Berry's Phase, TKN^2 Integers and All That: My work on Topology in Condensed Matter Physics 1983-1993

Barry Simon
IBM Professor of Mathematics and Theoretical Physics, Emeritus
California Institute of Technology
Pasadena, CA, U.S.A.

Introduction

I was a pioneer in the use of topology and geometry (mathematicians sometimes use "geometry" when there is an underlying distance and "topology" for those geometric object that don't rely on a distance) in NRQM.

Introduction

I was a pioneer in the use of topology and geometry (mathematicians sometimes use "geometry" when there is an underlying distance and "topology" for those geometric object that don't rely on a distance) in NRQM. In particular, Avron, Seiler and I realized that the approach of Thouless et al. to the quantum Hall effect (for which Thouless got the Nobel prize)

Introduction

I was a pioneer in the use of topology and geometry (mathematicians sometimes use "geometry" when there is an underlying distance and "topology" for those geometric object that don't rely on a distance) in NRQM. In particular, Avron, Seiler and I realized that the approach of Thouless et al. to the quantum Hall effect (for which Thouless got the Nobel prize) was basically an expression of the homotopy invariants (aka Chern integers) of a natural line bundle that arises in certain eigenvalue perturbation situations,

Introduction

I was a pioneer in the use of topology and geometry (mathematicians sometimes use "geometry" when there is an underlying distance and "topology" for those geometric object that don't rely on a distance) in NRQM. In particular, Avron, Seiler and I realized that the approach of Thouless et al. to the quantum Hall effect (for which Thouless got the Nobel prize) was basically an expression of the homotopy invariants (aka Chern integers) of a natural line bundle that arises in certain eigenvalue perturbation situations, and I realized that the phase that Berry found in the quantum adiabatic theorem is holonomy in this bundle

Introduction

I was a pioneer in the use of topology and geometry (mathematicians sometimes use "geometry" when there is an underlying distance and "topology" for those geometric object that don't rely on a distance) in NRQM. In particular, Avron, Seiler and I realized that the approach of Thouless et al. to the quantum Hall effect (for which Thouless got the Nobel prize) was basically an expression of the homotopy invariants (aka Chern integers) of a natural line bundle that arises in certain eigenvalue perturbation situations, and I realized that the phase that Berry found in the quantum adiabatic theorem is holonomy in this bundle and that the quantity Berry used to compute this phase (and which independently had been found by Avron et al), now called the *Berry curvature*,

Introduction

I was a pioneer in the use of topology and geometry (mathematicians sometimes use "geometry" when there is an underlying distance and "topology" for those geometric object that don't rely on a distance) in NRQM. In particular, Avron, Seiler and I realized that the approach of Thouless et al. to the quantum Hall effect (for which Thouless got the Nobel prize) was basically an expression of the homotopy invariants (aka Chern integers) of a natural line bundle that arises in certain eigenvalue perturbation situations, and I realized that the phase that Berry found in the quantum adiabatic theorem is holonomy in this bundle and that the quantity Berry used to compute this phase (and which independently had been found by Avron et al), now called the *Berry curvature*, is just the curvature in this line bundle.

Introduction

I emphasize that Thouless et al never mention "topology" and that Thouless learned they'd found a topological invariant, essentially the Chern class, from me.

Introduction

I emphasize that Thouless et al never mention "topology" and that Thouless learned they'd found a topological invariant, essentially the Chern class, from me. And the only mention of curvature or holonomy in Berry paper is where he remarks that

Introduction

I emphasize that Thouless et al never mention "topology" and that Thouless learned they'd found a topological invariant, essentially the Chern class, from me. And the only mention of curvature or holonomy in Berry paper is where he remarks that *Barry Simon, commenting on the original version of this paper, points out that the geometrical phase factor has a mathematical interpretation in terms of holonomy, with the phase two-form emerging naturally (in the form (7 b)) as the curvature (first Chern class) of a Hermitian line bundle*.

My Background

As a mathematician, I am mainly an analyst and most of my training and expertise is analytic so I should explain about how I came to know enough toplogy/geometry to realize its significance in NRQM.

My Background

As a mathematician, I am mainly an analyst and most of my training and expertise is analytic so I should explain about how I came to know enough toplogy/geometry to realize its significance in NRQM. As a freshman at Harvard, I took the celebrated Math 55 Advanced Calculus course whose first half did differential calculus in Banach spaces and second half integral calculus on manifolds. This was a dip into the sea of geometry but without any discussion of Riemannian metrics or curvature.

My Background

As a mathematician, I am mainly an analyst and most of my training and expertise is analytic so I should explain about how I came to know enough toplogy/geometry to realize its significance in NRQM. As a freshman at Harvard, I took the celebrated Math 55 Advanced Calculus course whose first half did differential calculus in Banach spaces and second half integral calculus on manifolds. This was a dip into the sea of geometry but without any discussion of Riemannian metrics or curvature. A key part of my education was a course on Algebraic Topology in senior year It was a wonderful course and I got into the subject,

My Background

As a mathematician, I am mainly an analyst and most of my training and expertise is analytic so I should explain about how I came to know enough toplogy/geometry to realize its significance in NRQM. As a freshman at Harvard, I took the celebrated Math 55 Advanced Calculus course whose first half did differential calculus in Banach spaces and second half integral calculus on manifolds. This was a dip into the sea of geometry but without any discussion of Riemannian metrics or curvature. A key part of my education was a course on Algebraic Topology in senior year It was a wonderful course and I got into the subject, so much that the instructor took me aside and tried to convince me to give up mathematical physics and switch to topology.

My Background

As a mathematician, I am mainly an analyst and most of my training and expertise is analytic so I should explain about how I came to know enough toplogy/geometry to realize its significance in NRQM. As a freshman at Harvard, I took the celebrated Math 55 Advanced Calculus course whose first half did differential calculus in Banach spaces and second half integral calculus on manifolds. This was a dip into the sea of geometry but without any discussion of Riemannian metrics or curvature. A key part of my education was a course on Algebraic Topology in senior year It was a wonderful course and I got into the subject, so much that the instructor took me aside and tried to convince me to give up mathematical physics and switch to topology. I was particularly taken with the homotopy group long exact sequence of a fibration.

Hopf Fibration

One of the simplest examples of fibrations of interest in physics is the Hopf fibration, a natural map of S^3 to S^2.

Hopf Fibration

One of the simplest examples of fibrations of interest in physics is the Hopf fibration, a natural map of S^3 to S^2. Let $\sigma_j; j = 1, 2, 3$ be the usual Pauli σ matrices.

Hopf Fibration

One of the simplest examples of fibrations of interest in physics is the Hopf fibration, a natural map of S^3 to S^2. Let $\sigma_j; j = 1, 2, 3$ be the usual Pauli σ matrices. If $\mathbf{a} = (a_0, a_1, a_2, a_3) = (a_0, \overrightarrow{a})$ is a unit vector in $\mathbb{R}^4$, then $U(\mathbf{a}) \equiv a_0\mathbf{1} + i\overrightarrow{a} \cdot \sigma$ is a unitary matrix with determinant 1 if and only if $\mathbf{a} \in S^3$.

Hopf Fibration

One of the simplest examples of fibrations of interest in physics is the Hopf fibration, a natural map of S^3 to S^2. Let $\sigma_j; j = 1, 2, 3$ be the usual Pauli σ matrices. If $\mathbf{a} = (a_0, a_1, a_2, a_3) = (a_0, \overrightarrow{a})$ is a unit vector in $\mathbb{R}^4$, then $U(\mathbf{a}) \equiv a_0\mathbf{1} + i\overrightarrow{a} \cdot \sigma$ is a unitary matrix with determinant 1 if and only if $\mathbf{a} \in S^3$. There is a rotation $R(\mathbf{a})$ on S^2 defined by $U(\mathbf{a})(\mathbf{b} \cdot \sigma)U(\mathbf{a})^{-1} = R(\mathbf{a})(\mathbf{b}) \cdot \sigma$, the Cayley-Klein parametrization, a map of $SU(2)$ onto $SO(3)$.

Hopf Fibration

One of the simplest examples of fibrations of interest in physics is the Hopf fibration, a natural map of S^3 to S^2. Let $\sigma_j; j = 1, 2, 3$ be the usual Pauli σ matrices. If $\mathbf{a} = (a_0, a_1, a_2, a_3) = (a_0, \overrightarrow{a})$ is a unit vector in $\mathbb{R}^4$, then $U(\mathbf{a}) \equiv a_0 \mathbf{1} + i\overrightarrow{a} \cdot \sigma$ is a unitary matrix with determinant 1 if and only if $\mathbf{a} \in S^3$. There is a rotation $R(\mathbf{a})$ on S^2 defined by $U(\mathbf{a})(\mathbf{b} \cdot \sigma)U(\mathbf{a})^{-1} = R(\mathbf{a})(\mathbf{b}) \cdot \sigma$, the Cayley-Klein parametrization, a map of $SU(2)$ onto $SO(3)$. If e_3 is a unit vector in the z direction, then $\mathbf{a} \mapsto R(\mathbf{a})e_3$ is the Hopf fibration, $H : S^3 \to S^2$.

Hopf Fibration

One of the simplest examples of fibrations of interest in physics is the Hopf fibration, a natural map of S^3 to S^2. Let $\sigma_j; j = 1, 2, 3$ be the usual Pauli σ matrices. If $\mathbf{a} = (a_0, a_1, a_2, a_3) = (a_0, \overrightarrow{a})$ is a unit vector in $\mathbb{R}^4$, then $U(\mathbf{a}) \equiv a_0\mathbf{1} + i\overrightarrow{a} \cdot \sigma$ is a unitary matrix with determinant 1 if and only if $\mathbf{a} \in S^3$. There is a rotation $R(\mathbf{a})$ on S^2 defined by $U(\mathbf{a})(\mathbf{b} \cdot \sigma)U(\mathbf{a})^{-1} = R(\mathbf{a})(\mathbf{b}) \cdot \sigma$, the Cayley-Klein parametrization, a map of $SU(2)$ onto $SO(3)$. If e_3 is a unit vector in the z direction, then $\mathbf{a} \mapsto R(\mathbf{a})e_3$ is the Hopf fibration, $H : S^3 \to S^2$. It is easy to see

Hopf Fibration

One of the simplest examples of fibrations of interest in physics is the Hopf fibration, a natural map of S^3 to S^2. Let $\sigma_j; j = 1, 2, 3$ be the usual Pauli σ matrices. If $\mathbf{a} = (a_0, a_1, a_2, a_3) = (a_0, \overrightarrow{a})$ is a unit vector in $\mathbb{R}^4$, then $U(\mathbf{a}) \equiv a_0\mathbf{1} + i\overrightarrow{a} \cdot \sigma$ is a unitary matrix with determinant 1 if and only if $\mathbf{a} \in S^3$. There is a rotation $R(\mathbf{a})$ on S^2 defined by $U(\mathbf{a})(\mathbf{b} \cdot \sigma)U(\mathbf{a})^{-1} = R(\mathbf{a})(\mathbf{b}) \cdot \sigma$, the Cayley-Klein parametrization, a map of $SU(2)$ onto $SO(3)$. If e_3 is a unit vector in the z direction, then $\mathbf{a} \mapsto R(\mathbf{a})e_3$ is the Hopf fibration, $H : S^3 \to S^2$. It is easy to see (for example, by looking at the inverse images of the north and south poles) that inverse images of distinct points under H are circles which are linked

Hopf Fibration

One of the simplest examples of fibrations of interest in physics is the Hopf fibration, a natural map of S^3 to S^2. Let $\sigma_j; j = 1, 2, 3$ be the usual Pauli σ matrices. If $\mathbf{a} = (a_0, a_1, a_2, a_3) = (a_0, \overrightarrow{a})$ is a unit vector in $\mathbb{R}^4$, then $U(\mathbf{a}) \equiv a_0\mathbf{1} + i\overrightarrow{a} \cdot \sigma$ is a unitary matrix with determinant 1 if and only if $\mathbf{a} \in S^3$. There is a rotation $R(\mathbf{a})$ on S^2 defined by $U(\mathbf{a})(\mathbf{b} \cdot \sigma)U(\mathbf{a})^{-1} = R(\mathbf{a})(\mathbf{b}) \cdot \sigma$, the Cayley-Klein parametrization, a map of $SU(2)$ onto $SO(3)$. If e_3 is a unit vector in the z direction, then $\mathbf{a} \mapsto R(\mathbf{a})e_3$ is the Hopf fibration, $H : S^3 \to S^2$. It is easy to see (for example, by looking at the inverse images of the north and south poles) that inverse images of distinct points under H are circles which are linked so the map is homotopically non-trivial proving that $\pi_3(S^2)$ is non-zero

Hopf Fibration

One of the simplest examples of fibrations of interest in physics is the Hopf fibration, a natural map of S^3 to S^2. Let $\sigma_j; j = 1, 2, 3$ be the usual Pauli σ matrices. If $\mathbf{a} = (a_0, a_1, a_2, a_3) = (a_0, \overrightarrow{a})$ is a unit vector in $\mathbb{R}^4$, then $U(\mathbf{a}) \equiv a_0\mathbf{1} + i\overrightarrow{a} \cdot \sigma$ is a unitary matrix with determinant 1 if and only if $\mathbf{a} \in S^3$. There is a rotation $R(\mathbf{a})$ on S^2 defined by $U(\mathbf{a})(\mathbf{b} \cdot \sigma)U(\mathbf{a})^{-1} = R(\mathbf{a})(\mathbf{b}) \cdot \sigma$, the Cayley-Klein parametrization, a map of $SU(2)$ onto $SO(3)$. If e_3 is a unit vector in the z direction, then $\mathbf{a} \mapsto R(\mathbf{a})e_3$ is the Hopf fibration, $H : S^3 \to S^2$. It is easy to see (for example, by looking at the inverse images of the north and south poles) that inverse images of distinct points under H are circles which are linked so the map is homotopically non-trivial proving that $\pi_3(S^2)$ is non-zero (in fact, this homotopy group is generated by H and is just $\mathbb{Z}$).

