}$ with period $L$, then gaps occur at energies where $k(E) = n/L$. So if we thought $V$ had period $L$ but really had a shorter period $L/p$, then the only gaps will be at $n$'s divisible by $p$.

# Hochstadt's Theorem



We recall, that if $V$ is a function on $\mathbb{R}$ with period $L$, then gaps occur at energies where $k(E) = n/L$. So if we thought $V$ had period $L$ but really had a shorter period $L/p$, then the only gaps will be at $n$'s divisible by $p$. In 1984, Hochstadt proved the following converse:

# Hochstadt's Theorem

We recall, that if $V$ is a function on $\mathbb{R}$ with period $L$, then gaps occur at energies where $k(E) = n/L$. So if we thought $V$ had period $L$ but really had a shorter period $L/p$, then the only gaps will be at $n$'s divisible by $p$. In 1984, Hochstadt proved the following converse:

**Theorem** *Let $V$ be a periodic function on $\mathbb{R}$ with period $L$ so that, for some integer $p$, $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R}, dx)$ has gaps in its spectrum only at some subset of the points where $k(E) = pn/L, n = 1, 2, \ldots$. Then $V$ has period $L/p$*

# Hochstadt's Theorem

We recall, that if $V$ is a function on $\mathbb{R}$ with period $L$, then gaps occur at energies where $k(E) = n/L$. So if we thought $V$ had period $L$ but really had a shorter period $L/p$, then the only gaps will be at $n$'s divisible by $p$. In 1984, Hochstadt proved the following converse:

**Theorem** *Let $V$ be a periodic function on $\mathbb{R}$ with period $L$ so that, for some integer $p$, $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R}, dx)$ has gaps in its spectrum only at some subset of the points where $k(E) = pn/L, n = 1, 2, \dots$. Then $V$ has period $L/p$*

This is a strengthening of Borg in that it implies a Borg's theorem (since no gaps means the hypothesis holds for all $p$, so $V(x + m/p) = V(x)$ for all rational $m/p$).

# Hochstadt's Theorem

We recall, that if $V$ is a function on $\mathbb{R}$ with period $L$, then gaps occur at energies where $k(E) = n/L$. So if we thought $V$ had period $L$ but really had a shorter period $L/p$, then the only gaps will be at $n$'s divisible by $p$. In 1984, Hochstadt proved the following converse:

**Theorem** *Let $V$ be a periodic function on $\mathbb{R}$ with period $L$ so that, for some integer $p$, $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R}, dx)$ has gaps in its spectrum only at some subset of the points where $k(E) = pn/L, n = 1, 2, \dots$. Then $V$ has period $L/p$*

This is a strengthening of Borg in that it implies a Borg's theorem (since no gaps means the hypothesis holds for all $p$, so $V(x + m/p) = V(x)$ for all rational $m/p$). There is a Jacobi matrix version of this theorem.

# The Bethe Sommerfeld Conjecture



Before leaving the discussion of results for gaps in the spectrum of periodic Schrödinger operators, I should mention

# The Bethe Sommerfeld Conjecture



Before leaving the discussion of results for gaps in the spectrum of periodic Schrödinger operators, I should mention

**Theorem** *If* $V$ *is a smooth, periodic function on* $\mathbb{R}^\nu, \nu \geq 2$*, then* $-\Delta + V$ *has only finitely many gaps in its spectrum*

# The Bethe Sommerfeld Conjecture

Before leaving the discussion of results for gaps in the spectrum of periodic Schrödinger operators, I should mention

**Theorem** *If* $V$ *is a smooth, periodic function on* $\mathbb{R}^\nu, \nu \geq 2$, *then* $-\Delta + V$ *has only finitely many gaps in its spectrum*

By periodic, we mean invariant under a $\nu$-dimension lattice of translations.

# The Bethe Sommerfeld Conjecture

Before leaving the discussion of results for gaps in the spectrum of periodic Schrödinger operators, I should mention

**Theorem** *If* $V$ *is a smooth, periodic function on* $\mathbb{R}^\nu, \nu \geq 2$, *then* $-\Delta + V$ *has only finitely many gaps in its spectrum*

By periodic, we mean invariant under a $\nu$-dimension lattice of translations. This result is generally called the Bethe-Sommerfeld conjecture after its occurrence in a 1933 monograph of those authors (Bethe was Sommerfeld's student).

# The Bethe Sommerfeld Conjecture

Before leaving the discussion of results for gaps in the spectrum of periodic Schrödinger operators, I should mention

**Theorem** *If* $V$ *is a smooth, periodic function on* $\mathbb{R}^\nu, \nu \geq 2$*, then* $-\Delta + V$ *has only finitely many gaps in its spectrum*

By periodic, we mean invariant under a $\nu$-dimension lattice of translations. This result is generally called the Bethe-Sommerfeld conjecture after its occurrence in a 1933 monograph of those authors (Bethe was Sommerfeld's student). It has a long involved literature. The definitive general result was proven by Parnovski in 2008.

# The Bethe Sommerfeld Conjecture



Before leaving the discussion of results for gaps in the spectrum of periodic Schrödinger operators, I should mention

**Theorem** *If* $V$ *is a smooth, periodic function on* $\mathbb{R}^\nu, \nu \geq 2$*, then* $-\Delta + V$ *has only finitely many gaps in its spectrum*

By periodic, we mean invariant under a $\nu$-dimension lattice of translations. This result is generally called the Bethe-Sommerfeld conjecture after its occurrence in a 1933 monograph of those authors (Bethe was Sommerfeld's student). It has a long involved literature. The definitive general result was proven by Parnovski in 2008. It says that 1D (which generically has infinitely many gaps) is very different from higher dimensions.

# Regular Trees



The rest of this talk focuses on Jacobi matrices on infinite trees.

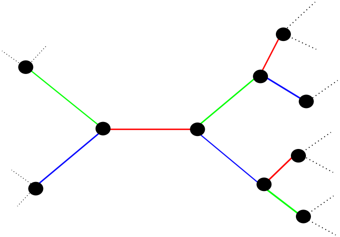
# Regular Trees

The rest of this talk focuses on Jacobi matrices on infinite trees. We will mainly consider the fixed degree tree like the following degree $3$ regular tree

# Regular Trees



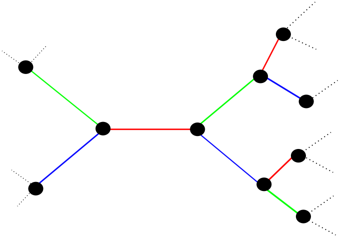
The rest of this talk focuses on Jacobi matrices on infinite trees. We will mainly consider the fixed degree tree like the following degree $3$ regular tree



# Regular Trees



The rest of this talk focuses on Jacobi matrices on infinite trees. We will mainly consider the fixed degree tree like the following degree $3$ regular tree



Among spectral theorists, about the only literature on such operators is on the random case (Klein, Aizenman-Warzel) and some results on rooted trees by Breuer and by Keller, Lenz and S. Warzel.

# Graph Theory Formalism



A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*.

# Graph Theory Formalism



A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*. Each edge has two ends which are vertices.

# Graph Theory Formalism

A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*. Each edge has two ends which are vertices. There is a natural topological space associated to a graph and we always suppose it is connected.

# Graph Theory Formalism



A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*. Each edge has two ends which are vertices. There is a natural topological space associated to a graph and we always suppose it is connected. We want to allow edges that start and end at the same vertex (aka *self–loops*) and definitely want to allow multiple edges between a given pair of vertices.

# Graph Theory Formalism



A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*. Each edge has two ends which are vertices. There is a natural topological space associated to a graph and we always suppose it is connected. We want to allow edges that start and end at the same vertex (aka *self–loops*) and definitely want to allow multiple edges between a given pair of vertices.

A graph which is simply connected is called a *tree*.

# Graph Theory Formalism

A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*. Each edge has two ends which are vertices. There is a natural topological space associated to a graph and we always suppose it is connected. We want to allow edges that start and end at the same vertex (aka *self–loops*) and definitely want to allow multiple edges between a given pair of vertices.

A graph which is simply connected is called a *tree*. The *degree* of a vertex is the number of edges with that vertex as an end.

# Graph Theory Formalism



A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*. Each edge has two ends which are vertices. There is a natural topological space associated to a graph and we always suppose it is connected. We want to allow edges that start and end at the same vertex (aka *self–loops*) and definitely want to allow multiple edges between a given pair of vertices.

A graph which is simply connected is called a *tree*. The *degree* of a vertex is the number of edges with that vertex as an end. A *leaf* is a vertex of degree one and we will normally only consider graphs with no leaves.

# Graph Theory Formalism

A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*. Each edge has two ends which are vertices. There is a natural topological space associated to a graph and we always suppose it is connected. We want to allow edges that start and end at the same vertex (aka *self–loops*) and definitely want to allow multiple edges between a given pair of vertices.

A graph which is simply connected is called a *tree*. The *degree* of a vertex is the number of edges with that vertex as an end. A *leaf* is a vertex of degree one and we will normally only consider graphs with no leaves. Thus, our trees are always infinite.

# Graph Theory Formalism



A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*. Each edge has two ends which are vertices. There is a natural topological space associated to a graph and we always suppose it is connected. We want to allow edges that start and end at the same vertex (aka *self–loops*) and definitely want to allow multiple edges between a given pair of vertices.

A graph which is simply connected is called a *tree*. The *degree* of a vertex is the number of edges with that vertex as an end. A *leaf* is a vertex of degree one and we will normally only consider graphs with no leaves. Thus, our trees are always infinite. Of course, trees have no self loops and at most one edge between two vertices.

# Graph Theory Formalism



A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*. Each edge has two ends which are vertices. There is a natural topological space associated to a graph and we always suppose it is connected. We want to allow edges that start and end at the same vertex (aka *self–loops*) and definitely want to allow multiple edges between a given pair of vertices.

A graph which is simply connected is called a *tree*. The *degree* of a vertex is the number of edges with that vertex as an end. A *leaf* is a vertex of degree one and we will normally only consider graphs with no leaves. Thus, our trees are always infinite. Of course, trees have no self loops and at most one edge between two vertices. A graph with constant degree is called *regular*.

# Graph Theory Formalism

A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*. Each edge has two ends which are vertices. There is a natural topological space associated to a graph and we always suppose it is connected. We want to allow edges that start and end at the same vertex (aka *self–loops*) and definitely want to allow multiple edges between a given pair of vertices.

A graph which is simply connected is called a *tree*. The *degree* of a vertex is the number of edges with that vertex as an end. A *leaf* is a vertex of degree one and we will normally only consider graphs with no leaves. Thus, our trees are always infinite. Of course, trees have no self loops and at most one edge between two vertices. A graph with constant degree is called *regular*.

We will most often consider regular graphs.

# Jacobi Matrices

A *Jacobi matrix on a graph*, $\mathcal{G}$, is associated to a set of real numbers $\{b_j\}_{j\in V}$ assigned to each vertex and strictly positive reals $\{a_\alpha\}_{\alpha\in E}$ assigned to each edge.

# Jacobi Matrices

A *Jacobi matrix on a graph*, $\mathcal{G}$, is associated to a set of real numbers $\{b_j\}_{j\in V}$ assigned to each vertex and strictly positive reals $\{a_\alpha\}_{\alpha\in E}$ assigned to each edge. Because we will only consider finite graphs or infinite trees with periodic parameters, the $a$'s and $b$'s are bounded sets.

# Jacobi Matrices

A *Jacobi matrix on a graph*, $\mathcal{G}$, is associated to a set of real numbers $\{b_j\}_{j\in V}$ assigned to each vertex and strictly positive reals $\{a_\alpha\}_{\alpha\in E}$ assigned to each edge. Because we will only consider finite graphs or infinite trees with periodic parameters, the $a$'s and $b$'s are bounded sets. The Jacobi matrix acts on $\ell^2(V) \equiv \mathcal{H}(\mathcal{G})$, the vector space of square summable sequences indexed by the vertices of the graph.

# Jacobi Matrices

A *Jacobi matrix on a graph*, $\mathcal{G}$, is associated to a set of real numbers $\{b_j\}_{j\in V}$ assigned to each vertex and strictly positive reals $\{a_\alpha\}_{\alpha\in E}$ assigned to each edge. Because we will only consider finite graphs or infinite trees with periodic parameters, the $a$'s and $b$'s are bounded sets. The Jacobi matrix acts on $\ell^2(V) \equiv \mathcal{H}(\mathcal{G})$, the vector space of square summable sequences indexed by the vertices of the graph. It has matrix elements

$$H_{jk} = \begin{cases} b_j, & \text{if } j = k; \\ & \\ & \end{cases}$$

# Jacobi Matrices

A *Jacobi matrix on a graph*, $\mathcal{G}$, is associated to a set of real numbers $\{b_j\}_{j\in V}$ assigned to each vertex and strictly positive reals $\{a_\alpha\}_{\alpha\in E}$ assigned to each edge. Because we will only consider finite graphs or infinite trees with periodic parameters, the $a$'s and $b$'s are bounded sets. The Jacobi matrix acts on $\ell^2(V) \equiv \mathcal{H}(\mathcal{G})$, the vector space of square summable sequences indexed by the vertices of the graph. It has matrix elements

$$H_{jk} = \begin{cases} b_j, & \text{if } j = k; \\ \sum_\alpha a_\alpha, & \text{if } j \neq k \text{ are ends of one or more edges} \\ & \qquad \alpha \text{ which we sum over;} \end{cases}$$

# Jacobi Matrices



A *Jacobi matrix on a graph*, $\mathcal{G}$, is associated to a set of real numbers $\{b_j\}_{j\in V}$ assigned to each vertex and strictly positive reals $\{a_\alpha\}_{\alpha\in E}$ assigned to each edge. Because we will only consider finite graphs or infinite trees with periodic parameters, the $a$'s and $b$'s are bounded sets. The Jacobi matrix acts on $\ell^2(V) \equiv \mathcal{H}(\mathcal{G})$, the vector space of square summable sequences indexed by the vertices of the graph. It has matrix elements

$$H_{jk} = \begin{cases} b_j, & \text{if } j = k; \\ \sum_\alpha a_\alpha, & \text{if } j \neq k \text{ are ends of one or more edges} \\ & \qquad\qquad \alpha \text{ which we sum over;} \\ 0, & \text{if no edges have } i \text{ and } j \text{ as ends.} \end{cases}$$

# Jacobi Matrices

A *Jacobi matrix on a graph*, $\mathcal{G}$, is associated to a set of real numbers $\{b_j\}_{j\in V}$ assigned to each vertex and strictly positive reals $\{a_\alpha\}_{\alpha\in E}$ assigned to each edge. Because we will only consider finite graphs or infinite trees with periodic parameters, the $a$'s and $b$'s are bounded sets. The Jacobi matrix acts on $\ell^2(V) \equiv \mathcal{H}(\mathcal{G})$, the vector space of square summable sequences indexed by the vertices of the graph. It has matrix elements

$$H_{jk} = \begin{cases} b_j, & \text{if } j = k; \\ \sum_\alpha a_\alpha, & \text{if } j \neq k \text{ are ends of one or more edges} \\ & \qquad \alpha \text{ which we sum over;} \\ 0, & \text{if no edges have } i \text{ and } j \text{ as ends.} \end{cases}$$

If there are self–loops, one needs to modify this.

# Covering Space Formalism



Let $\mathcal{G}$ be a finite graph (with no leaves).

# Covering Space Formalism



Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree

# Covering Space Formalism



Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree and if $\mathcal{G}$ has constant degree, so does $\mathcal{T}$, i.e. it is a *regular tree*.

# Covering Space Formalism



Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree and if $\mathcal{G}$ has constant degree, so does $\mathcal{T}$, i.e. it is a *regular tree*.

Now let $J$ be a Jacobi matrix on $\mathcal{G}$. There is a unique Jacobi matrix, $H$, on $\mathcal{T}$

# Covering Space Formalism

Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree and if $\mathcal{G}$ has constant degree, so does $\mathcal{T}$, i.e. it is a *regular tree*.

Now let $J$ be a Jacobi matrix on $\mathcal{G}$. There is a unique Jacobi matrix, $H$, on $\mathcal{T}$ so that if $\Xi : \mathcal{T} \to \mathcal{G}$ is the covering map and $B_j, A_\alpha$ the Jacobi parameters of $J$ and $b_j, a_\alpha$ of $H$,

# Covering Space Formalism



Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree and if $\mathcal{G}$ has constant degree, so does $\mathcal{T}$, i.e. it is a *regular tree*.

Now let $J$ be a Jacobi matrix on $\mathcal{G}$. There is a unique Jacobi matrix, $H$, on $\mathcal{T}$ so that if $\Xi : \mathcal{T} \to \mathcal{G}$ is the covering map and $B_j, A_\alpha$ the Jacobi parameters of $J$ and $b_j, a_\alpha$ of $H$, then $b_j = B_{\Xi(j)}, a_\alpha = A_{\Xi(\alpha)}$.

# Covering Space Formalism

Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree and if $\mathcal{G}$ has constant degree, so does $\mathcal{T}$, i.e. it is a *regular tree*.

Now let $J$ be a Jacobi matrix on $\mathcal{G}$. There is a unique Jacobi matrix, $H$, on $\mathcal{T}$ so that if $\Xi : \mathcal{T} \to \mathcal{G}$ is the covering map and $B_j, A_\alpha$ the Jacobi parameters of $J$ and $b_j, a_\alpha$ of $H$, then $b_j = B_{\Xi(j)}, a_\alpha = A_{\Xi(\alpha)}$. Any deck transformation, $G \in \Gamma$, the set of deck transformations on $\mathcal{T}$, induces a unitary on $\mathcal{H}(\mathcal{T})$

# Covering Space Formalism

Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree and if $\mathcal{G}$ has constant degree, so does $\mathcal{T}$, i.e. it is a *regular tree*.

Now let $J$ be a Jacobi matrix on $\mathcal{G}$. There is a unique Jacobi matrix, $H$, on $\mathcal{T}$ so that if $\Xi : \mathcal{T} \to \mathcal{G}$ is the covering map and $B_j, A_\alpha$ the Jacobi parameters of $J$ and $b_j, a_\alpha$ of $H$, then $b_j = B_{\Xi(j)}, a_\alpha = A_{\Xi(\alpha)}$. Any deck transformation, $G \in \Gamma$, the set of deck transformations on $\mathcal{T}$, induces a unitary on $\mathcal{H}(\mathcal{T})$ and these unitaries all commute with $H$.

# Covering Space Formalism

Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree and if $\mathcal{G}$ has constant degree, so does $\mathcal{T}$, i.e. it is a *regular tree*.