Witten

Geometry in the naive sense was present, even central, to some of my work in 1970's,

Witten

Geometry in the naive sense was present, even central, to some of my work in 1970's, for example my work on phase space methods in N-body NQRM and I had even discussed that the Agmon metric, which I named, was the geodesic distance in a suitable Riemann metric

Witten

Geometry in the naive sense was present, even central, to some of my work in 1970's, for example my work on phase space methods in N-body NQRM and I had even discussed that the Agmon metric, which I named, was the geodesic distance in a suitable Riemann metric but if one thinks of "real" geometry needing curvature and "real" topology needing homology or homotopy invariants, I'd not used them in my research in the '70's.

Witten

Geometry in the naive sense was present, even central, to some of my work in 1970's, for example my work on phase space methods in N-body NQRM and I had even discussed that the Agmon metric, which I named, was the geodesic distance in a suitable Riemann metric but if one thinks of "real" geometry needing curvature and "real" topology needing homology or homotopy invariants, I'd not used them in my research in the '70's.

In the early 1980's I was motivated by Witten's seminal paper on the supersymmetry proof of the Morse inequalities and index theorem.

Witten

Geometry in the naive sense was present, even central, to some of my work in 1970's, for example my work on phase space methods in N-body NQRM and I had even discussed that the Agmon metric, which I named, was the geodesic distance in a suitable Riemann metric but if one thinks of "real" geometry needing curvature and "real" topology needing homology or homotopy invariants, I'd not used them in my research in the '70's.

In the early 1980's I was motivated by Witten's seminal paper on the supersymmetry proof of the Morse inequalities and index theorem. This paper has been celebrated not only for the results itself but because of the bridge it opened up between high energy theorists studying gauge (and later string) theories and topologists

Witten

Geometry in the naive sense was present, even central, to some of my work in 1970's, for example my work on phase space methods in N-body NQRM and I had even discussed that the Agmon metric, which I named, was the geodesic distance in a suitable Riemann metric but if one thinks of "real" geometry needing curvature and "real" topology needing homology or homotopy invariants, I'd not used them in my research in the '70's.

In the early 1980's I was motivated by Witten's seminal paper on the supersymmetry proof of the Morse inequalities and index theorem. This paper has been celebrated not only for the results itself but because of the bridge it opened up between high energy theorists studying gauge (and later string) theories and topologists but it also impacted me in leading me to consider certain geometric ideas that I needed in the work I'll describe later.

Witten

This is not so much in those of my papers directly motivated by Witten but through other mathematics motivated by it.

Witten

This is not so much in those of my papers directly motivated by Witten but through other mathematics motivated by it. For Witten motivated several reworkings of the proof of the Atiyah-Singer Index theorem, in particular, a preprint of Getzler which caught my attention in the period just after I gave the Bayreuth lectures which eventually appeared as a book coauthored with Cycon, Froese and Kirsch.

Witten

This is not so much in those of my papers directly motivated by Witten but through other mathematics motivated by it. For Witten motivated several reworkings of the proof of the Atiyah-Singer Index theorem, in particular, a preprint of Getzler which caught my attention in the period just after I gave the Bayreuth lectures which eventually appeared as a book coauthored with Cycon, Froese and Kirsch. I had lectured on Witten's proof of the Morse inequalities there and decided to add a chapter on this further extension

Witten

This is not so much in those of my papers directly motivated by Witten but through other mathematics motivated by it. For Witten motivated several reworkings of the proof of the Atiyah-Singer Index theorem, in particular, a preprint of Getzler which caught my attention in the period just after I gave the Bayreuth lectures which eventually appeared as a book coauthored with Cycon, Froese and Kirsch. I had lectured on Witten's proof of the Morse inequalities there and decided to add a chapter on this further extension (the chapter, chapter 12, was actually the only chapter I wrote in that book - the other chapters were written by my coauthors based on and usually expanding the lectures I'd given).

Gauss Bonnet

For pedagogical reasons, I decided to give details only in the special, indeed, classical case of the Gauss-Bonnet theorem

Gauss Bonnet

For pedagogical reasons, I decided to give details only in the special, indeed, classical case of the Gauss-Bonnet theorem where it turns out that Getzler's proof is essentially one found in 1971 by Patoldi who didn't know that he was speaking supersymmetry!

Gauss Bonnet

For pedagogical reasons, I decided to give details only in the special, indeed, classical case of the Gauss-Bonnet theorem where it turns out that Getzler's proof is essentially one found in 1971 by Patoldi who didn't know that he was speaking supersymmetry! While I'd heard of the Gauss-Bonnet theorem, I hadn't known exactly what it said until following up on Witten taught me all about it.

Gauss Bonnet

For pedagogical reasons, I decided to give details only in the special, indeed, classical case of the Gauss-Bonnet theorem where it turns out that Getzler's proof is essentially one found in 1971 by Patoldi who didn't know that he was speaking supersymmetry! While I'd heard of the Gauss-Bonnet theorem, I hadn't known exactly what it said until following up on Witten taught me all about it. Since it will explain some of my later work, let me say a little about this theorem (and also holonomy) in the case of S^2, the sphere of radius R embedded in $\mathbb{R}^3$.

Gauss Bonnet

For pedagogical reasons, I decided to give details only in the special, indeed, classical case of the Gauss-Bonnet theorem where it turns out that Getzler's proof is essentially one found in 1971 by Patoldi who didn't know that he was speaking supersymmetry! While I'd heard of the Gauss-Bonnet theorem, I hadn't known exactly what it said until following up on Witten taught me all about it. Since it will explain some of my later work, let me say a little about this theorem (and also holonomy) in the case of S^2, the sphere of radius R embedded in $\mathbb{R}^3$. At each point, the Gaussian curvature is $1/R^2$ so, if K is the curvature and $d^2\omega$ the surface area, we have that

Gauss Bonnet

For pedagogical reasons, I decided to give details only in the special, indeed, classical case of the Gauss-Bonnet theorem where it turns out that Getzler's proof is essentially one found in 1971 by Patoldi who didn't know that he was speaking supersymmetry! While I'd heard of the Gauss-Bonnet theorem, I hadn't known exactly what it said until following up on Witten taught me all about it. Since it will explain some of my later work, let me say a little about this theorem (and also holonomy) in the case of S^2, the sphere of radius R embedded in $\mathbb{R}^3$. At each point, the Gaussian curvature is $1/R^2$ so, if K is the curvature and $d^2\omega$ the surface area, we have that

$$\frac{1}{2\pi}\int K\,d\omega = \frac{1}{2\pi}\frac{1}{R^2}4\pi R^2 = 2$$

Gauss Bonnet

The remarkable fact is that if you deform the sphere to another surface, say, an ellipsoid, then the curvature is no longer constant but the integral above is still 2.

Gauss Bonnet

The remarkable fact is that if you deform the sphere to another surface, say, an ellipsoid, then the curvature is no longer constant but the integral above is still 2. But this is not true for the torus. The integral is still independent of the underlying metric needed to define K, but it is 0, as can been seen by looking at the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ with the Euclidean metric on $\mathbb{R}^2$ (which cannot be isometrically embedded into $\mathbb{R}^3$ but can in $\mathbb{R}^4$).

Gauss Bonnet

The remarkable fact is that if you deform the sphere to another surface, say, an ellipsoid, then the curvature is no longer constant but the integral above is still 2. But this is not true for the torus. The integral is still independent of the underlying metric needed to define K, but it is 0, as can been seen by looking at the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ with the Euclidean metric on $\mathbb{R}^2$ (which cannot be isometrically embedded into $\mathbb{R}^3$ but can in $\mathbb{R}^4$). In fact, for any surface in $\mathbb{R}^3$ (and for hypersurfaces in general dimension) the integral is the Euler characteristic of the surface (Euler-Poincaré characteristic in higher dimension).

Gauss Bonnet

The remarkable fact is that if you deform the sphere to another surface, say, an ellipsoid, then the curvature is no longer constant but the integral above is still 2. But this is not true for the torus. The integral is still independent of the underlying metric needed to define K, but it is 0, as can been seen by looking at the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ with the Euclidean metric on $\mathbb{R}^2$ (which cannot be isometrically embedded into $\mathbb{R}^3$ but can in $\mathbb{R}^4$). In fact, for any surface in $\mathbb{R}^3$ (and for hypersurfaces in general dimension) the integral is the Euler characteristic of the surface (Euler-Poincaré characteristic in higher dimension). This is the Gauss-Bonnet theorem. It says that the integral of a natural geometric quantity lies in a discrete set and is determined by topological invariants.

Holonomy

To explain holonomy, consider someone carrying a spear around the earth trying at all times to keep the spear tangent to the sphere and parallel to the direction it was pointing.

Holonomy

To explain holonomy, consider someone carrying a spear around the earth trying at all times to keep the spear tangent to the sphere and parallel to the direction it was pointing. Imagine, going along the equator through one quarter of the earth, turning left, going to the north pole,

Holonomy

To explain holonomy, consider someone carrying a spear around the earth trying at all times to keep the spear tangent to the sphere and parallel to the direction it was pointing. Imagine, going along the equator through one quarter of the earth, turning left, going to the north pole, turning left and going back to the original point.

Holonomy

To explain holonomy, consider someone carrying a spear around the earth trying at all times to keep the spear tangent to the sphere and parallel to the direction it was pointing. Imagine, going along the equator through one quarter of the earth, turning left, going to the north pole, turning left and going back to the original point. Suppose the spear is parallel to the equator at the start.

Holonomy

To explain holonomy, consider someone carrying a spear around the earth trying at all times to keep the spear tangent to the sphere and parallel to the direction it was pointing. Imagine, going along the equator through one quarter of the earth, turning left, going to the north pole, turning left and going back to the original point. Suppose the spear is parallel to the equator at the start. The person turns to move along a line of longitude, but being careful not to turn the spear, it will point directly to his right.

Holonomy

To explain holonomy, consider someone carrying a spear around the earth trying at all times to keep the spear tangent to the sphere and parallel to the direction it was pointing. Imagine, going along the equator through one quarter of the earth, turning left, going to the north pole, turning left and going back to the original point. Suppose the spear is parallel to the equator at the start. The person turns to move along a line of longitude, but being careful not to turn the spear, it will point directly to his right. After the next turn, the spear will point backwards.

Holonomy

To explain holonomy, consider someone carrying a spear around the earth trying at all times to keep the spear tangent to the sphere and parallel to the direction it was pointing. Imagine, going along the equator through one quarter of the earth, turning left, going to the north pole, turning left and going back to the original point. Suppose the spear is parallel to the equator at the start. The person turns to move along a line of longitude, but being careful not to turn the spear, it will point directly to his right. After the next turn, the spear will point backwards. So despite having tried to keep it parallel , upon return, it has rotated by $90°$, i.e. $\pi/2$ radians.