Now let $J$ be a Jacobi matrix on $\mathcal{G}$. There is a unique Jacobi matrix, $H$, on $\mathcal{T}$ so that if $\Xi : \mathcal{T} \to \mathcal{G}$ is the covering map and $B_j, A_\alpha$ the Jacobi parameters of $J$ and $b_j, a_\alpha$ of $H$, then $b_j = B_{\Xi(j)}, a_\alpha = A_{\Xi(\alpha)}$. Any deck transformation, $G \in \Gamma$, the set of deck transformations on $\mathcal{T}$, induces a unitary on $\mathcal{H}(\mathcal{T})$ and these unitaries all commute with $H$. We call $H$ a *periodic Jacobi matrix*

# Covering Space Formalism

Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree and if $\mathcal{G}$ has constant degree, so does $\mathcal{T}$, i.e. it is a *regular tree*.

Now let $J$ be a Jacobi matrix on $\mathcal{G}$. There is a unique Jacobi matrix, $H$, on $\mathcal{T}$ so that if $\Xi : \mathcal{T} \to \mathcal{G}$ is the covering map and $B_j, A_\alpha$ the Jacobi parameters of $J$ and $b_j, a_\alpha$ of $H$, then $b_j = B_{\Xi(j)}, a_\alpha = A_{\Xi(\alpha)}$. Any deck transformation, $G \in \Gamma$, the set of deck transformations on $\mathcal{T}$, induces a unitary on $\mathcal{H}(\mathcal{T})$ and these unitaries all commute with $H$. We call $H$ a *periodic Jacobi matrix* and set $p$, the number of vertices of $\mathcal{G}$ to be its period,

# Covering Space Formalism

Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree and if $\mathcal{G}$ has constant degree, so does $\mathcal{T}$, i.e. it is a *regular tree*.

Now let $J$ be a Jacobi matrix on $\mathcal{G}$. There is a unique Jacobi matrix, $H$, on $\mathcal{T}$ so that if $\Xi : \mathcal{T} \to \mathcal{G}$ is the covering map and $B_j, A_\alpha$ the Jacobi parameters of $J$ and $b_j, a_\alpha$ of $H$, then $b_j = B_{\Xi(j)}, a_\alpha = A_{\Xi(\alpha)}$. Any deck transformation, $G \in \Gamma$, the set of deck transformations on $\mathcal{T}$, induces a unitary on $\mathcal{H}(\mathcal{T})$ and these unitaries all commute with $H$. We call $H$ a *periodic Jacobi matrix* and set $p$, the number of vertices of $\mathcal{G}$ to be its period, although, as I'll explain, there is some question if this is the right definition of period!

# Covering Space Formalism



Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree and if $\mathcal{G}$ has constant degree, so does $\mathcal{T}$, i.e. it is a *regular tree*.

Now let $J$ be a Jacobi matrix on $\mathcal{G}$. There is a unique Jacobi matrix, $H$, on $\mathcal{T}$ so that if $\Xi : \mathcal{T} \to \mathcal{G}$ is the covering map and $B_j, A_\alpha$ the Jacobi parameters of $J$ and $b_j, a_\alpha$ of $H$, then $b_j = B_{\Xi(j)}, a_\alpha = A_{\Xi(\alpha)}$. Any deck transformation, $G \in \Gamma$, the set of deck transformations on $\mathcal{T}$, induces a unitary on $\mathcal{H}(\mathcal{T})$ and these unitaries all commute with $H$. We call $H$ a *periodic Jacobi matrix* and set $p$, the number of vertices of $\mathcal{G}$ to be its period, although, as I'll explain, there is some question if this is the right definition of period! We let $q$ be the number of edges.

# Free Groups



If $\mathcal{G}$ has $m$ independent loops

# Free Groups

If $\mathcal{G}$ has $m$ independent loops (equivalently, one can drop $m$ edges and turn $\mathcal{G}$ into a connected finite tree),

# Free Groups



If $\mathcal{G}$ has $m$ independent loops (equivalently, one can drop $m$ edges and turn $\mathcal{G}$ into a connected finite tree), then the fundamental group of $\mathcal{G}$ is the free nonabelian group with $m$ generators, $\mathcal{F}_m$.

# Free Groups

If $\mathcal{G}$ has $m$ independent loops (equivalently, one can drop $m$ edges and turn $\mathcal{G}$ into a connected finite tree), then the fundamental group of $\mathcal{G}$ is the free nonabelian group with $m$ generators, $\mathcal{F}_m$. So that is the natural symmetry of our periodic trees.

# Free Groups



If $\mathcal{G}$ has $m$ independent loops (equivalently, one can drop $m$ edges and turn $\mathcal{G}$ into a connected finite tree), then the fundamental group of $\mathcal{G}$ is the free nonabelian group with $m$ generators, $\mathcal{F}_m$. So that is the natural symmetry of our periodic trees.

The *free Jacobi matrix* on a tree is the one with all $b$'s 0 and all $a$'s 1.

# Free Groups

If $\mathcal{G}$ has $m$ independent loops (equivalently, one can drop $m$ edges and turn $\mathcal{G}$ into a connected finite tree), then the fundamental group of $\mathcal{G}$ is the free nonabelian group with $m$ generators, $\mathcal{F}_m$. So that is the natural symmetry of our periodic trees.

The *free Jacobi matrix* on a tree is the one with all $b$'s 0 and all $a$'s 1. In this regard, there is a strange distinction between regular trees of constant degree $d$ depending on whether $d$ is even or odd!

# Free Groups



If $\mathcal{G}$ has $m$ independent loops (equivalently, one can drop $m$ edges and turn $\mathcal{G}$ into a connected finite tree), then the fundamental group of $\mathcal{G}$ is the free nonabelian group with $m$ generators, $\mathcal{F}_m$. So that is the natural symmetry of our periodic trees.

The *free Jacobi matrix* on a tree is the one with all $b$'s 0 and all $a$'s 1. In this regard, there is a strange distinction between regular trees of constant degree $d$ depending on whether $d$ is even or odd! The graph with one vertex and $k$ self loops has degree $d = 2k$.

# Free Groups

If $\mathcal{G}$ has $m$ independent loops (equivalently, one can drop $m$ edges and turn $\mathcal{G}$ into a connected finite tree), then the fundamental group of $\mathcal{G}$ is the free nonabelian group with $m$ generators, $\mathcal{F}_m$. So that is the natural symmetry of our periodic trees.

The *free Jacobi matrix* on a tree is the one with all $b$'s 0 and all $a$'s 1. In this regard, there is a strange distinction between regular trees of constant degree $d$ depending on whether $d$ is even or odd! The graph with one vertex and $k$ self loops has degree $d = 2k$. Its universal cover is the regular graph of degree $d = 2k$ and its free Laplacian is a period $1$ Jacobi matrix.

# Free Groups



If $\mathcal{G}$ has $m$ independent loops (equivalently, one can drop $m$ edges and turn $\mathcal{G}$ into a connected finite tree), then the fundamental group of $\mathcal{G}$ is the free nonabelian group with $m$ generators, $\mathcal{F}_m$. So that is the natural symmetry of our periodic trees.

The *free Jacobi matrix* on a tree is the one with all $b$'s 0 and all $a$'s 1. In this regard, there is a strange distinction between regular trees of constant degree $d$ depending on whether $d$ is even or odd! The graph with one vertex and $k$ self loops has degree $d = 2k$. Its universal cover is the regular graph of degree $d = 2k$ and its free Laplacian is a period $1$ Jacobi matrix. But there is no graph with a single vertex of odd degree,

# Free Groups

If $\mathcal{G}$ has $m$ independent loops (equivalently, one can drop $m$ edges and turn $\mathcal{G}$ into a connected finite tree), then the fundamental group of $\mathcal{G}$ is the free nonabelian group with $m$ generators, $\mathcal{F}_m$. So that is the natural symmetry of our periodic trees.

The *free Jacobi matrix* on a tree is the one with all $b$'s 0 and all $a$'s 1. In this regard, there is a strange distinction between regular trees of constant degree $d$ depending on whether $d$ is even or odd! The graph with one vertex and $k$ self loops has degree $d = 2k$. Its universal cover is the regular graph of degree $d = 2k$ and its free Laplacian is a period $1$ Jacobi matrix. But there is no graph with a single vertex of odd degree, so, with our definition, the free Jacobi matrix on an odd degree homogenous tree is of period $2$!!!

# Free Groups



The point is the free group with $k$ generators acts freely (i.e. no fixed point for non-identity elements) and transitively on the degree $2k$ regular tree.

# Free Groups



The point is the free group with $k$ generators acts freely (i.e. no fixed point for non-identity elements) and transitively on the degree $2k$ regular tree. There is no such symmetry group on any odd degree regular tree

# Free Groups

The point is the free group with $k$ generators acts freely (i.e. no fixed point for non-identity elements) and transitively on the degree $2k$ regular tree. There is no such symmetry group on any odd degree regular tree although by looking at the cover of the two vertex, no self loop, $d$ edge graph, one sees that $\mathcal{F}_{d-1}$ acts freely on the degree $d$ regular tree but with two orbits rather than transitively. One can add an extra generator to get a transitive symmetry group but the action is no longer free.

# Example 1: Free Jacobi Matrix on a Homogeneous Tree

It is illuminating to consider those few cases where we can compute the Green's function ($G_{jk}(z) = \langle \delta_j, (H-z)^{-1}\delta_k \rangle$) especially the diagonal case which is the Stieltjes transform of the spectral measure, $d\mu_j$.