Holonomy

To explain holonomy, consider someone carrying a spear around the earth trying at all times to keep the spear tangent to the sphere and parallel to the direction it was pointing. Imagine, going along the equator through one quarter of the earth, turning left, going to the north pole, turning left and going back to the original point. Suppose the spear is parallel to the equator at the start. The person turns to move along a line of longitude, but being careful not to turn the spear, it will point directly to his right. After the next turn, the spear will point backwards. So despite having tried to keep it parallel , upon return, it has rotated by 90°, i.e. $\pi/2$ radians. This rotation after parallel transport is *holonomy*.

Holonomy

To explain holonomy, consider someone carrying a spear around the earth trying at all times to keep the spear tangent to the sphere and parallel to the direction it was pointing. Imagine, going along the equator through one quarter of the earth, turning left, going to the north pole, turning left and going back to the original point. Suppose the spear is parallel to the equator at the start. The person turns to move along a line of longitude, but being careful not to turn the spear, it will point directly to his right. After the next turn, the spear will point backwards. So despite having tried to keep it parallel , upon return, it has rotated by 90°, i.e. $\pi/2$ radians. This rotation after parallel transport is *holonomy*. The path encloses one eighth of the earth, a area of $4\pi R^2/8 = \pi R^2/2$ so the integral of the curvature over the enclosed area is the holonomy!

Feynman

Perhaps relevant to my work is the following amusing story.

Feynman

Perhaps relevant to my work is the following amusing story. Avron and I were talking in my office when Dick Feynman burst in and exclaimed "how do you compute the homotopy groups of spheres?"

Feynman

Perhaps relevant to my work is the following amusing story. Avron and I were talking in my office when Dick Feynman burst in and exclaimed "how do you compute the homotopy groups of spheres?" They had appeared in the high energy literature and he was puzzled why the higher homotopy groups were not trivial.

Feynman

Perhaps relevant to my work is the following amusing story. Avron and I were talking in my office when Dick Feynman burst in and exclaimed "how do you compute the homotopy groups of spheres?" They had appeared in the high energy literature and he was puzzled why the higher homotopy groups were not trivial. I told him about the Hopf fibration and then retrieved from my memory the exact sequence of a fibration.

Feynman

Perhaps relevant to my work is the following amusing story. Avron and I were talking in my office when Dick Feynman burst in and exclaimed "how do you compute the homotopy groups of spheres?" They had appeared in the high energy literature and he was puzzled why the higher homotopy groups were not trivial. I told him about the Hopf fibration and then retrieved from my memory the exact sequence of a fibration. When I finished Avron looked at me and said: "Barry, I didn't realize you knew anything about that".

Feynman

Perhaps relevant to my work is the following amusing story. Avron and I were talking in my office when Dick Feynman burst in and exclaimed "how do you compute the homotopy groups of spheres?" They had appeared in the high energy literature and he was puzzled why the higher homotopy groups were not trivial. I told him about the Hopf fibration and then retrieved from my memory the exact sequence of a fibration. When I finished Avron looked at me and said: "Barry, I didn't realize you knew anything about that". Before I could answer, Dick with a huge grin on his face turned around waved his hands at my rather full bookshelves and exclaimed in his trademark New York accent:

Feynman

Perhaps relevant to my work is the following amusing story. Avron and I were talking in my office when Dick Feynman burst in and exclaimed "how do you compute the homotopy groups of spheres?" They had appeared in the high energy literature and he was puzzled why the higher homotopy groups were not trivial. I told him about the Hopf fibration and then retrieved from my memory the exact sequence of a fibration. When I finished Avron looked at me and said: "Barry, I didn't realize you knew anything about that". Before I could answer, Dick with a huge grin on his face turned around waved his hands at my rather full bookshelves and exclaimed in his trademark New York accent: "Whadya mean? He's a Professor, of course he knows it!"

Feynman

Perhaps relevant to my work is the following amusing story. Avron and I were talking in my office when Dick Feynman burst in and exclaimed “how do you compute the homotopy groups of spheres?” They had appeared in the high energy literature and he was puzzled why the higher homotopy groups were not trivial. I told him about the Hopf fibration and then retrieved from my memory the exact sequence of a fibration. When I finished Avron looked at me and said: “Barry, I didn't realize you knew anything about that”. Before I could answer, Dick with a huge grin on his face turned around waved his hands at my rather full bookshelves and exclaimed in his trademark New York accent: “Whadya mean? He's a Professor, of course he knows it!” When I needed homotopy and the exact sequence of a fibration several months later, it helped that I'd had this interaction!

TKNN

In early 1983, Yosi Avron told me about the Phys Rev Letters paper of Thouless and his group (Kohmoto, Nightingale and den Nijs)

TKNN

In early 1983, Yosi Avron told me about the Phys Rev Letters paper of Thouless and his group (Kohmoto, Nightingale and den Nijs) which gave a novel explanation of the quantum Hall effect, a subject that had fascinated Yosi.

TKNN

In early 1983, Yosi Avron told me about the Phys Rev Letters paper of Thouless and his group (Kohmoto, Nightingale and den Nijs) which gave a novel explanation of the quantum Hall effect, a subject that had fascinated Yosi. The striking aspect of that effect is that a resistance was quantized.

TKNN

In early 1983, Yosi Avron told me about the Phys Rev Letters paper of Thouless and his group (Kohmoto, Nightingale and den Nijs) which gave a novel explanation of the quantum Hall effect, a subject that had fascinated Yosi. The striking aspect of that effect is that a resistance was quantized. In the TKNN approach (we quickly came up with that abbreviation, especially TKNN integers, a name which has stuck),

TKNN

In early 1983, Yosi Avron told me about the Phys Rev Letters paper of Thouless and his group (Kohmoto, Nightingale and den Nijs) which gave a novel explanation of the quantum Hall effect, a subject that had fascinated Yosi. The striking aspect of that effect is that a resistance was quantized. In the TKNN approach (we quickly came up with that abbreviation, especially TKNN integers, a name which has stuck), this arose because, using the Kubo formula,

TKNN

In early 1983, Yosi Avron told me about the Phys Rev Letters paper of Thouless and his group (Kohmoto, Nightingale and den Nijs) which gave a novel explanation of the quantum Hall effect, a subject that had fascinated Yosi. The striking aspect of that effect is that a resistance was quantized. In the TKNN approach (we quickly came up with that abbreviation, especially TKNN integers, a name which has stuck), this arose because, using the Kubo formula, they got the resistance (in a certain idealized situation) was given by an integral over a torus that turned out to be an integer (in suitable units)

Our Work

We quickly realized that their integers were associated to a single band which was assumed non-degenerate

Our Work

We quickly realized that their integers were associated to a single band which was assumed non-degenerate (i.e. at every point in the Brillouin zone, the eigenstate for that band is simple)

Our Work

We quickly realized that their integers were associated to a single band which was assumed non-degenerate (i.e. at every point in the Brillouin zone, the eigenstate for that band is simple) and their integrand involved the change of eigenfunction.

Our Work

We quickly realized that their integers were associated to a single band which was assumed non-degenerate (i.e. at every point in the Brillouin zone, the eigenstate for that band is simple) and their integrand involved the change of eigenfunction. We also realized that since the integrand was an integer it had to be invariant under continuous change and so an indication of a homotopy invariant of maps from the two dimension torus T^2 to unit vectors in Hilbert space mod phases (equivalently a continuous assignment of a one dimensional subspace in the Hilbert space to each point in T^2).

Our Work

We quickly realized that their integers were associated to a single band which was assumed non-degenerate (i.e. at every point in the Brillouin zone, the eigenstate for that band is simple) and their integrand involved the change of eigenfunction. We also realized that since the integrand was an integer it had to be invariant under continuous change and so an indication of a homotopy invariant of maps from the two dimension torus T^2 to unit vectors in Hilbert space mod phases (equivalently a continuous assignment of a one dimensional subspace in the Hilbert space to each point in T^2). After more thought and study, we learned that the homotopy class of maps from T^2 could be classified by maps from S^1 and S^2 and so the underlying homotopy groups of $\mathbb{P}(\infty)$, the one dimensional subspaces of a Hilbert space.

Our Work

We also considered that there might be non-trivial homotopy invariants depending on several bands so what we wanted to consider was the homotopy groups of the set, $\mathcal{N}$, of compact operators with non-degenerate eigenvalues.

Our Work

We also considered that there might be non-trivial homotopy invariants depending on several bands so what we wanted to consider was the homotopy groups of the set, $\mathcal{N}$, of compact operators with non-degenerate eigenvalues. We got excited since if, for example, we found a non-trivial π_3, there would be new topological invariants for the physically relevant three-dimensional torus!

Our Work

We also considered that there might be non-trivial homotopy invariants depending on several bands so what we wanted to consider was the homotopy groups of the set, $\mathcal{N}$, of compact operators with non-degenerate eigenvalues. We got excited since if, for example, we found a non-trivial π_3, there would be new topological invariants for the physically relevant three-dimensional torus! By a continuous deformation, we could consider maps to a fixed set of simple eigenvalues but variable eigenspaces.

Our Work

We also considered that there might be non-trivial homotopy invariants depending on several bands so what we wanted to consider was the homotopy groups of the set, $\mathcal{N}$, of compact operators with non-degenerate eigenvalues. We got excited since if, for example, we found a non-trivial π_3, there would be new topological invariants for the physically relevant three-dimensional torus! By a continuous deformation, we could consider maps to a fixed set of simple eigenvalues but variable eigenspaces. Given the phase change this was the same as the quotient of all unitary maps by the diagonal unitary maps $\mathcal{U}(\mathcal{H})/D\mathcal{U}(\mathcal{H})$.

Our Work

So these homotopy groups might be computable via the exact sequence of the fibration that my talk with Feynman had reminded me about!

Our Work

So these homotopy groups might be computable via the exact sequence of the fibration that my talk with Feynman had reminded me about! Indeed, since it was known that the set of all unitaries $\mathcal{U}(\mathcal{H})$ is contractible,

Our Work

So these homotopy groups might be computable via the exact sequence of the fibration that my talk with Feynman had reminded me about! Indeed, since it was known that the set of all unitaries $\mathcal{U}(\mathcal{H})$ is contractible, it has no homotopy, i.e. all homotopy groups are trivial and, thus, by the exact sequence of the fibration, we knew that $\pi_j(\mathcal{N}) = \pi_{j-1}(D\mathcal{U}(\mathcal{H}))$.

Our Work

So these homotopy groups might be computable via the exact sequence of the fibration that my talk with Feynman had reminded me about! Indeed, since it was known that the set of all unitaries $\mathcal{U}(\mathcal{H})$ is contractible, it has no homotopy, i.e. all homotopy groups are trivial and, thus, by the exact sequence of the fibration, we knew that $\pi_j(\mathcal{N}) = \pi_{j-1}(D\mathcal{U}(\mathcal{H}))$. Since the diagonal unitaries is just an infinite product of circles, T^1, and $\pi_j(T^1)$ is $\mathbb{Z}$ for $j = 1$ and 0 for all other j,

Our Work

So these homotopy groups might be computable via the exact sequence of the fibration that my talk with Feynman had reminded me about! Indeed, since it was known that the set of all unitaries $\mathcal{U}(\mathcal{H})$ is contractible, it has no homotopy, i.e. all homotopy groups are trivial and, thus, by the exact sequence of the fibration, we knew that $\pi_j(\mathcal{N}) = \pi_{j-1}(D\mathcal{U}(\mathcal{H}))$. Since the diagonal unitaries is just an infinite product of circles, T^1, and $\pi_j(T^1)$ is $\mathbb{Z}$ for $j = 1$ and 0 for all other j, we had discovered that the *only* non-trivial homotopy group of $\mathcal{N}$ was π_2, that the same was true for $\mathbb{P}(\infty)$ and that $\pi_2(\mathcal{N})$ was just an infinite product of $\pi_2(\mathbb{P}(\infty))$'s.

Our Work

So these homotopy groups might be computable via the exact sequence of the fibration that my talk with Feynman had reminded me about! Indeed, since it was known that the set of all unitaries $\mathcal{U}(\mathcal{H})$ is contractible, it has no homotopy, i.e. all homotopy groups are trivial and, thus, by the exact sequence of the fibration, we knew that $\pi_j(\mathcal{N}) = \pi_{j-1}(D\mathcal{U}(\mathcal{H}))$. Since the diagonal unitaries is just an infinite product of circles, T^1, and $\pi_j(T^1)$ is $\mathbb{Z}$ for $j = 1$ and 0 for all other j, we had discovered that the *only* non-trivial homotopy group of $\mathcal{N}$ was π_2, that the same was true for $\mathbb{P}(\infty)$ and that $\pi_2(\mathcal{N})$ was just an infinite product of $\pi_2(\mathbb{P}(\infty))$'s. In other words, the only homotopy invariants were the TKNN integers.

Our Work

We added Reudi Seiler, whom Yosi had been consulting, to the authors and published this negative result in *Physical Review Letters*.

Our Work

We added Reudi Seiler, whom Yosi had been consulting, to the authors and published this negative result in *Physical Review Letters*. We made a big deal of our new result that if two non-degenerate bands with TKNN integers n_1 and n_2 went through a degeneracy as parameters were varied so that afterwards they were again non-degenerate with TKNN integers n_3 and n_4, then $n_1 + n_2 = n_3 + n_4$.

Our Work

We added Reudi Seiler, whom Yosi had been consulting, to the authors and published this negative result in *Physical Review Letters*. We made a big deal of our new result that if two non-degenerate bands with TKNN integers n_1 and n_2 went through a degeneracy as parameters were varied so that afterwards they were again non-degenerate with TKNN integers n_3 and n_4, then $n_1 + n_2 = n_3 + n_4$. But there were results that were more important although only noted in passing.

Our Work

We added Reudi Seiler, whom Yosi had been consulting, to the authors and published this negative result in *Physical Review Letters*. We made a big deal of our new result that if two non-degenerate bands with TKNN integers n_1 and n_2 went through a degeneracy as parameters were varied so that afterwards they were again non-degenerate with TKNN integers n_3 and n_4, then $n_1 + n_2 = n_3 + n_4$. But there were results that were more important although only noted in passing. Most basic was the new one that the TKNN integers were homotopy invariants, something that would be clarified by my work on Berry's phase which I turn to shortly.

The Curvature

We also found two compact formulae for the integrand that eventually became commonly used in further work.

The Curvature

We also found two compact formulae for the integrand that eventually became commonly used in further work. First if that ψ_j is the eigenstate of band j, then the corresponding TKNN integer, n_j, is given by

The Curvature

We also found two compact formulae for the integrand that eventually became commonly used in further work. First if that ψ_j is the eigenstate of band j, then the corresponding TKNN integer, n_j, is given by

$$n_j = \frac{1}{2\pi} \int_{T^2} K_j; \qquad K_j = i\langle d\psi_j, d\psi_j \rangle$$

The Curvature

We also found two compact formulae for the integrand that eventually became commonly used in further work. First if that ψ_j is the eigenstate of band j, then the corresponding TKNN integer, n_j, is given by

$$n_j = \frac{1}{2\pi} \int_{T^2} K_j; \qquad K_j = i\langle d\psi_j, d\psi_j \rangle$$

We were especially fond of a second formula, that if P_j is the projection onto ψ_j, then

The Curvature

We also found two compact formulae for the integrand that eventually became commonly used in further work. First if that ψ_j is the eigenstate of band j, then the corresponding TKNN integer, n_j, is given by

$$n_j = \frac{1}{2\pi} \int_{T^2} K_j; \qquad K_j = i\langle d\psi_j, d\psi_j \rangle$$

We were especially fond of a second formula, that if P_j is the projection onto ψ_j, then

$$K_j = i\mathrm{Tr}(dP_j P_j dP_j)$$

The Curvature

We also found two compact formulae for the integrand that eventually became commonly used in further work. First if that ψ_j is the eigenstate of band j, then the corresponding TKNN integer, n_j, is given by

$$n_j = \frac{1}{2\pi} \int_{T^2} K_j; \qquad K_j = i\langle d\psi_j, d\psi_j \rangle$$

We were especially fond of a second formula, that if P_j is the projection onto ψ_j, then

$$K_j = i\mathrm{Tr}(dP_j P_j dP_j)$$

We liked this because while the first formula requires a choice of phase in each space, the second is manifestly phase invariant.

The Curvature

We also found two compact formulae for the integrand that eventually became commonly used in further work. First if that ψ_j is the eigenstate of band j, then the corresponding TKNN integer, n_j, is given by

$$n_j = \frac{1}{2\pi}\int_{T^2} K_j; \qquad K_j = i\langle d\psi_j, d\psi_j\rangle$$

We were especially fond of a second formula, that if P_j is the projection onto ψ_j, then

$$K_j = i\mathrm{Tr}(dP_j P_j dP_j)$$

We liked this because while the first formula requires a choice of phase in each space, the second is manifestly phase invariant. The operator d in the last two expressions is the exterior derivative and there is an implicit wedge product.

The Curvature

You might worry that because $df \wedge df = 0$, if there were no trace and P_j were a function, the quantity *would* be 0.

The Curvature

You might worry that because $df \wedge df = 0$, if there were no trace and P_j were a function, the quantity *would* be 0. But because P_j is operator valued, it is not 0.

The Curvature

You might worry that because $df \wedge df = 0$, if there were no trace and P_j were a function, the quantity *would* be 0. But because P_j is operator valued, it is not 0. Indeed,

$$K = i \sum_{k,\ell} \mathrm{Tr} \left(\frac{\partial P_j}{\partial x_k} P_j \frac{\partial P_j}{\partial x_\ell} \right) dx_k \wedge dx_\ell$$

The Curvature

You might worry that because $df \wedge df = 0$, if there were no trace and P_j were a function, the quantity *would* be 0. But because P_j is operator valued, it is not 0. Indeed,

$$K = i \sum_{k,\ell} \operatorname{Tr}\left(\frac{\partial P_j}{\partial x_k} P_j \frac{\partial P_j}{\partial x_\ell}\right) dx_k \wedge dx_\ell$$

$$= \frac{i}{2} \sum_{k,\ell} \operatorname{Tr}\left(P_j \left[\frac{\partial P_j}{\partial x_k}, \frac{\partial P_j}{\partial x_\ell}\right]\right) dx_k \wedge dx_\ell$$

The Curvature

You might worry that because $df \wedge df = 0$, if there were no trace and P_j were a function, the quantity *would* be 0. But because P_j is operator valued, it is not 0. Indeed,

$$K = i \sum_{k,\ell} \mathrm{Tr}\left(\frac{\partial P_j}{\partial x_k} P_j \frac{\partial P_j}{\partial x_\ell} \right) dx_k \wedge dx_\ell$$

$$= \frac{i}{2} \sum_{k,\ell} \mathrm{Tr}\left(P_j \left[\frac{\partial P_j}{\partial x_k}, \frac{\partial P_j}{\partial x_\ell} \right] \right) dx_k \wedge dx_\ell$$

$$= \frac{i}{2} \sum_{k,\ell} \left\langle \psi_j, \left[\frac{\partial P_j}{\partial x_k}, \frac{\partial P_j}{\partial x_\ell} \right] \psi_j \right\rangle dx_k \wedge dx_\ell$$

The Curvature

You might worry that because $df \wedge df = 0$, if there were no trace and P_j were a function, the quantity *would* be 0. But because P_j is operator valued, it is not 0. Indeed,

$$K = i \sum_{k,\ell} \mathrm{Tr}\left(\frac{\partial P_j}{\partial x_k} P_j \frac{\partial P_j}{\partial x_\ell}\right) dx_k \wedge dx_\ell$$

$$= \frac{i}{2} \sum_{k,\ell} \mathrm{Tr}\left(P_j \left[\frac{\partial P_j}{\partial x_k}, \frac{\partial P_j}{\partial x_\ell}\right]\right) dx_k \wedge dx_\ell$$

$$= \frac{i}{2} \sum_{k,\ell} \left\langle \psi_j, \left[\frac{\partial P_j}{\partial x_k}, \frac{\partial P_j}{\partial x_\ell}\right] \psi_j \right\rangle dx_k \wedge dx_\ell$$

where $[\cdot,\cdot]$ is commutator and we used the antisymmetry of $dx_k \wedge dx_\ell$.

Australia!

The next part of this story took place in Australia,

Australia!

The next part of this story took place in Australia, so I should mention that trip in the summer of 1983 (well, the winter in Australia!) almost didn't happen!

Australia!

The next part of this story took place in Australia, so I should mention that trip in the summer of 1983 (well, the winter in Australia!) almost didn't happen! My fourth child, Aryeh, was born in Dec., 1982 and given the time to get his birth certificate and passport, it was only the end of April that I was able to contact the Australian consul in Los Angeles to get visas for all of us including a work visa for me.

Australia!

The next part of this story took place in Australia, so I should mention that trip in the summer of 1983 (well, the winter in Australia!) almost didn't happen! My fourth child, Aryeh, was born in Dec., 1982 and given the time to get his birth certificate and passport, it was only the end of April that I was able to contact the Australian consul in Los Angeles to get visas for all of us including a work visa for me. He sent a long medical form for me requiring a new general exam from a doctor and xray.

Australia!

The next part of this story took place in Australia, so I should mention that trip in the summer of 1983 (well, the winter in Australia!) almost didn't happen! My fourth child, Aryeh, was born in Dec., 1982 and given the time to get his birth certificate and passport, it was only the end of April that I was able to contact the Australian consul in Los Angeles to get visas for all of us including a work visa for me. He sent a long medical form for me requiring a new general exam from a doctor and xray. I'd had them 3 months before at Kaiser Health Services but was told by the consul that I had to do them over. I've been raised to avoid unnecessary xrays and I wasn't sure Kaiser would agree to a second exam.

Australia!

The next part of this story took place in Australia, so I should mention that trip in the summer of 1983 (well, the winter in Australia!) almost didn't happen! My fourth child, Aryeh, was born in Dec., 1982 and given the time to get his birth certificate and passport, it was only the end of April that I was able to contact the Australian consul in Los Angeles to get visas for all of us including a work visa for me. He sent a long medical form for me requiring a new general exam from a doctor and xray. I'd had them 3 months before at Kaiser Health Services but was told by the consul that I had to do them over. I've been raised to avoid unnecessary xrays and I wasn't sure Kaiser would agree to a second exam. As far as I could tell, this was a restriction put in place to make it difficult for Asians to come and work and I tried to use my invitation from the Australian Academy of Sciences to get a waiver.

Australia!

The consul was uncooperative, almost nasty.

Australia!

The consul was uncooperative, almost nasty. This was not only pre-Skype but email was almost non-existent and intercontinental phone calls were very expensive,

Australia!

The consul was uncooperative, almost nasty. This was not only pre-Skype but email was almost non-existent and intercontinental phone calls were very expensive, so I sent a telex to my host, Derek Robinson, explaining that, because of visa issues, I would probably have to cancel my trip.

Australia!

The consul was uncooperative, almost nasty. This was not only pre-Skype but email was almost non-existent and intercontinental phone calls were very expensive, so I sent a telex to my host, Derek Robinson, explaining that, because of visa issues, I would probably have to cancel my trip. The next day, he called me, which impressed me given the cost of international calls, telling me to stay calm and he'd fix it.

Australia!

The consul was uncooperative, almost nasty. This was not only pre-Skype but email was almost non-existent and intercontinental phone calls were very expensive, so I sent a telex to my host, Derek Robinson, explaining that, because of visa issues, I would probably have to cancel my trip. The next day, he called me, which impressed me given the cost of international calls, telling me to stay calm and he'd fix it. I didn't know that Derek was the secretary of the Australian Academy of Sciences.

Australia!

The consul was uncooperative, almost nasty. This was not only pre-Skype but email was almost non-existent and intercontinental phone calls were very expensive, so I sent a telex to my host, Derek Robinson, explaining that, because of visa issues, I would probably have to cancel my trip. The next day, he called me, which impressed me given the cost of international calls, telling me to stay calm and he'd fix it. I didn't know that Derek was the secretary of the Australian Academy of Sciences. But three days later, I get a call from the consul saying

Australia!