# Example 1: Free Jacobi Matrix on a Homogeneous Tree



It is illuminating to consider those few cases where we can compute the Green's function ($G_{jk}(z) = \langle \delta_j, (H-z)^{-1}\delta_k \rangle$) especially the diagonal case which is the Stieltjes transform of the spectral measure, $d\mu_j$. The IDS, $k(E)$, is defined by averaging the spectral measures over a unit cell (one point from each orbit) and integrating it from $-\infty$ to $E$.

# Example 1: Free Jacobi Matrix on a Homogeneous Tree

It is illuminating to consider those few cases where we can compute the Green's function ($G_{jk}(z) = \langle \delta_j, (H-z)^{-1}\delta_k \rangle$) especially the diagonal case which is the Stieltjes transform of the spectral measure, $d\mu_j$. The IDS, $k(E)$, is defined by averaging the spectral measures over a unit cell (one point from each orbit) and integrating it from $-\infty$ to $E$. The simplest example is the $d$ regular tree with all $a = 1$ and all $b = 0$. It has been extensively studied.

# Example 1: Free Jacobi Matrix on a Homogeneous Tree

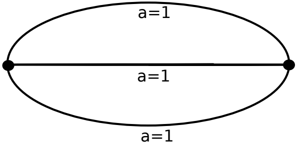


It is illuminating to consider those few cases where we can compute the Green's function ($G_{jk}(z) = \langle \delta_j, (H-z)^{-1}\delta_k \rangle$) especially the diagonal case which is the Stieltjes transform of the spectral measure, $d\mu_j$. The IDS, $k(E)$, is defined by averaging the spectral measures over a unit cell (one point from each orbit) and integrating it from $-\infty$ to $E$. The simplest example is the $d$ regular tree with all $a = 1$ and all $b = 0$. It has been extensively studied. Here is the underlying graph when $d = 3$ (set $a_1 = a_2 = a_3 = 1$)

# Example 1: Free Jacobi Matrix on a Homogeneous Tree



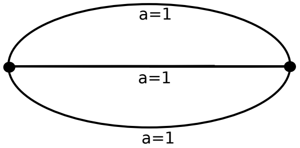
It is illuminating to consider those few cases where we can compute the Green's function ($G_{jk}(z) = \langle \delta_j, (H-z)^{-1}\delta_k \rangle$) especially the diagonal case which is the Stieltjes transform of the spectral measure, $d\mu_j$. The IDS, $k(E)$, is defined by averaging the spectral measures over a unit cell (one point from each orbit) and integrating it from $-\infty$ to $E$. The simplest example is the $d$ regular tree with all $a = 1$ and all $b = 0$. It has been extensively studied. Here is the underlying graph when $d = 3$ (set $a_1 = a_2 = a_3 = 1$)



# Example 1: Free Jacobi Matrix on a Homogeneous Tree



It is illuminating to consider those few cases where we can compute the Green's function ($G_{jk}(z) = \langle \delta_j, (H-z)^{-1}\delta_k \rangle$) especially the diagonal case which is the Stieltjes transform of the spectral measure, $d\mu_j$. The IDS, $k(E)$, is defined by averaging the spectral measures over a unit cell (one point from each orbit) and integrating it from $-\infty$ to $E$. The simplest example is the $d$ regular tree with all $a = 1$ and all $b = 0$. It has been extensively studied. Here is the underlying graph when $d = 3$ (set $a_1 = a_2 = a_3 = 1$)




There are related $m$-functions which we don't have time to discuss here although they will appear in some calculations.

# Example 1: Free Jacobi Matrix on a Homogeneous Tree



The equation for $m$, which is independent of vertex and edge, is

$$m = \frac{1}{-z - (d-1)m} \Rightarrow m = \frac{-z + \sqrt{z^2 - 4(d-1)}}{2(d-1)}$$

# Example 1: Free Jacobi Matrix on a Homogeneous Tree



The equation for $m$, which is independent of vertex and edge, is

$$m = \frac{1}{-z-(d-1)m} \Rightarrow m = \frac{-z+\sqrt{z^2-4(d-1)}}{2(d-1)}$$

We take the plus sign on the square root to go to zero at $\infty$.

# Example 1: Free Jacobi Matrix on a Homogeneous Tree



The equation for $m$, which is independent of vertex and edge, is

$$m = \frac{1}{-z-(d-1)m} \Rightarrow m = \frac{-z+\sqrt{z^2-4(d-1)}}{2(d-1)}$$

We take the plus sign on the square root to go to zero at $\infty$. Thus $\mathrm{spec}(H) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$.

# Example 1: Free Jacobi Matrix on a Homogeneous Tree



The equation for $m$, which is independent of vertex and edge, is

$$m = \frac{1}{-z - (d-1)m} \Rightarrow m = \frac{-z + \sqrt{z^2 - 4(d-1)}}{2(d-1)}$$

We take the plus sign on the square root to go to zero at $\infty$. Thus $\mathrm{spec}(H) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$. The formula for $G$, which is independent of vertex, is $(q \equiv d-1)$

# Example 1: Free Jacobi Matrix on a Homogeneous Tree



The equation for $m$, which is independent of vertex and edge, is

$$m = \frac{1}{-z-(d-1)m} \Rightarrow m = \frac{-z+\sqrt{z^2-4(d-1)}}{2(d-1)}$$

We take the plus sign on the square root to go to zero at $\infty$. Thus $\mathrm{spec}(H) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$. The formula for $G$, which is independent of vertex, is ($q \equiv d-1$)

$$G(z) = \frac{-(d-2)z + d\sqrt{z^2-4q}}{2(d^2-z^2)}$$

# Example 1: Free Jacobi Matrix on a Homogeneous Tree



The equation for $m$, which is independent of vertex and edge, is

$$m = \frac{1}{-z-(d-1)m} \Rightarrow m = \frac{-z+\sqrt{z^2-4(d-1)}}{2(d-1)}$$

We take the plus sign on the square root to go to zero at $\infty$. Thus $\operatorname{spec}(H) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$. The formula for $G$, which is independent of vertex, is ($q \equiv d-1$)

$$G(z) = \frac{-(d-2)z + d\sqrt{z^2-4q}}{2(d^2-z^2)} \Rightarrow \frac{dk}{dE} = \frac{d\sqrt{4q-E^2}}{2\pi(d^2-E^2)}$$

# Example 1: Free Jacobi Matrix on a Homogeneous Tree



The equation for $m$, which is independent of vertex and edge, is

$$m = \frac{1}{-z-(d-1)m} \Rightarrow m = \frac{-z+\sqrt{z^2-4(d-1)}}{2(d-1)}$$

We take the plus sign on the square root to go to zero at $\infty$. Thus $\operatorname{spec}(H) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$. The formula for $G$, which is independent of vertex, is $(q \equiv d-1)$

$$G(z) = \frac{-(d-2)z + d\sqrt{z^2-4q}}{2(d^2-z^2)} \Rightarrow \frac{dk}{dE} = \frac{d\sqrt{4q-E^2}}{2\pi(d^2-E^2)}$$

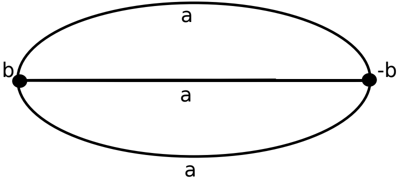
the famed Kesten–McKay distribution, which arose first in random graph models, as the DOS for a large random degree $d$ graph.

# Example 2: Bipartite Degree 3

Consider a graph with two vertices and three edges between them.

# Example 2: Bipartite Degree 3

Consider a graph with two vertices and three edges between them. All the $a = 1$ and the two $b$'s are $b$ and $-b$ as shown here

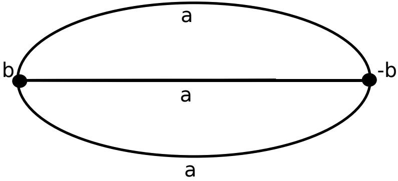



There are two $m$-functions, $m_\pm$. A direct calculation gets equations they each obey which are quadratic in the $m$ and quartic in $z$ and one finds that

# Example 2: Bipartite Degree 3



Consider a graph with two vertices and three edges between them. All the $a = 1$ and the two $b$'s are $b$ and $-b$ as shown here




There are two $m$-functions, $m_\pm$. A direct calculation gets equations they each obey which are quadratic in the $m$ and quartic in $z$ and one finds that

$$m_\pm(z) = -\frac{(z^2 - b^2) - \sqrt{(z^2 - b^2)^2 - 8(z^2 - b^2)}}{4(z \mp b)}$$

# Example 2: Bipartite Degree 3



If $P(z)$ is the polynomial in the square root, one find that $P$ vanishes at $z = \pm b, z = \pm\sqrt{b^2 + 8}$

# Example 2: Bipartite Degree 3



If $P(z)$ is the polynomial in the square root, one find that $P$ vanishes at $z = \pm b, z = \pm\sqrt{b^2+8}$ so

$$\operatorname{spec}(H) = \left[-\sqrt{b^2+8}, -b\right] \cup \left[b, \sqrt{b^2+8}\right]$$

# Example 2: Bipartite Degree 3

If $P(z)$ is the polynomial in the square root, one find that $P$ vanishes at $z = \pm b, z = \pm\sqrt{b^2+8}$ so

$$\operatorname{spec}(H) = \left[-\sqrt{b^2+8}, -b\right] \cup \left[b, \sqrt{b^2+8}\right]$$

If $b \neq 0$, there is a single gap which is *always* open.