The consul was uncooperative, almost nasty. This was not only pre-Skype but email was almost non-existent and intercontinental phone calls were very expensive, so I sent a telex to my host, Derek Robinson, explaining that, because of visa issues, I would probably have to cancel my trip. The next day, he called me, which impressed me given the cost of international calls, telling me to stay calm and he'd fix it. I didn't know that Derek was the secretary of the Australian Academy of Sciences. But three days later, I get a call from the consul saying "Sir, I am anxious to issue your visas, but I need you to return the forms I sent you."

Australia!

The consul was uncooperative, almost nasty. This was not only pre-Skype but email was almost non-existent and intercontinental phone calls were very expensive, so I sent a telex to my host, Derek Robinson, explaining that, because of visa issues, I would probably have to cancel my trip. The next day, he called me, which impressed me given the cost of international calls, telling me to stay calm and he'd fix it. I didn't know that Derek was the secretary of the Australian Academy of Sciences. But three days later, I get a call from the consul saying "Sir, I am anxious to issue your visas, but I need you to return the forms I sent you." I replied "But what about the medical form."

Australia!

The consul was uncooperative, almost nasty. This was not only pre-Skype but email was almost non-existent and intercontinental phone calls were very expensive, so I sent a telex to my host, Derek Robinson, explaining that, because of visa issues, I would probably have to cancel my trip. The next day, he called me, which impressed me given the cost of international calls, telling me to stay calm and he'd fix it. I didn't know that Derek was the secretary of the Australian Academy of Sciences. But three days later, I get a call from the consul saying "Sir, I am anxious to issue your visas, but I need you to return the forms I sent you." I replied "But what about the medical form." "Oh, you don't need that, sir."

Australia!

The consul was uncooperative, almost nasty. This was not only pre-Skype but email was almost non-existent and intercontinental phone calls were very expensive, so I sent a telex to my host, Derek Robinson, explaining that, because of visa issues, I would probably have to cancel my trip. The next day, he called me, which impressed me given the cost of international calls, telling me to stay calm and he'd fix it. I didn't know that Derek was the secretary of the Australian Academy of Sciences. But three days later, I get a call from the consul saying "Sir, I am anxious to issue your visas, but I need you to return the forms I sent you." I replied "But what about the medical form." "Oh, you don't need that, sir." According to the current vogue, I should feel guilty for having used my white privilege, but given how important this visit turned out to be, I am glad.

Derek was actually away for the first two weeks of my visit

Berry

Derek was actually away for the first two weeks of my visit but Brian Davies had also just arrived so we collaborated together on what became the work on ultracontractivity.

Berry

Derek was actually away for the first two weeks of my visit but Brian Davies had also just arrived so we collaborated together on what became the work on ultracontractivity. Midway through my visit, I heard that Michael Berry, whom I'd meet several years before at Joel Lebowitz' seminar,

Berry

Derek was actually away for the first two weeks of my visit but Brian Davies had also just arrived so we collaborated together on what became the work on ultracontractivity. Midway through my visit, I heard that Michael Berry, whom I'd meet several years before at Joel Lebowitz' seminar, was visiting physics at Australian National University where Derek was in mathematics and where I was visiting.

Berry

Derek was actually away for the first two weeks of my visit but Brian Davies had also just arrived so we collaborated together on what became the work on ultracontractivity. Midway through my visit, I heard that Michael Berry, whom I'd meet several years before at Joel Lebowitz' seminar, was visiting physics at Australian National University where Derek was in mathematics and where I was visiting. He'd given a seminar, but before I'd learned he was there,

Berry

Derek was actually away for the first two weeks of my visit but Brian Davies had also just arrived so we collaborated together on what became the work on ultracontractivity. Midway through my visit, I heard that Michael Berry, whom I'd meet several years before at Joel Lebowitz' seminar, was visiting physics at Australian National University where Derek was in mathematics and where I was visiting. He'd given a seminar, but before I'd learned he was there, so I called and asked him for a private version which he kindly agreed to.

Berry

He explained to me his work on an extra phase he'd found in the adiabatic theorem and gave me a copy of the manuscript that he'd recently submitted to Proc. Roy. Soc.

Berry

He explained to me his work on an extra phase he'd found in the adiabatic theorem and gave me a copy of the manuscript that he'd recently submitted to Proc. Roy. Soc. He mentioned that Bernard Souillard, when he heard about Berry's work, told Berry that he thought it might have something to do with the paper of Thouless on TKNN integers

Berry

He explained to me his work on an extra phase he'd found in the adiabatic theorem and gave me a copy of the manuscript that he'd recently submitted to Proc. Roy. Soc. He mentioned that Bernard Souillard, when he heard about Berry's work, told Berry that he thought it might have something to do with the paper of Thouless on TKNN integers but then Berry added that when he asked Thouless about it,

Berry

He explained to me his work on an extra phase he'd found in the adiabatic theorem and gave me a copy of the manuscript that he'd recently submitted to Proc. Roy. Soc. He mentioned that Bernard Souillard, when he heard about Berry's work, told Berry that he thought it might have something to do with the paper of Thouless on TKNN integers but then Berry added that when he asked Thouless about it, Thouless said that he doubted there was any connection (no pun intended).

Berry

He explained to me his work on an extra phase he'd found in the adiabatic theorem and gave me a copy of the manuscript that he'd recently submitted to Proc. Roy. Soc. He mentioned that Bernard Souillard, when he heard about Berry's work, told Berry that he thought it might have something to do with the paper of Thouless on TKNN integers but then Berry added that when he asked Thouless about it, Thouless said that he doubted there was any connection (no pun intended). I replied I thought there probably was

Berry

He explained to me his work on an extra phase he'd found in the adiabatic theorem and gave me a copy of the manuscript that he'd recently submitted to Proc. Roy. Soc. He mentioned that Bernard Souillard, when he heard about Berry's work, told Berry that he thought it might have something to do with the paper of Thouless on TKNN integers but then Berry added that when he asked Thouless about it, Thouless said that he doubted there was any connection (no pun intended). I replied I thought there probably was and that night, I figured out all the main points that appeared in my Phys. Rev Lett (which was published earlier than Berry's paper)!

The Adiabatic Theorem

Berry's paper dealt with the quantum adiabatic theorem.

The Adiabatic Theorem

Berry's paper dealt with the quantum adiabatic theorem. This theorem deals with a time dependent Hamiltonian $H(s); 0 \leq s \leq 1$ and considers T large and $H(s/T)$ so one is looking at very slow changes.

The Adiabatic Theorem

Berry's paper dealt with the quantum adiabatic theorem. This theorem deals with a time dependent Hamiltonian $H(s); 0 \leq s \leq 1$ and considers T large and $H(s/T)$ so one is looking at very slow changes.

$\varphi_T(s) \equiv \widetilde{U}_T(s)\varphi;\ 0 \leq s \leq T$ solves
$\dot{\varphi}_T(s) = -iH(s/T)\varphi_T(s); \varphi_T(0) = \varphi$.

The Adiabatic Theorem

Berry's paper dealt with the quantum adiabatic theorem. This theorem deals with a time dependent Hamiltonian $H(s); 0 \leq s \leq 1$ and considers T large and $H(s/T)$ so one is looking at very slow changes.
$\varphi_T(s) \equiv \widetilde{U}_T(s)\varphi;\ 0 \leq s \leq T$ solves
$\dot{\varphi}_T(s) = -iH(s/T)\varphi_T(s); \varphi_T(0) = \varphi$. Let $E(s)$ be an isolated, simple eigenvalue of $H(s)$ and let $P(s)$ be the projection onto the corresponding eigenspace.

The Adiabatic Theorem

Berry's paper dealt with the quantum adiabatic theorem. This theorem deals with a time dependent Hamiltonian $H(s); 0 \leq s \leq 1$ and considers T large and $H(s/T)$ so one is looking at very slow changes. $\varphi_T(s) \equiv \widetilde{U}_T(s)\varphi$; $0 \leq s \leq T$ solves $\dot{\varphi}_T(s) = -iH(s/T)\varphi_T(s); \varphi_T(0) = \varphi$. Let $E(s)$ be an isolated, simple eigenvalue of $H(s)$ and let $P(s)$ be the projection onto the corresponding eigenspace. The adiabatic theorem says that if $P(0)\varphi = \varphi$,

The Adiabatic Theorem

Berry's paper dealt with the quantum adiabatic theorem. This theorem deals with a time dependent Hamiltonian $H(s); 0 \leq s \leq 1$ and considers T large and $H(s/T)$ so one is looking at very slow changes. $\varphi_T(s) \equiv \widetilde{U}_T(s)\varphi$; $0 \leq s \leq T$ solves $\dot{\varphi}_T(s) = -iH(s/T)\varphi_T(s); \varphi_T(0) = \varphi$. Let $E(s)$ be an isolated, simple eigenvalue of $H(s)$ and let $P(s)$ be the projection onto the corresponding eigenspace. The adiabatic theorem says that if $P(0)\varphi = \varphi$, then $\lim_{T\to\infty}(\mathbf{1} - P(s/T))\varphi_T(s/T) = 0$, i.e. if you start in an eigenspace you stay in it adiabatically.

The Adiabatic Theorem

Berry asked and answered the question, what happens if $H(1) = H(0)$ so you end where you start.

The Adiabatic Theorem

Berry asked and answered the question, what happens if $H(1) = H(0)$ so you end where you start. What is the limiting phase of $\varphi_T(T)$.

The Adiabatic Theorem

Berry asked and answered the question, what happens if $H(1) = H(0)$ so you end where you start. What is the limiting phase of $\varphi_T(T)$. The surprise he found

The Adiabatic Theorem

Berry asked and answered the question, what happens if $H(1) = H(0)$ so you end where you start. What is the limiting phase of $\varphi_T(T)$. The surprise he found (it turned out that in 1956 Pancharatnam had done the same thing, but it had been forgotten)

The Adiabatic Theorem

Berry asked and answered the question, what happens if $H(1) = H(0)$ so you end where you start. What is the limiting phase of $\varphi_T(T)$. The surprise he found (it turned out that in 1956 Pancharatnam had done the same thing, but it had been forgotten) is that the naive guess that $\varphi_T(T) \sim e^{-iT\int_0^1 E(s)ds}\varphi$ is wrong

The Adiabatic Theorem

Berry asked and answered the question, what happens if $H(1) = H(0)$ so you end where you start. What is the limiting phase of $\varphi_T(T)$. The surprise he found (it turned out that in 1956 Pancharatnam had done the same thing, but it had been forgotten) is that the naive guess that $\varphi_T(T) \sim e^{-iT\int_0^1 E(s)ds}\varphi$ is wrong but that there is an additional phase, $e^{i\Gamma}$.

The Adiabatic Theorem

Berry asked and answered the question, what happens if $H(1) = H(0)$ so you end where you start. What is the limiting phase of $\varphi_T(T)$. The surprise he found (it turned out that in 1956 Pancharatnam had done the same thing, but it had been forgotten) is that the naive guess that $\varphi_T(T) \sim e^{-iT\int_0^1 E(s)ds}\varphi$ is wrong but that there is an additional phase, $e^{i\Gamma}$. In my paper, I gave Γ the name it is known by - *Berry's phase*.

The Adiabatic Theorem

Berry originally wrote Γ as a line integral but, then,

The Adiabatic Theorem

Berry originally wrote Γ as a line integral but, then, assuming that the family $H(s)$ was a closed curve in a parameter space, he used Stokes theorem to write Γ as the integral over a surface, S, in parameter space whose boundary was the closed curve in the form

The Adiabatic Theorem

Berry originally wrote Γ as a line integral but, then, assuming that the family $H(s)$ was a closed curve in a parameter space, he used Stokes theorem to write Γ as the integral over a surface, S, in parameter space whose boundary was the closed curve in the form

$$\Gamma = \int_S K(\omega)\, d\omega$$

The Adiabatic Theorem

Berry originally wrote Γ as a line integral but, then, assuming that the family $H(s)$ was a closed curve in a parameter space, he used Stokes theorem to write Γ as the integral over a surface, S, in parameter space whose boundary was the closed curve in the form

$$\Gamma = \int_S K(\omega)\, d\omega$$

$$K = \operatorname{Im} \sum_{m \neq 0} \frac{\langle \varphi_m(\omega), \nabla H(\omega) \varphi_0(\omega) \rangle \times \langle \varphi_0(\omega), \nabla H(\omega) \varphi_m(\omega) \rangle}{(E_m(\omega) - E_0(\omega))^2}$$

The Adiabatic Theorem

Berry originally wrote Γ as a line integral but, then, assuming that the family $H(s)$ was a closed curve in a parameter space, he used Stokes theorem to write Γ as the integral over a surface, S, in parameter space whose boundary was the closed curve in the form

$$\Gamma = \int_S K(\omega)\, d\omega$$

$$K = \operatorname{Im} \sum_{m \neq 0} \frac{\langle \varphi_m(\omega), \nabla H(\omega)\varphi_0(\omega)\rangle \times \langle \varphi_0(\omega), \nabla H(\omega)\varphi_m(\omega)\rangle}{(E_m(\omega) - E_0(\omega))^2}$$

where he supposed the interpolating Hamiltonian $H(\omega)$ had a complete set $\{\varphi_m\}_m$ of simple eigenfunctions with $H(\omega)\varphi_m(\omega) = E_m(\omega)\varphi_m(\omega)$ and $P(\omega)\varphi_0(\omega) = \varphi_0(\omega)$; $E(\omega) = E_0(\omega)$.

My Work

What I did in my paper is realize that what Berry was doing was simple and standard geometry

My Work

What I did in my paper is realize that what Berry was doing was simple and standard geometry in the exact same setting as TKNN.

My Work

What I did in my paper is realize that what Berry was doing was simple and standard geometry in the exact same setting as TKNN. I'd learned in the meantime that the TKNN integers were called the Chern invariant and the curvature K was called the Chern class

My Work

What I did in my paper is realize that what Berry was doing was simple and standard geometry in the exact same setting as TKNN. I'd learned in the meantime that the TKNN integers were called the Chern invariant and the curvature K was called the Chern class and used those names for the first time in this context.

My Work

What I did in my paper is realize that what Berry was doing was simple and standard geometry in the exact same setting as TKNN. I'd learned in the meantime that the TKNN integers were called the Chern invariant and the curvature K was called the Chern class and used those names for the first time in this context. The adiabatic theorem defines a connection, i.e. a way of doing parallel transport and Berry's phase was nothing but the holonomy in this connection.

My Work

What I did in my paper is realize that what Berry was doing was simple and standard geometry in the exact same setting as TKNN. I'd learned in the meantime that the TKNN integers were called the Chern invariant and the curvature K was called the Chern class and used those names for the first time in this context. The adiabatic theorem defines a connection, i.e. a way of doing parallel transport and Berry's phase was nothing but the holonomy in this connection. Berry had used our first, not explicitly phase covariant, formula as an intermediate formula in his paper but didn't have the phase invariant formula of Avron-Seiler-Simon.

My Work

Despite the fact that our independent work was earlier (dates of submission for our paper is May 31, 1983 and his June 13, 1983)

My Work

Despite the fact that our independent work was earlier (dates of submission for our paper is May 31, 1983 and his June 13, 1983) and that the geometric ideas were in our paper (and more explicitly with the name curvature in my Berry phase paper),

My Work

Despite the fact that our independent work was earlier (dates of submission for our paper is May 31, 1983 and his June 13, 1983) and that the geometric ideas were in our paper (and more explicitly with the name curvature in my Berry phase paper), K is universally known as the *Berry curvature*.

My Work

Despite the fact that our independent work was earlier (dates of submission for our paper is May 31, 1983 and his June 13, 1983) and that the geometric ideas were in our paper (and more explicitly with the name curvature in my Berry phase paper), K is universally known as the *Berry curvature*.

Berry also realized that in situations where the parameter space could be interpolated into higher dimensions, that eigenvalue degeneracies were sources of curvature, a theme I developed in my paper.

Some Related Work

In the vast literature related to these issues, I should mention two especially illuminating points.

Some Related Work

In the vast literature related to these issues, I should mention two especially illuminating points. The first involves the fact that the first mathematically precise and, in many ways, still the best proof of the quantum adiabatic theorem is Kato's 1950 proof!

Some Related Work

In the vast literature related to these issues, I should mention two especially illuminating points. The first involves the fact that the first mathematically precise and, in many ways, still the best proof of the quantum adiabatic theorem is Kato's 1950 proof! Without loss, one can suppose E(s)=0 (otherwise replace $H(s)$ by $H(s) - E(s)\mathbf{1}$).

Some Related Work

In the vast literature related to these issues, I should mention two especially illuminating points. The first involves the fact that the first mathematically precise and, in many ways, still the best proof of the quantum adiabatic theorem is Kato's 1950 proof! Without loss, one can suppose E(s)=0 (otherwise replace $H(s)$ by $H(s) - E(s)\mathbf{1}$). Kato constructs a comparison dynamics solving

$$\frac{d}{ds}W(s) = iA(s)W(s), \quad 0 \leq s \leq 1; \qquad W(0) = \mathbf{1}$$

$$iA(s) \equiv [P'(s), P(s)]$$

Some Related Work

for which

$$W(s)^{-1}P(s)W(s) = P(0)$$

Some Related Work

for which

$$W(s)^{-1}P(s)W(s) = P(0)$$

by an explicit calculation and he proves that

$$\|W(s)P(0) - U_T(s)P(0)\| = \mathsf{O}(1/T)$$

Some Related Work

for which

$$W(s)^{-1}P(s)W(s) = P(0)$$

by an explicit calculation and he proves that

$$\|W(s)P(0) - U_T(s)P(0)\| = \mathsf{O}(1/T)$$

The relevant point here is that $W(s)$ defines a connection whose differential is $[P, dP]$ so that its differential, the curvature, is given by an earlier formula.

Some Related Work

for which

$$W(s)^{-1}P(s)W(s) = P(0)$$

by an explicit calculation and he proves that

$$\|W(s)P(0) - U_T(s)P(0)\| = \mathrm{O}(1/T)$$

The relevant point here is that $W(s)$ defines a connection whose differential is $[P, dP]$ so that its differential, the curvature, is given by an earlier formula. Thus the Avron-Simon-Seiler formula for the Berry curvature was almost in Kato's paper nearly 35 years before!

Some Related Work

for which

$$W(s)^{-1}P(s)W(s) = P(0)$$

by an explicit calculation and he proves that

$$\|W(s)P(0) - U_T(s)P(0)\| = \mathrm{O}(1/T)$$

The relevant point here is that $W(s)$ defines a connection whose differential is $[P, dP]$ so that its differential, the curvature, is given by an earlier formula. Thus the Avron-Simon-Seiler formula for the Berry curvature was almost in Kato's paper nearly 35 years before!

Secondly, as noted in my Berry phase paper, when the Hilbert space is $\mathbb{C}^n$, this connection appeared a 1965 paper of Bott-Chern.

Some Related Work

for which

$$W(s)^{-1}P(s)W(s) = P(0)$$

by an explicit calculation and he proves that

$$\|W(s)P(0) - U_T(s)P(0)\| = \mathrm{O}(1/T)$$

The relevant point here is that $W(s)$ defines a connection whose differential is $[P, dP]$ so that its differential, the curvature, is given by an earlier formula. Thus the Avron-Simon-Seiler formula for the Berry curvature was almost in Kato's paper nearly 35 years before!

Secondly, as noted in my Berry phase paper, when the Hilbert space is $\mathbb{C}^n$, this connection appeared a 1965 paper of Bott-Chern. As noted later by Aharonov-Anadan, this connection is induced by a Riemannian metric going back to Fubini and Study at the start of the twentieth century.

Time Reversal Covariance

I returned to the subject of the quantum Hall effect and Berry's phase twice.

Time Reversal Covariance

I returned to the subject of the quantum Hall effect and Berry's phase twice. As background, I note that from Berry's paper onwards, a key observation was that Berry's phase is zero if all the $H(s)$ can be taken simultaneously real

Time Reversal Covariance

I returned to the subject of the quantum Hall effect and Berry's phase twice. As background, I note that from Berry's paper onwards, a key observation was that Berry's phase is zero if all the $H(s)$ can be taken simultaneously real (indeed, Berry tells the story that prior to this work, he noted a curiosity in eigenvalue perturbation theory;

Time Reversal Covariance

I returned to the subject of the quantum Hall effect and Berry's phase twice. As background, I note that from Berry's paper onwards, a key observation was that Berry's phase is zero if all the $H(s)$ can be taken simultaneously real (indeed, Berry tells the story that prior to this work, he noted a curiosity in eigenvalue perturbation theory; if one has real matrices depending on two parameters with an eigenvalue degeneracy only at $0 \in \mathbb{R}^2$, then going around the degeneracy causes a sign flip in the eigenvector.

Time Reversal Covariance

I returned to the subject of the quantum Hall effect and Berry's phase twice. As background, I note that from Berry's paper onwards, a key observation was that Berry's phase is zero if all the $H(s)$ can be taken simultaneously real (indeed, Berry tells the story that prior to this work, he noted a curiosity in eigenvalue perturbation theory; if one has real matrices depending on two parameters with an eigenvalue degeneracy only at $0 \in \mathbb{R}^2$, then going around the degeneracy causes a sign flip in the eigenvector. In this case, because eigenvectors are chosen real, there is only a $\pm$ degeneracy and so a unique way of continuing.

Time Reversal Covariance

I returned to the subject of the quantum Hall effect and Berry's phase twice. As background, I note that from Berry's paper onwards, a key observation was that Berry's phase is zero if all the $H(s)$ can be taken simultaneously real (indeed, Berry tells the story that prior to this work, he noted a curiosity in eigenvalue perturbation theory; if one has real matrices depending on two parameters with an eigenvalue degeneracy only at $0 \in \mathbb{R}^2$, then going around the degeneracy causes a sign flip in the eigenvector. In this case, because eigenvectors are chosen real, there is only a $\pm$ degeneracy and so a unique way of continuing. He talked about this result and someone asked him what happened in the complex case and he replied, there was no difference.

Time Reversal Covariance

But after the talk, he realized that in the complex case, phase ambiguity meant there was no unique way to continue under just perturbation of parameters and then, that the adiabatic theorem did give a way of continuing which in the complex case could lead to a non-trivial phase).

Time Reversal Covariance

But after the talk, he realized that in the complex case, phase ambiguity meant there was no unique way to continue under just perturbation of parameters and then, that the adiabatic theorem did give a way of continuing which in the complex case could lead to a non-trivial phase).

Since the curvature must be real, the Im in the formulae for curvature show if all the P's are real then $K = 0$ and there is no Berry phase.

Time Reversal Covariance

But after the talk, he realized that in the complex case, phase ambiguity meant there was no unique way to continue under just perturbation of parameters and then, that the adiabatic theorem did give a way of continuing which in the complex case could lead to a non-trivial phase).

Since the curvature must be real, the Im in the formulae for curvature show if all the P's are real then $K = 0$ and there is no Berry phase. For spinless particles, time reversal just complex conjugates the wave function so the mantra became "time reversal invariance kills Berry's phase".

Time Reversal Covariance

But after the talk, he realized that in the complex case, phase ambiguity meant there was no unique way to continue under just perturbation of parameters and then, that the adiabatic theorem did give a way of continuing which in the complex case could lead to a non-trivial phase).

Since the curvature must be real, the Im in the formulae for curvature show if all the P's are real then $K = 0$ and there is no Berry phase. For spinless particles, time reversal just complex conjugates the wave function so the mantra became "time reversal invariance kills Berry's phase". Magnetic fields destroy reality of the operators (and are not time reversal invariant).

Time Reversal Covariance

But after the talk, he realized that in the complex case, phase ambiguity meant there was no unique way to continue under just perturbation of parameters and then, that the adiabatic theorem did give a way of continuing which in the complex case could lead to a non-trivial phase).

Since the curvature must be real, the Im in the formulae for curvature show if all the P's are real then $K = 0$ and there is no Berry phase. For spinless particles, time reversal just complex conjugates the wave function so the mantra became "time reversal invariance kills Berry's phase". Magnetic fields destroy reality of the operators (and are not time reversal invariant). Indeed, the basic example is to take a constant magnetic field, $\mathbf{B} \in \mathbb{R}^3$ and $H(\mathbf{B}) = \mathbf{B} \cdot \sigma$ where σ is a spin s spin. The curvature is then $(2s+1)\mathbf{B}/B^3$.