# Example 2: Bipartite Degree 3

If $P(z)$ is the polynomial in the square root, one find that $P$ vanishes at $z = \pm b, z = \pm\sqrt{b^2+8}$ so

$$\operatorname{spec}(H) = \left[-\sqrt{b^2+8}, -b\right] \cup \left[b, \sqrt{b^2+8}\right]$$

If $b \neq 0$, there is a single gap which is *always* open. This is a strong hint that something like Borg's Theorem might hold.

# Example 3: The rg Model

Wigner von Neumann
Generic Periodic 1D
Borg's Theorem
Periodic Jacobi Matrices on Trees
Examples
Big 3 Theorems
Borg Conjectures
Figà-Talamanca Steger Model
Possible Conjectures

Aomoto found a very interesting example which we have dubbed the rg model.

# Example 3: The rg Model



Aomoto found a very interesting example which we have dubbed the rg model. $r$ and $g$ are two positive integers. The underlying finite graph has $r+g$ vertices which we thing of as $r$ red vertices and $g$ green.

# Example 3: The rg Model



Aomoto found a very interesting example which we have dubbed the rg model. $r$ and $g$ are two positive integers. The underlying finite graph has $r + g$ vertices which we thing of as $r$ red vertices and $g$ green. There are $rg$ edges one between each pair of vertices of different colors, so the red vertices have degree $g$ and the green degree $r$. $b \equiv 0$ and $a = 1$ on all edges.

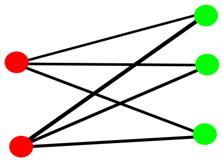
# Example 3: The rg Model

Aomoto found a very interesting example which we have dubbed the rg model. $r$ and $g$ are two positive integers. The underlying finite graph has $r+g$ vertices which we thing of as $r$ red vertices and $g$ green. There are $rg$ edges one between each pair of vertices of different colors, so the red vertices have degree $g$ and the green degree $r$. $b \equiv 0$ and $a = 1$ on all edges. Here is the graph if $r = 2, g = 3$.

# Example 3: The rg Model



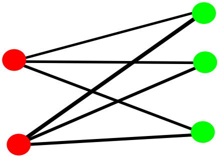
Aomoto found a very interesting example which we have dubbed the rg model. $r$ and $g$ are two positive integers. The underlying finite graph has $r+g$ vertices which we thing of as $r$ red vertices and $g$ green. There are $rg$ edges one between each pair of vertices of different colors, so the red vertices have degree $g$ and the green degree $r$. $b \equiv 0$ and $a = 1$ on all edges. Here is the graph if $r = 2, g = 3$.



# Example 3: The rg Model



Aomoto found a very interesting example which we have dubbed the rg model. $r$ and $g$ are two positive integers. The underlying finite graph has $r + g$ vertices which we thing of as $r$ red vertices and $g$ green. There are $rg$ edges one between each pair of vertices of different colors, so the red vertices have degree $g$ and the green degree $r$. $b \equiv 0$ and $a = 1$ on all edges. Here is the graph if $r = 2, g = 3$.



Aomoto showed that if $r \neq g$, this model always has an eigenvalue at $E = 0$. He analyzed some Green's function equations he had, to prove there must be a pole.

# Example 3: The rg Model

Avni, Breuer and I wrote down an explicit eigenfunction that is illuminating.

# Example 3: The rg Model



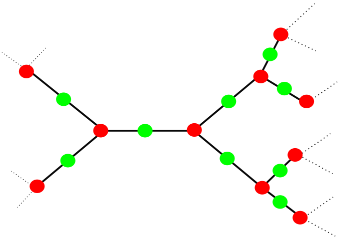
Avni, Breuer and I wrote down an explicit eigenfunction that is illuminating. One needs to display an explicit $\ell^2$ function with the property that for each vertex, the sum of the values at all the neighbors is 0.

# Example 3: The rg Model

Avni, Breuer and I wrote down an explicit eigenfunction that is illuminating. One needs to display an explicit $\ell^2$ function with the property that for each vertex, the sum of the values at all the neighbors is 0. In the above case, here is the tree

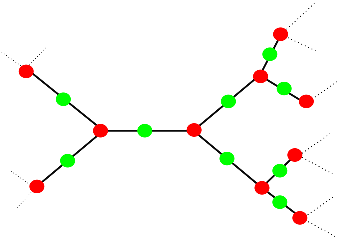
# Example 3: The rg Model

Avni, Breuer and I wrote down an explicit eigenfunction that is illuminating. One needs to display an explicit $\ell^2$ function with the property that for each vertex, the sum of the values at all the neighbors is 0. In the above case, here is the tree



# Example 3: The rg Model

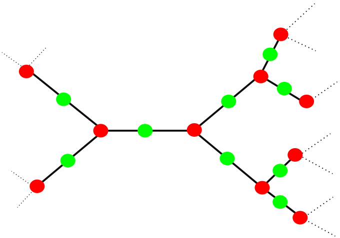
Avni, Breuer and I wrote down an explicit eigenfunction that is illuminating. One needs to display an explicit $\ell^2$ function with the property that for each vertex, the sum of the values at all the neighbors is 0. In the above case, here is the tree



The function is zero at all red vertices so the eigenfunction equation holds trivially at every green vertex.

# Example 3: The rg Model

Avni, Breuer and I wrote down an explicit eigenfunction that is illuminating. One needs to display an explicit $\ell^2$ function with the property that for each vertex, the sum of the values at all the neighbors is 0. In the above case, here is the tree



The function is zero at all red vertices so the eigenfunction equation holds trivially at every green vertex. The value at the green vertices depends only on the distance from the central vertex.

# Example 3: The rg Model



It has the value $1$ at the center and must have the value $-1/2$ are the vertices a distance $2$ away and inductively $(-1/2)^k$ are vertices a distance $2k$ from the center. The number of such vertices is $2(2)^k$ so the $\ell^2$ norm is $1 + \sum_{k=1}^{\infty} 2^{k+1}(1/2)^{2k} < \infty$.

# Example 3: The rg Model

It has the value $1$ at the center and must have the value $-1/2$ are the vertices a distance $2$ away and inductively $(-1/2)^k$ are vertices a distance $2k$ from the center. The number of such vertices is $2(2)^k$ so the $\ell^2$ norm is $1 + \sum_{k=1}^{\infty} 2^{k+1}(1/2)^{2k} < \infty$. Avni-Breuer-Simon found explicit formulae for the Green's function and showed that $0$ is an isolated point of the spectrum which if $r < g$ has DOS weight $g - r/g + r$. There are two symmetric bands each of whose DOS weight is $r/r + g$.

# Example 3: The rg Model



It has the value $1$ at the center and must have the value $-1/2$ are the vertices a distance $2$ away and inductively $(-1/2)^k$ are vertices a distance $2k$ from the center. The number of such vertices is $2(2)^k$ so the $\ell^2$ norm is $1 + \sum_{k=1}^{\infty} 2^{k+1}(1/2)^{2k} < \infty$. Avni-Breuer-Simon found explicit formulae for the Green's function and showed that $0$ is an isolated point of the spectrum which if $r < g$ has DOS weight $g - r/g + r$. There are two symmetric bands each of whose DOS weight is $r/r + g$.

Christiansen-Simon-Zinchenko (in prep) have analyzed this further and showed that the function I described and its translates span the eigenspace.

# Sunda's Theorem



So far there are three big theorems known for these families of interesting operators

# Sunda's Theorem



So far there are three big theorems known for these families of interesting operators

**Theorem 1** (Sunada, 1992) *For a period $p$ periodic Jacobi matrix on a tree, $k(E)$ in any gap has a value which is a multiple of $1/p$. This implies the spectrum has at most $p$ bands.*

# Sunda's Theorem



So far there are three big theorems known for these families of interesting operators

**Theorem 1** (Sunada, 1992) *For a period $p$ periodic Jacobi matrix on a tree, $k(E)$ in any gap has a value which is a multiple of $1/p$. This implies the spectrum has at most $p$ bands.*

Sunada doesn't discuss discrete models explicitly but instead discusss continuum models on hyperbolic manifolds and remarks *A discrete (graph-theoretical) analogue of periodic Schrödinger operators can be treated in much the same way.*

# Sunda's Theorem



So far there are three big theorems known for these families of interesting operators

**Theorem 1** (Sunada, 1992) *For a period $p$ periodic Jacobi matrix on a tree, $k(E)$ in any gap has a value which is a multiple of $1/p$. This implies the spectrum has at most $p$ bands.*

Sunada doesn't discuss discrete models explicitly but instead discusss continuum models on hyperbolic manifolds and remarks *A discrete (graph-theoretical) analogue of periodic Schrödinger operators can be treated in much the same way.* Recently (2020 preprint), Garza-Vargas and Kulkarni found an alternate proof of Sunada's theorem using free probability.

# Aomoto's Theorem

**Theorem 2** (Aomoto, 1991) *A period $p$ periodic Jacobi matrix on a homogeneous tree has no eigenvalues*

# Aomoto's Theorem

**Theorem 2** (Aomoto, 1991) *A period $p$ periodic Jacobi matrix on a homogeneous tree has no eigenvalues*

This is, of course, of especial interest because in the same paper, Aomoto described what we call the $rg$ model and proved that it did have eigenvalues.

# Aomoto's Theorem



**Theorem 2** (Aomoto, 1991) *A period $p$ periodic Jacobi matrix on a homogeneous tree has no eigenvalues*

This is, of course, of especial interest because in the same paper, Aomoto described what we call the $rg$ model and proved that it did have eigenvalues. We find Aomoto's proof extremely mysterious. He has several strange looking calculations which in the end lead to an equality that implies the result.

# No Singular Continuous Spectrum



**Theorem 3** (Avni-Breuer-Simon, 2020) *All period $p$ periodic Jacobi matrices on trees have no singular continuous spectrum*

# No Singular Continuous Spectrum

**Theorem 3** (Avni-Breuer-Simon, 2020) *All period $p$ periodic Jacobi matrices on trees have no singular continuous spectrum*

We prove this by showing that the Green's functions are algebraic functions (i.e. near infinity they solve $P(G(z), z) = 0$ for a polynomial in two variables).

# No Singular Continuous Spectrum

**Theorem 3** (Avni-Breuer-Simon, 2020) *All period $p$ periodic Jacobi matrices on trees have no singular continuous spectrum*

We prove this by showing that the Green's functions are algebraic functions (i.e. near infinity they solve $P(G(z), z) = 0$ for a polynomial in two variables). It follows that as they approach the real axis, the only possible singular points are finitely many poles and/or branch points.

# No Singular Continuous Spectrum



**Theorem 3** (Avni-Breuer-Simon, 2020) *All period $p$ periodic Jacobi matrices on trees have no singular continuous spectrum*

We prove this by showing that the Green's functions are algebraic functions (i.e. near infinity they solve $P(G(z), z) = 0$ for a polynomial in two variables). It follows that as they approach the real axis, the only possible singular points are finitely many poles and/or branch points. While he makes no mention of singular continuous spectrum, in several papers, Aomoto claims that the Green's functions are algebraic

# No Singular Continuous Spectrum



**Theorem 3** (Avni-Breuer-Simon, 2020) *All period $p$ periodic Jacobi matrices on trees have no singular continuous spectrum*

We prove this by showing that the Green's functions are algebraic functions (i.e. near infinity they solve $P(G(z), z) = 0$ for a polynomial in two variables). It follows that as they approach the real axis, the only possible singular points are finitely many poles and/or branch points. While he makes no mention of singular continuous spectrum, in several papers, Aomoto claims that the Green's functions are algebraic because he finds a series of $p$ linked algebraic equations in $z$ and the $p$ diagonal Green's functions (with square roots) that they obey.

# No Singular Continuous Spectrum

**Theorem 3** (Avni-Breuer-Simon, 2020) *All period $p$ periodic Jacobi matrices on trees have no singular continuous spectrum*

We prove this by showing that the Green's functions are algebraic functions (i.e. near infinity they solve $P(G(z), z) = 0$ for a polynomial in two variables). It follows that as they approach the real axis, the only possible singular points are finitely many poles and/or branch points. While he makes no mention of singular continuous spectrum, in several papers, Aomoto claims that the Green's functions are algebraic because he finds a series of $p$ linked algebraic equations in $z$ and the $p$ diagonal Green's functions (with square roots) that they obey. But there is a gap in the proof in that he never proves that they are independent.

# First Guess

There is an obvious first guess of how one might guess Borg's Theorem extends to trees.

# First Guess

There is an obvious first guess of how one might guess Borg's Theorem extends to trees.

**Conjecture** *If a periodic Jacobi matrix has no gaps in its spectrum, then $a$ and $b$ are each constant*

# First Guess

There is an obvious first guess of how one might guess Borg's Theorem extends to trees.

**Conjecture** *If a periodic Jacobi matrix has no gaps in its spectrum, then* $a$ *and* $b$ *are each constant*

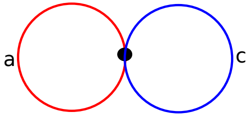
We've been working on these problems for about 5 years and for a while we thought this was a reasonable conjecture,

# First Guess

There is an obvious first guess of how one might guess Borg's Theorem extends to trees.

**Conjecture** *If a periodic Jacobi matrix has no gaps in its spectrum, then $a$ and $b$ are each constant*

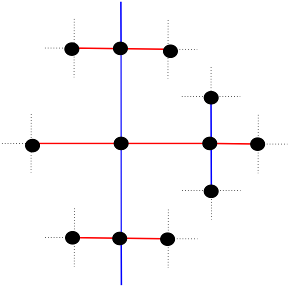
We've been working on these problems for about 5 years and for a while we thought this was a reasonable conjecture, but then we realized that




has period 1!

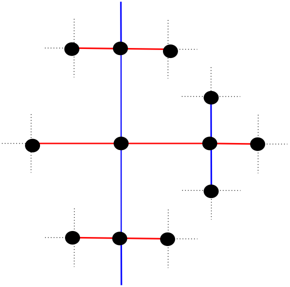
# First Guess

So the tree



# First Guess

So the tree

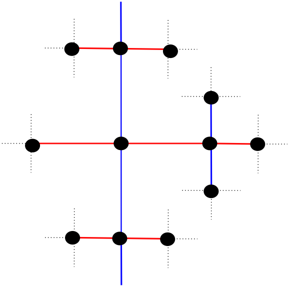


which definitely has non-constant $a$ also has period 1 and so no gap.

# First Guess



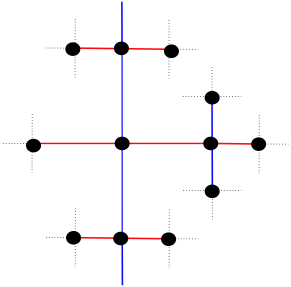
So the tree



which definitely has non-constant $a$ also has period 1 and so no gap. Clearly, a similar phenomenon works on any homogeneous tree with even degree.

# First Guess

So the tree



which definitely has non-constant $a$ also has period 1 and so no gap. Clearly, a similar phenomenon works on any homogeneous tree with even degree. If $b = 0$ and the $2k$ values of $a$ are equal in pairs, we have period 1 and no gap!

# The Conjectures



After thinking about this, we decided it was a decent guess that this was the only counterexample

# The Conjectures

After thinking about this, we decided it was a decent guess that this was the only counterexample so our published paper has the following

# The Conjectures



After thinking about this, we decided it was a decent guess that this was the only counterexample so our published paper has the following

**Conjecture 1**. *Let $\mathcal{T}$ be a regular tree of odd degree.*

# The Conjectures

After thinking about this, we decided it was a decent guess that this was the only counterexample so our published paper has the following

**Conjecture 1**. *Let $\mathcal{T}$ be a regular tree of odd degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then $b$ is constant and $a$ is constant.*

# The Conjectures

After thinking about this, we decided it was a decent guess that this was the only counterexample so our published paper has the following

**Conjecture 1**. *Let $\mathcal{T}$ be a regular tree of odd degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then $b$ is constant and $a$ is constant.*

**Conjecture 2**. *Let $\mathcal{T}$ be a regular tree of even degree.*

# The Conjectures



After thinking about this, we decided it was a decent guess that this was the only counterexample so our published paper has the following

**Conjecture 1**. *Let $\mathcal{T}$ be a regular tree of odd degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then $b$ is constant and $a$ is constant.*

**Conjecture 2**. *Let $\mathcal{T}$ be a regular tree of even degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then the period is $1$.*

# The Conjectures

After thinking about this, we decided it was a decent guess that this was the only counterexample so our published paper has the following

**Conjecture 1**. *Let $\mathcal{T}$ be a regular tree of odd degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then $b$ is constant and $a$ is constant.*

**Conjecture 2**. *Let $\mathcal{T}$ be a regular tree of even degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then the period is $1$.*

That means, $\mathcal{G}$ has a single $b$ and $\deg(\mathcal{T})/2$ self loops.

# The Conjectures

After thinking about this, we decided it was a decent guess that this was the only counterexample so our published paper has the following

**Conjecture 1**. *Let $\mathcal{T}$ be a regular tree of odd degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then $b$ is constant and $a$ is constant.*

**Conjecture 2**. *Let $\mathcal{T}$ be a regular tree of even degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then the period is $1$.*

That means, $\mathcal{G}$ has a single $b$ and $\deg(\mathcal{T})/2$ self loops.

**Conjecture 3**. *Let $\mathcal{T}$ be a tree which is not regular.*

# The Conjectures

After thinking about this, we decided it was a decent guess that this was the only counterexample so our published paper has the following

**Conjecture 1**. *Let $\mathcal{T}$ be a regular tree of odd degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then $b$ is constant and $a$ is constant.*

**Conjecture 2**. *Let $\mathcal{T}$ be a regular tree of even degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then the period is $1$.*

That means, $\mathcal{G}$ has a single $b$ and $\deg(\mathcal{T})/2$ self loops.

**Conjecture 3**. *Let $\mathcal{T}$ be a tree which is not regular. If $H(\mathcal{T})$ is a periodic Jacobi matrix, then it must have gaps in its spectrum.*

# The Conjectures

After thinking about this, we decided it was a decent guess that this was the only counterexample so our published paper has the following

**Conjecture 1**. *Let $\mathcal{T}$ be a regular tree of odd degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then $b$ is constant and $a$ is constant.*

**Conjecture 2**. *Let $\mathcal{T}$ be a regular tree of even degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then the period is $1$.*
That means, $\mathcal{G}$ has a single $b$ and $\deg(\mathcal{T})/2$ self loops.