Avron-Sadun-Seigert-Simon

In work with Avron and two then postdocs Sadun and Seigert in 1988,

Avron-Sadun-Seigert-Simon

In work with Avron and two then postdocs Sadun and Seigert in 1988, I discovered that for fermions you could have a non-zero Berry phase even with time reversal invariance and that there was a remarkable underlying quaternionic structure relevant to their study.

Avron-Sadun-Seigert-Simon

In work with Avron and two then postdocs Sadun and Seigert in 1988, I discovered that for fermions you could have a non-zero Berry phase even with time reversal invariance and that there was a remarkable underlying quaternionic structure relevant to their study. The underlying issue goes back to a 1932 paper of Wigner on time reversal invariance, T, in quantum mechanics. He first proved his famous theorem that symmetries in quantum mechanics are given by either unitary or anti-unitary operators and then argued that T was always antiunitary with $T^2 = \mathbf{1}$ for bosons and $T^2 = -\mathbf{1}$ for fermions.

Avron-Sadun-Seigert-Simon

In the Bose case, that means T acts like a complex conjugate and so the argument of no Berry's phase applies but not in the fermion case.

Avron-Sadun-Seigert-Simon

In the Bose case, that means T acts like a complex conjugate and so the argument of no Berry's phase applies but not in the fermion case. Instead $J \equiv T$ and, I, the map of multiplication by i are two anticommuting operators whose squares are each $-\mathbf{1}$, so they and $K = IJ$ turn the underlying vector space into one over the quaternions!

Avron-Sadun-Seigert-Simon

In the Bose case, that means T acts like a complex conjugate and so the argument of no Berry's phase applies but not in the fermion case. Instead $J \equiv T$ and, I, the map of multiplication by i are two anticommuting operators whose squares are each $-\mathbf{1}$, so they and $K = IJ$ turn the underlying vector space into one over the quaternions!

Just as the simplest example of Berry's phase is a spin $1/2$ magnetic dipole, our simple example is a spin $3/2$ electric quadrupole. An interesting feature concerns the fact that eigenspaces are never simple but always even complex dimension.

Avron-Sadun-Seigert-Simon

In the Bose case, that means T acts like a complex conjugate and so the argument of no Berry's phase applies but not in the fermion case. Instead $J \equiv T$ and, I, the map of multiplication by i are two anticommuting operators whose squares are each $-\mathbf{1}$, so they and $K = IJ$ turn the underlying vector space into one over the quaternions!

Just as the simplest example of Berry's phase is a spin $1/2$ magnetic dipole, our simple example is a spin $3/2$ electric quadrupole. An interesting feature concerns the fact that eigenspaces are never simple but always even complex dimension. This is known as Kramers degeneracy - one point of Wigner is to explain this as a result of fermion time reversal symmetry.

Avron-Sadun-Seigert-Simon

In the Bose case, that means T acts like a complex conjugate and so the argument of no Berry's phase applies but not in the fermion case. Instead $J \equiv T$ and, I, the map of multiplication by i are two anticommuting operators whose squares are each $-\mathbf{1}$, so they and $K = IJ$ turn the underlying vector space into one over the quaternions!

Just as the simplest example of Berry's phase is a spin $1/2$ magnetic dipole, our simple example is a spin $3/2$ electric quadrupole. An interesting feature concerns the fact that eigenspaces are never simple but always even complex dimension. This is known as Kramers degeneracy - one point of Wigner is to explain this as a result of fermion time reversal symmetry. Thus one looks for holonomy in systems with quaternionicly simple eigenvalues, i.e. eigenvalues of fixed complex multiplicity 2.

Avron-Sadun-Seigert-Simon

In the Bose case, that means T acts like a complex conjugate and so the argument of no Berry's phase applies but not in the fermion case. Instead $J \equiv T$ and, I, the map of multiplication by i are two anticommuting operators whose squares are each $-\mathbf{1}$, so they and $K = IJ$ turn the underlying vector space into one over the quaternions!

Just as the simplest example of Berry's phase is a spin $1/2$ magnetic dipole, our simple example is a spin $3/2$ electric quadrupole. An interesting feature concerns the fact that eigenspaces are never simple but always even complex dimension. This is known as Kramers degeneracy - one point of Wigner is to explain this as a result of fermion time reversal symmetry. Thus one looks for holonomy in systems with quaternionicly simple eigenvalues, i.e. eigenvalues of fixed complex multiplicity 2. The Berry phase is thus a 2×2 unitary matrix.

Avron-Sadun-Seigert-Simon

One noteworthy aspect of our paper is its abstract which reads in full:

Avron-Sadun-Seigert-Simon

One noteworthy aspect of our paper is its abstract which reads in full: *Yes, but some parts are reasonably concrete*.

Avron-Sadun-Seigert-Simon

One noteworthy aspect of our paper is its abstract which reads in full: *Yes, but some parts are reasonably concrete*. While I had introduced topological ideas, I was somewhat dismayed about all the terribly fancy stuff that appeared in the math physics literature, especially throwing around the term "fiber bundle".

Avron-Sadun-Seigert-Simon

One noteworthy aspect of our paper is its abstract which reads in full: *Yes, but some parts are reasonably concrete*. While I had introduced topological ideas, I was somewhat dismayed about all the terribly fancy stuff that appeared in the math physics literature, especially throwing around the term "fiber bundle". Yosi and I used to joke that some people seemed to suffer from bundle fibrosis. So we were concerned about some of the abstruse language in our paper and decided to work out several examples in full as a counterweight.

Avron-Sadun-Seigert-Simon

One noteworthy aspect of our paper is its abstract which reads in full: *Yes, but some parts are reasonably concrete*. While I had introduced topological ideas, I was somewhat dismayed about all the terribly fancy stuff that appeared in the math physics literature, especially throwing around the term "fiber bundle". Yosi and I used to joke that some people seemed to suffer from bundle fibrosis. So we were concerned about some of the abstruse language in our paper and decided to work out several examples in full as a counterweight. We liked our abstract, but getting it into the journal was not easy, an interesting story that I'll not include.

Avron-Sadun-Seigert-Simon

One noteworthy aspect of our paper is its abstract which reads in full: *Yes, but some parts are reasonably concrete*. While I had introduced topological ideas, I was somewhat dismayed about all the terribly fancy stuff that appeared in the math physics literature, especially throwing around the term "fiber bundle". Yosi and I used to joke that some people seemed to suffer from bundle fibrosis. So we were concerned about some of the abstruse language in our paper and decided to work out several examples in full as a counterweight. We liked our abstract, but getting it into the journal was not easy, an interesting story that I'll not include. Almost twenty five years after our paper, the abstract earned a fan blog post entitled *Abstract Snark* that declared our abstract and one other "almost Zen in their simplicity and perfection".

Pairs of Projections

My other work in this area is three related papers that I wrote with Avron and Seiler in 1990 that followed up on an alternate approach to the quantum Hall effect due to Bellisard in which topology entered as an index in C^*-algebraic K-theory.

Pairs of Projections

My other work in this area is three related papers that I wrote with Avron and Seiler in 1990 that followed up on an alternate approach to the quantum Hall effect due to Bellisard in which topology entered as an index in C^*-algebraic K-theory. We developed an index theory for the simpler case where certain subsidiary operators were Fredholm. To me, some of the mathematics we developed was most fascinating.

Pairs of Projections

My other work in this area is three related papers that I wrote with Avron and Seiler in 1990 that followed up on an alternate approach to the quantum Hall effect due to Bellisard in which topology entered as an index in C^*-algebraic K-theory. We developed an index theory for the simpler case where certain subsidiary operators were Fredholm. To me, some of the mathematics we developed was most fascinating. In particular we proved

Pairs of Projections

My other work in this area is three related papers that I wrote with Avron and Seiler in 1990 that followed up on an alternate approach to the quantum Hall effect due to Bellisard in which topology entered as an index in C^*-algebraic K-theory. We developed an index theory for the simpler case where certain subsidiary operators were Fredholm. To me, some of the mathematics we developed was most fascinating. In particular we proved

Theorem. *Let P and Q be two orthogonal projections so that $P - Q$ is trace class. Then $\mathrm{Tr}(P - Q)$ is an integer.*

Pairs of Projections

Remarks 1. This is a result that begs to be proven by Goldberger's method

Pairs of Projections

Remarks 1. This is a result that begs to be proven by Goldberger's method (Murph Goldberger was one of my professors at Princeton and, in his day, a famous theoretical physicist

Pairs of Projections

Remarks 1. This is a result that begs to be proven by Goldberger's method (Murph Goldberger was one of my professors at Princeton and, in his day, a famous theoretical physicist who used to joke about things that just had to be true:

Pairs of Projections

Remarks 1. This is a result that begs to be proven by Goldberger's method (Murph Goldberger was one of my professors at Princeton and, in his day, a famous theoretical physicist who used to joke about things that just had to be true: *oh, you just use Goldberger's method which is a proof by reductio ad absurdum.*

Pairs of Projections

Remarks 1. This is a result that begs to be proven by Goldberger's method (Murph Goldberger was one of my professors at Princeton and, in his day, a famous theoretical physicist who used to joke about things that just had to be true: *oh, you just use Goldberger's method which is a proof by reductio ad absurdum. Suppose it's false; why that's absurd!*).

Pairs of Projections

Remarks 1. This is a result that begs to be proven by Goldberger's method (Murph Goldberger was one of my professors at Princeton and, in his day, a famous theoretical physicist who used to joke about things that just had to be true: *oh, you just use Goldberger's method which is a proof by reductio ad absurdum. Suppose it's false; why that's absurd!*).
2. Slightly earlier, this result was proven by different methods by Effros.

Pairs of Projections

Remarks 1. This is a result that begs to be proven by Goldberger's method (Murph Goldberger was one of my professors at Princeton and, in his day, a famous theoretical physicist who used to joke about things that just had to be true: *oh, you just use Goldberger's method which is a proof by reductio ad absurdum. Suppose it's false; why that's absurd!*).

2. Slightly earlier, this result was proven by different methods by Effros. I found another proof using the Krein spectral shift which is sketched in my Operator Theory Book.

Pairs of Projections

Remarks 1. This is a result that begs to be proven by Goldberger's method (Murph Goldberger was one of my professors at Princeton and, in his day, a famous theoretical physicist who used to joke about things that just had to be true: *oh, you just use Goldberger's method which is a proof by reductio ad absurdum. Suppose it's false; why that's absurd!*).

2. Slightly earlier, this result was proven by different methods by Effros. I found another proof using the Krein spectral shift which is sketched in my Operator Theory Book. Amrein-Sinha have a fourth proof.

Pairs of Projections

Remarks 1. This is a result that begs to be proven by Goldberger's method (Murph Goldberger was one of my professors at Princeton and, in his day, a famous theoretical physicist who used to joke about things that just had to be true: *oh, you just use Goldberger's method which is a proof by reductio ad absurdum. Suppose it's false; why that's absurd!*).
2. Slightly earlier, this result was proven by different methods by Effros. I found another proof using the Krein spectral shift which is sketched in my Operator Theory Book. Amrein-Sinha have a fourth proof.
3. There is a huge literature on pairs of projections. I have several much more recent papers on pairs of projections.

The Proof

Our proof relied on two operators used extensively by Kato in his book, $A = P - Q$ and $B = 1 - P - Q$

The Proof

Our proof relied on two operators used extensively by Kato in his book, $A = P - Q$ and $B = 1 - P - Q$ which he showed obeyed $A^2 + B^2 = \mathbf{1}$.

The Proof

Our proof relied on two operators used extensively by Kato in his book, $A = P - Q$ and $B = 1 - P - Q$ which he showed obeyed $A^2 + B^2 = \mathbf{1}$. We noted that one also had the supersymmetry relation $AB + BA = 0$.