**Conjecture 3**. *Let $\mathcal{T}$ be a tree which is not regular. If $H(\mathcal{T})$ is a periodic Jacobi matrix, then it must have gaps in its spectrum.*

Actually, these are a single conjecture that no gaps implies period $1$!

# The Conjectures



But we wish to emphasize the different forms

# The Conjectures

But we wish to emphasize the different forms and the proofs may be different.

# The Conjectures



But we wish to emphasize the different forms and the proofs may be different. As for all gaps open, we made two conjectures.

# The Conjectures

But we wish to emphasize the different forms and the proofs may be different. As for all gaps open, we made two conjectures. Let $\mathcal{G}$ be a finite graph. Let $\mathcal{P}(\mathcal{G})$ be the set of allowed Jacobi parameters.

# The Conjectures

But we wish to emphasize the different forms and the proofs may be different. As for all gaps open, we made two conjectures. Let $\mathcal{G}$ be a finite graph. Let $\mathcal{P}(\mathcal{G})$ be the set of allowed Jacobi parameters. It is an open orthant of $\mathbb{R}^{p+q}$ since $p+q$ is the number of vertices plus the number of edges.

# The Conjectures



But we wish to emphasize the different forms and the proofs may be different. As for all gaps open, we made two conjectures. Let $\mathcal{G}$ be a finite graph. Let $\mathcal{P}(\mathcal{G})$ be the set of allowed Jacobi parameters. It is an open orthant of $\mathbb{R}^{p+q}$ since $p+q$ is the number of vertices plus the number of edges. We say a period $p$ Jacobi matrix has all gaps open if the spectrum has $p$ bands.

# The Conjectures



But we wish to emphasize the different forms and the proofs may be different. As for all gaps open, we made two conjectures. Let $\mathcal{G}$ be a finite graph. Let $\mathcal{P}(\mathcal{G})$ be the set of allowed Jacobi parameters. It is an open orthant of $\mathbb{R}^{p+q}$ since $p+q$ is the number of vertices plus the number of edges. We say a period $p$ Jacobi matrix has all gaps open if the spectrum has $p$ bands. It is easy to see the set of Jacobi parameters for which all gaps are open is an open set.

# The Conjectures

But we wish to emphasize the different forms and the proofs may be different. As for all gaps open, we made two conjectures. Let $\mathcal{G}$ be a finite graph. Let $\mathcal{P}(\mathcal{G})$ be the set of allowed Jacobi parameters. It is an open orthant of $\mathbb{R}^{p+q}$ since $p+q$ is the number of vertices plus the number of edges. We say a period $p$ Jacobi matrix has all gaps open if the spectrum has $p$ bands. It is easy to see the set of Jacobi parameters for which all gaps are open is an open set.

**Conjecture 4.** *The set of parameters with all gaps open is a dense open set in the set of allowed parameters.*

# The Conjectures

But we wish to emphasize the different forms and the proofs may be different. As for all gaps open, we made two conjectures. Let $\mathcal{G}$ be a finite graph. Let $\mathcal{P}(\mathcal{G})$ be the set of allowed Jacobi parameters. It is an open orthant of $\mathbb{R}^{p+q}$ since $p+q$ is the number of vertices plus the number of edges. We say a period $p$ Jacobi matrix has all gaps open if the spectrum has $p$ bands. It is easy to see the set of Jacobi parameters for which all gaps are open is an open set.

**Conjecture 4.** *The set of parameters with all gaps open is a dense open set in the set of allowed parameters.*

We at least know the set is non-empty, for if all $b$ are different and $\sum a < \min_{i\neq j} |b_i - b_j|$, then all gaps are open.

# The Conjectures

But we wish to emphasize the different forms and the proofs may be different. As for all gaps open, we made two conjectures. Let $\mathcal{G}$ be a finite graph. Let $\mathcal{P}(\mathcal{G})$ be the set of allowed Jacobi parameters. It is an open orthant of $\mathbb{R}^{p+q}$ since $p+q$ is the number of vertices plus the number of edges. We say a period $p$ Jacobi matrix has all gaps open if the spectrum has $p$ bands. It is easy to see the set of Jacobi parameters for which all gaps are open is an open set.

**Conjecture 4**. *The set of parameters with all gaps open is a dense open set in the set of allowed parameters.*

We at least know the set is non-empty, for if all $b$ are different and $\sum a < \min_{i\neq j} |b_i - b_j|$, then all gaps are open. Thinking Wigner-von Neumann, we conjectured

# The Conjectures

But we wish to emphasize the different forms and the proofs may be different. As for all gaps open, we made two conjectures. Let $\mathcal{G}$ be a finite graph. Let $\mathcal{P}(\mathcal{G})$ be the set of allowed Jacobi parameters. It is an open orthant of $\mathbb{R}^{p+q}$ since $p+q$ is the number of vertices plus the number of edges. We say a period $p$ Jacobi matrix has all gaps open if the spectrum has $p$ bands. It is easy to see the set of Jacobi parameters for which all gaps are open is an open set.

**Conjecture 4**. *The set of parameters with all gaps open is a dense open set in the set of allowed parameters.*

We at least know the set is non-empty, for if all $b$ are different and $\sum a < \min_{i \neq j} |b_i - b_j|$, then all gaps are open. Thinking Wigner-von Neumann, we conjectured

**Conjecture 5** *The set of parameters where all gaps are not open is a variety of codimension 2.*

# An interesting email

I'm sure you've heard lots of complaints about it taking too long from acceptance of a paper to publication.

# An interesting email

I'm sure you've heard lots of complaints about it taking too long from acceptance of a paper to publication. But I doubt you've heard a complaint about it taking too little time.

# An interesting email

I'm sure you've heard lots of complaints about it taking too long from acceptance of a paper to publication. But I doubt you've heard a complaint about it taking too little time. It was literally 11 days between submission of the final version of our accepted paper for *Advances in Math* and the appearance of proofs in our mailbox.

# An interesting email



I'm sure you've heard lots of complaints about it taking too long from acceptance of a paper to publication. But I doubt you've heard a complaint about it taking too little time. It was literally 11 days between submission of the final version of our accepted paper for *Advances in Math* and the appearance of proofs in our mailbox. Not surprisingly, there were no changes in our paper so we returned proofs rapidly and 2 days after we received proofs, the paper appeared "published" online.

# An interesting email

I'm sure you've heard lots of complaints about it taking too long from acceptance of a paper to publication. But I doubt you've heard a complaint about it taking too little time. It was literally 11 days between submission of the final version of our accepted paper for *Advances in Math* and the appearance of proofs in our mailbox. Not surprisingly, there were no changes in our paper so we returned proofs rapidly and 2 days after we received proofs, the paper appeared "published" online. And 16 days after that, we received an email from two graduate students at Berkeley, Jorge Garza Vargas and Achit Kulkarni,

# An interesting email

I'm sure you've heard lots of complaints about it taking too long from acceptance of a paper to publication. But I doubt you've heard a complaint about it taking too little time. It was literally 11 days between submission of the final version of our accepted paper for *Advances in Math* and the appearance of proofs in our mailbox. Not surprisingly, there were no changes in our paper so we returned proofs rapidly and 2 days after we received proofs, the paper appeared "published" online. And 16 days after that, we received an email from two graduate students at Berkeley, Jorge Garza Vargas and Achit Kulkarni, with counter examples to several of our recently published conjectures!!!!

# An interesting email



Their manuscript was a second version of a preprint that they posted in the arXiv, a few weeks after our paper (about 7 months before).

# An interesting email



Their manuscript was a second version of a preprint that they posted in the arXiv, a few weeks after our paper (about 7 months before). That original version didn't know of our work nor of Sunada's much earlier work.

# An interesting email



Their manuscript was a second version of a preprint that they posted in the arXiv, a few weeks after our paper (about 7 months before). That original version didn't know of our work nor of Sunada's much earlier work. They had rediscovered Sunada's gap labelling theorem with a new proof using free probability theory.

# An interesting email



Their manuscript was a second version of a preprint that they posted in the arXiv, a few weeks after our paper (about 7 months before). That original version didn't know of our work nor of Sunada's much earlier work. They had rediscovered Sunada's gap labelling theorem with a new proof using free probability theory. They found our preprint in the meantime, realized their main result wasn't new and found our Borg conjectures which they determined were false!

# FTS Model

For counterexamples to Conjectures 1 and 2, the heavy lifting had been done earlier.

# FTS Model



For counterexamples to Conjectures 1 and 2, the heavy lifting had been done earlier. Jacobi matrices on the degree $d$ homogeneous tree with $b = 0$ and the each vertex with the same three $a$ values are connected to random walks on certain groups if the $a$'s from a single vertex sum to $1$.

# FTS Model

For counterexamples to Conjectures 1 and 2, the heavy lifting had been done earlier. Jacobi matrices on the degree $d$ homogeneous tree with $b = 0$ and the each vertex with the same three $a$ values are connected to random walks on certain groups if the $a$'s from a single vertex sum to $1$. In 1985, Tim Steeger proved the main result we'll need in his PhD thesis

# FTS Model

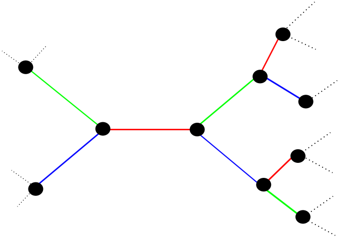
For counterexamples to Conjectures 1 and 2, the heavy lifting had been done earlier. Jacobi matrices on the degree $d$ homogeneous tree with $b = 0$ and the each vertex with the same three $a$ values are connected to random walks on certain groups if the $a$'s from a single vertex sum to $1$. In 1985, Tim Steeger proved the main result we'll need in his PhD thesis and the thesis only appeared in print in a 1994 as an AMS Memoir jointly with his thesis advisor Alessandro Figà-Talamanca.