The Proof

Our proof relied on two operators used extensively by Kato in his book, $A = P - Q$ and $B = 1 - P - Q$ which he showed obeyed $A^2 + B^2 = \mathbf{1}$. We noted that one also had the supersymmetry relation $AB + BA = 0$. Since A is trace class and self-adjoint, using a basis of eigenfunctions and the Hilbert-Schmidt Theorem shows that

The Proof

Our proof relied on two operators used extensively by Kato in his book, $A = P - Q$ and $B = 1 - P - Q$ which he showed obeyed $A^2 + B^2 = \mathbf{1}$. We noted that one also had the supersymmetry relation $AB + BA = 0$. Since A is trace class and self-adjoint, using a basis of eigenfunctions and the Hilbert-Schmidt Theorem shows that

$$\mathrm{Tr}(A) = \sum_{\lambda} \lambda d_{\lambda}$$

The Proof

Our proof relied on two operators used extensively by Kato in his book, $A = P - Q$ and $B = 1 - P - Q$ which he showed obeyed $A^2 + B^2 = \mathbf{1}$. We noted that one also had the supersymmetry relation $AB + BA = 0$. Since A is trace class and self-adjoint, using a basis of eigenfunctions and the Hilbert-Schmidt Theorem shows that

$$\mathrm{Tr}(A) = \sum_{\lambda} \lambda d_\lambda$$

where we sum over eigenvalues and $d_\lambda = \dim(\mathcal{H}_\lambda)$ with $\mathcal{H}_\lambda = \{\varphi \mid A\varphi = \lambda\varphi\}$.

The Proof

Our proof relied on two operators used extensively by Kato in his book, $A = P - Q$ and $B = 1 - P - Q$ which he showed obeyed $A^2 + B^2 = \mathbf{1}$. We noted that one also had the supersymmetry relation $AB + BA = 0$. Since A is trace class and self-adjoint, using a basis of eigenfunctions and the Hilbert-Schmidt Theorem shows that

$$\mathrm{Tr}(A) = \sum_{\lambda} \lambda d_\lambda$$

where we sum over eigenvalues and $d_\lambda = \dim(\mathcal{H}_\lambda)$ with $\mathcal{H}_\lambda = \{\varphi \mid A\varphi = \lambda\varphi\}$. The supersymmetry implies that $\psi \in \mathcal{H}_\lambda \Rightarrow B\psi \in \mathcal{H}_{-\lambda}$.

The Proof

Our proof relied on two operators used extensively by Kato in his book, $A = P - Q$ and $B = 1 - P - Q$ which he showed obeyed $A^2 + B^2 = \mathbf{1}$. We noted that one also had the supersymmetry relation $AB + BA = 0$. Since A is trace class and self-adjoint, using a basis of eigenfunctions and the Hilbert-Schmidt Theorem shows that

$$\mathrm{Tr}(A) = \sum_{\lambda} \lambda d_\lambda$$

where we sum over eigenvalues and $d_\lambda = \dim(\mathcal{H}_\lambda)$ with $\mathcal{H}_\lambda = \{\varphi \mid A\varphi = \lambda\varphi\}$. The supersymmetry implies that $\psi \in \mathcal{H}_\lambda \Rightarrow B\psi \in \mathcal{H}_{-\lambda}$. Since $B^2 \restriction \mathcal{H}_\lambda = (1 - \lambda^2)\mathbf{1}$, we see if $\lambda \neq \pm 1$, then B is a bijection of $\mathcal{H}_\lambda$ and $\mathcal{H}_{-\lambda}$

The Proof

Our proof relied on two operators used extensively by Kato in his book, $A = P - Q$ and $B = 1 - P - Q$ which he showed obeyed $A^2 + B^2 = \mathbf{1}$. We noted that one also had the supersymmetry relation $AB + BA = 0$. Since A is trace class and self-adjoint, using a basis of eigenfunctions and the Hilbert-Schmidt Theorem shows that

$$\mathrm{Tr}(A) = \sum_{\lambda} \lambda d_{\lambda}$$

where we sum over eigenvalues and $d_\lambda = \dim(\mathcal{H}_\lambda)$ with $\mathcal{H}_\lambda = \{\varphi \mid A\varphi = \lambda\varphi\}$. The supersymmetry implies that $\psi \in \mathcal{H}_\lambda \Rightarrow B\psi \in \mathcal{H}_{-\lambda}$. Since $B^2 \restriction \mathcal{H}_\lambda = (1 - \lambda^2)\mathbf{1}$, we see if $\lambda \neq \pm 1$, then B is a bijection of $\mathcal{H}_\lambda$ and $\mathcal{H}_{-\lambda}$ so, for such λ, we have that $d_\lambda = d_{-\lambda}$.

The Proof

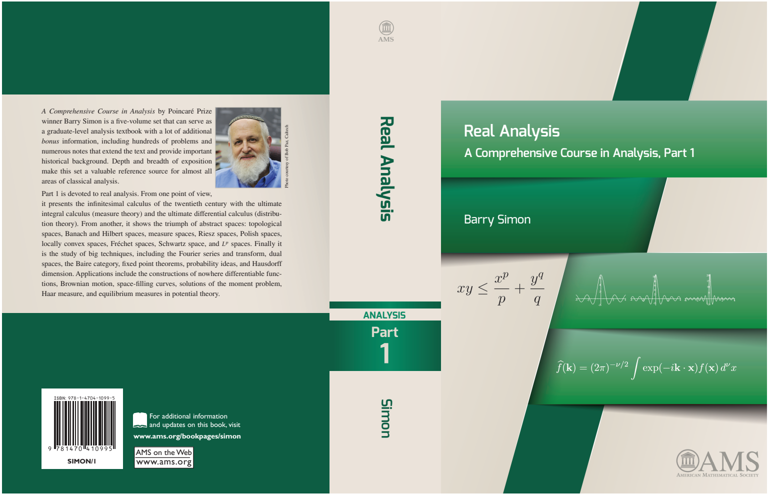
Our proof relied on two operators used extensively by Kato in his book, $A = P - Q$ and $B = 1 - P - Q$ which he showed obeyed $A^2 + B^2 = \mathbf{1}$. We noted that one also had the supersymmetry relation $AB + BA = 0$. Since A is trace class and self-adjoint, using a basis of eigenfunctions and the Hilbert-Schmidt Theorem shows that

$$\mathrm{Tr}(A) = \sum_{\lambda} \lambda d_\lambda$$

where we sum over eigenvalues and $d_\lambda = \dim(\mathcal{H}_\lambda)$ with $\mathcal{H}_\lambda = \{\varphi \mid A\varphi = \lambda\varphi\}$. The supersymmetry implies that $\psi \in \mathcal{H}_\lambda \Rightarrow B\psi \in \mathcal{H}_{-\lambda}$. Since $B^2 \restriction \mathcal{H}_\lambda = (1 - \lambda^2)\mathbf{1}$, we see if $\lambda \neq \pm 1$, then B is a bijection of $\mathcal{H}_\lambda$ and $\mathcal{H}_{-\lambda}$ so, for such λ, we have that $d_\lambda = d_{-\lambda}$. Thus $\mathrm{Tr}(A) = d_1 - d_{-1} \in \mathbb{Z}$

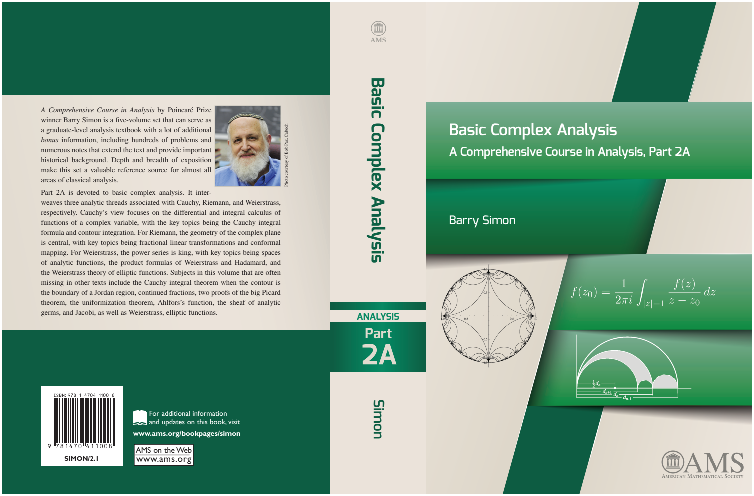
And Now a Word from Our Sponsor

And Now a Word from Our Sponsor



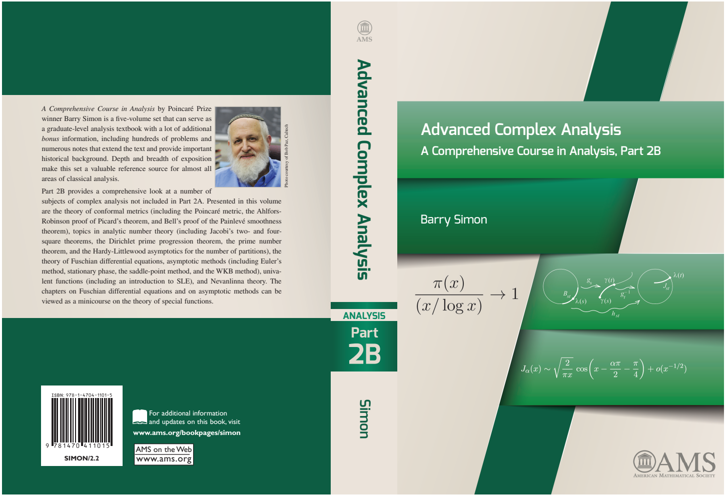
Google *simon comprehensive course preview*

And Now a Word from Our Sponsor



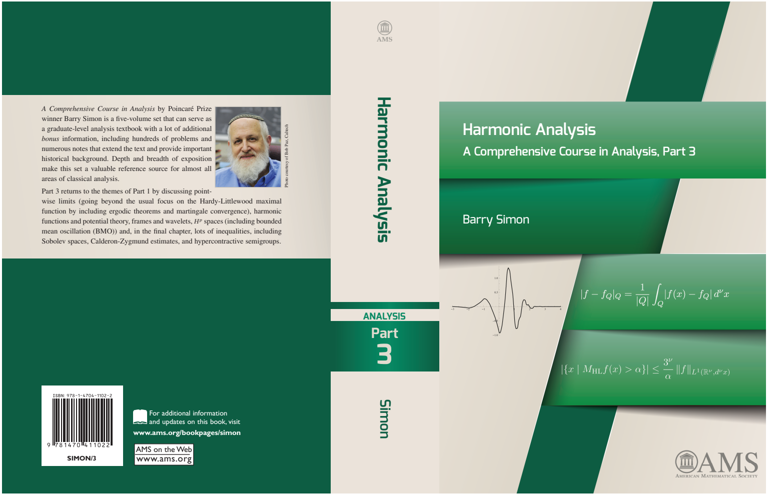
Google *simon comprehensive course preview*

And Now a Word from Our Sponsor



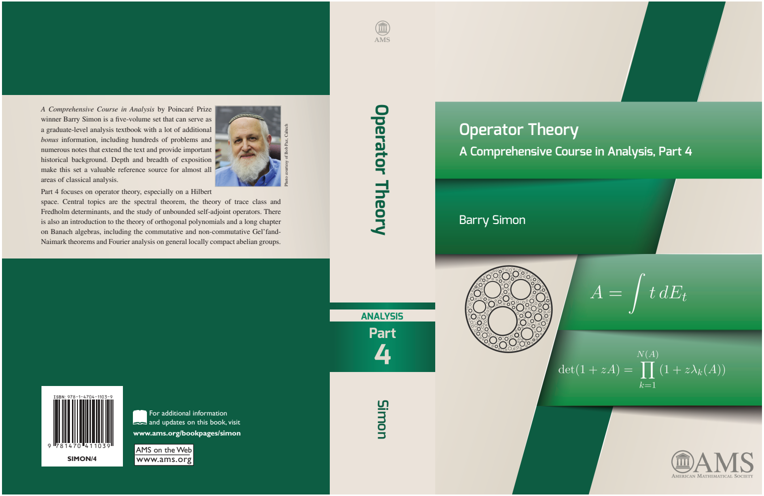
Google *simon comprehensive course preview*

And Now a Word from Our Sponsor



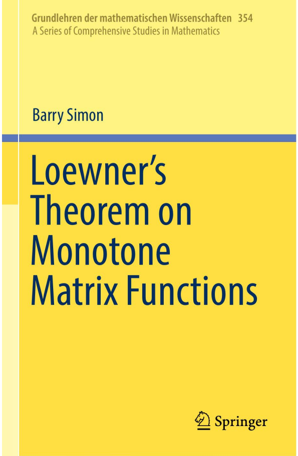
Google *simon comprehensive course preview*

And Now a Word from Our Sponsor



Google *simon comprehensive course preview*

And Now a Word from Our Sponsor



And tada, the latest book