# FTS Model

For counterexamples to Conjectures 1 and 2, the heavy lifting had been done earlier. Jacobi matrices on the degree $d$ homogeneous tree with $b = 0$ and the each vertex with the same three $a$ values are connected to random walks on certain groups if the $a$'s from a single vertex sum to $1$. In 1985, Tim Steeger proved the main result we'll need in his PhD thesis and the thesis only appeared in print in a 1994 as an AMS Memoir jointly with his thesis advisor Alessandro Figà-Talamanca. First their model when $d = 3$

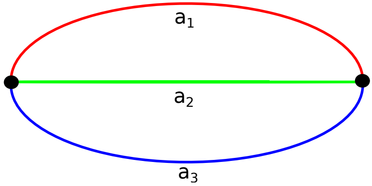
# FTS Model

For counterexamples to Conjectures 1 and 2, the heavy lifting had been done earlier. Jacobi matrices on the degree $d$ homogeneous tree with $b = 0$ and the each vertex with the same three $a$ values are connected to random walks on certain groups if the $a$'s from a single vertex sum to $1$. In 1985, Tim Steeger proved the main result we'll need in his PhD thesis and the thesis only appeared in print in a 1994 as an AMS Memoir jointly with his thesis advisor Alessandro Figà-Talamanca. First their model when $d = 3$



# FTS Model

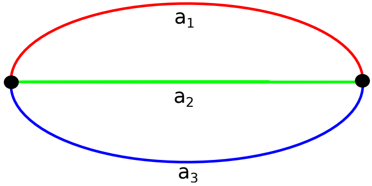
This model is the lift of the $2$ point



# FTS Model



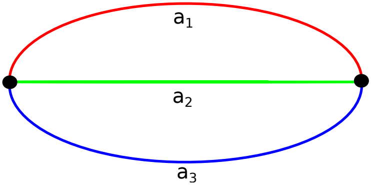
This model is the lift of the $2$ point




so the period is $2$ and there is at most one gap open.

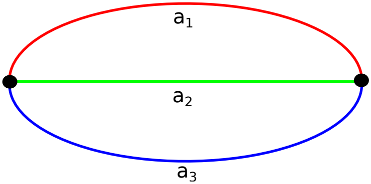
# FTS Model

This model is the lift of the $2$ point




so the period is $2$ and there is at most one gap open. Their important result is

This model is the lift of the $2$ point




so the period is $2$ and there is at most one gap open. Their important result is

**Theorem** (Figà-Talamanca-Steger, 1985/1994) *Let $H$ be the Jacobi matrix on the degree $d$ homogeneous tree with $b = 0$ and $a_1 \geq a_2 \ldots \geq a_d$ at each vertex. Then $0 \in \mathrm{spec}(H)$ if and only if*

$$a_1^2 \leq \sum_{j=2}^{d} a_j^2$$

# FTS Model



GVK realized that because the unitary, $U$, that flips signs at odd vertices obeys $UHU^{-1} = -H$, if there is a gap, then $0$ must be in the gap, so not in the spectrum

# FTS Model

GVK realized that because the unitary, $U$, that flips signs at odd vertices obeys $UHU^{-1} = -H$, if there is a gap, then $0$ must be in the gap, so not in the spectrum and conversely, if there is no gap, then the spectrum is $[-c, c]$, so $0 \in \operatorname{spec}(H)$

# FTS Model



GVK realized that because the unitary, $U$, that flips signs at odd vertices obeys $UHU^{-1} = -H$, if there is a gap, then $0$ must be in the gap, so not in the spectrum and conversely, if there is no gap, then the spectrum is $[-c, c]$, so $0 \in \mathrm{spec}(H)$ so the last equation gives a necessary and sufficient condition for there to be no gap.

# FTS Model

GVK realized that because the unitary, $U$, that flips signs at odd vertices obeys $UHU^{-1} = -H$, if there is a gap, then $0$ must be in the gap, so not in the spectrum and conversely, if there is no gap, then the spectrum is $[-c, c]$, so $0 \in \mathrm{spec}(H)$ so the last equation gives a necessary and sufficient condition for there to be no gap. Thus the gap is closed for example if $d-1$ $a$'s have the value $1$ and the other $a$ has the value $\beta$ with $0 < \beta \leq \sqrt{d-1}$ providing lots of examples where there is no gap for a period $2$ model with $a$ not constant.

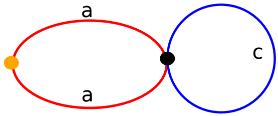
# FTS Model



GVK realized that because the unitary, $U$, that flips signs at odd vertices obeys $UHU^{-1} = -H$, if there is a gap, then $0$ must be in the gap, so not in the spectrum and conversely, if there is no gap, then the spectrum is $[-c, c]$, so $0 \in \mathrm{spec}(H)$ so the last equation gives a necessary and sufficient condition for there to be no gap. Thus the gap is closed for example if $d-1$ $a$'s have the value $1$ and the other $a$ has the value $\beta$ with $0 < \beta \leq \sqrt{d-1}$ providing lots of examples where there is no gap for a period $2$ model with $a$ not constant. Moreover, one has $p = 2$ and $q = d$, so noting that the gap remains closed if the equation holds and $b_1 = b_2$, we get a set with no gap of codimension $1$ contradicting Conjecture 5!

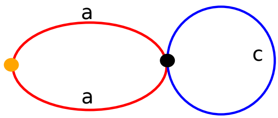
# GVK Model

GVK also showed that the model with $p = 2$, $q = 3$ with graph



# GVK Model

GVK also showed that the model with $p = 2$, $q = 3$ with graph




has no gap providing a counter example to Conjecture 3 (since one vertex has degree 2 and one has degree 4).

# Latest Borg Attempt



All the counterexamples we know have constant $b$.

# Latest Borg Attempt



All the counterexamples we know have constant $b$. The most conservative conjecture to make is

# Latest Borg Attempt

All the counterexamples we know have constant $b$. The most conservative conjecture to make is

**Conjecture 6** *Let $H$ be a periodic Jacobi matrix on the degree $d$ homogeneous tree with all $a = 1$. If $H$ has no gaps in its spectrum, then $b$ is constant*

# Latest Borg Attempt

All the counterexamples we know have constant $b$. The most conservative conjecture to make is

**Conjecture 6** *Let $H$ be a periodic Jacobi matrix on the degree $d$ homogeneous tree with all $a = 1$. If $H$ has no gaps in its spectrum, then $b$ is constant*

Braver is to conjecture a stronger result

# Latest Borg Attempt

All the counterexamples we know have constant $b$. The most conservative conjecture to make is

**Conjecture 6** *Let $H$ be a periodic Jacobi matrix on the degree $d$ homogeneous tree with all $a = 1$. If $H$ has no gaps in its spectrum, then $b$ is constant*

Braver is to conjecture a stronger result

**Conjecture 7** *Let $H$ be a periodic Jacobi matrix on some tree. If $H$ has no gaps in its spectrum, then $b$ is constant*

# Latest Borg Attempt



All the counterexamples we know have constant $b$. The most conservative conjecture to make is

**Conjecture 6** *Let $H$ be a periodic Jacobi matrix on the degree $d$ homogeneous tree with all $a = 1$. If $H$ has no gaps in its spectrum, then $b$ is constant*

Braver is to conjecture a stronger result

**Conjecture 7** *Let $H$ be a periodic Jacobi matrix on some tree. If $H$ has no gaps in its spectrum, then $b$ is constant*

Note that Borg only considered the potential whose analog is $b$ so one can claim that all along this is the correct analog of Borg.

# What about Wigner-von Neumann

Once we accept the fact that gap opening depends on $b$ and $a$'s don't help, one can understand why Conjecture $5$ failed for $p = 2$. Given that adding a constant to all $b's$ doesn't change which gaps are open, when $p = 2$, there are only two $b$'s and one free parameter so, of course, only codimension 1.

# What about Wigner-von Neumann



Once we accept the fact that gap opening depends on $b$ and $a$'s don't help, one can understand why Conjecture $5$ failed for $p = 2$. Given that adding a constant to all $b's$ doesn't change which gaps are open, when $p = 2$, there are only two $b$'s and one free parameter so, of course, only codimension 1. So we can be really brave and

**Conjecture 8** *If the period $p \geq 3$, the set of parameters where all gaps are not open is a variety of codimension 2.*

# What about Wigner-von Neumann



Once we accept the fact that gap opening depends on $b$ and $a$'s don't help, one can understand why Conjecture $5$ failed for $p = 2$. Given that adding a constant to all $b's$ doesn't change which gaps are open, when $p = 2$, there are only two $b$'s and one free parameter so, of course, only codimension 1. So we can be really brave and

**Conjecture 8** *If the period $p \geq 3$, the set of parameters where all gaps are not open is a variety of codimension 2.*

The viewer should decide if this like grasping at straws or random walking towards successful conjectures!