Universality and Clock Behavior for Ergodic Jacobi Matrices

Barry Simon
California Institute of Technology

Joint work with Artur Avila and Yoram Last
The zeros of OPs are of intrinsic interest; also Gauss quadrature (Riemann integrals and zeros of Legendre) and eigenvalues in a box
Consider the zeros of OPUC with $\alpha_n = (\frac{1}{2})^{n+1}$. From my book; check Mhaskar–Saff; clock name.
Clock Behavior
Non-Circular Clock

Density of states; $d\nu_n = \frac{1}{n}$ counting measure of zeros

$$d\nu_n \rightarrow d\rho = \rho(x) \, dx \quad \text{typically}$$

Often, $d\rho$ is potential theoretic equilibrium measure

$$x_n(x), \text{ zeros with } \ldots < x_{-1} < x \leq x_0 < x_1 < \ldots$$

Clock. $n(x_{j+1} - x_j) \rightarrow \frac{1}{\rho(x)}$

Weak Clock. $\frac{x_{j-1} - x_j}{x_1 - x_0} \rightarrow 1 \text{ (does not require DOS to exist)}$

On $[-1, 1]$, $d\rho_e(x) = \frac{1}{\pi \sqrt{1 - x^2}} \, dx$
History of Clock Behavior

- Jacobi polynomials, classical (Szegő’s book)
- Erdös–Turán, 1940: on $[-1, 1]$ regular a.c. weight
- Freud, 1971: universality and clock
- Simon, 2005: series of papers
- Last–Simon, 2006: bounded variation perturbation of free $\Rightarrow$ clock
- Lubinsky, 2007: first universality (Lubinsky inequality), approach of $[-1, 1]$
- Levin–Lubinsky, 2007: rediscovered Freud and applied
- Simon & Totik, 2007: first Lubinsky approach, general $\epsilon$
- Lubinsky, 2008: second approach to universality
- Avila–Last–Simon, 2008: used second Lubinsky approach for ergodic Jacobi matrices
CD Kernel

d\mu(x) = w(x) \, dx + d\mu_s(x) \quad \text{measure "on" } e \subset \mathbb{R}

p_n(x) \quad \text{orthonormal polynomials}

K_n(x, y) = \sum_{j=0}^{n} p_j(x) p_j(y) \quad \text{CD kernel}

CD formula:

K_n(x, y) = \frac{a_{n+1} \left( p_{n+1}(x) p_n(y) - p_{n+1}(y) p_n(x) \right)}{\bar{x} - y}
Universality

\[
\frac{K_n(x + \frac{a}{n}, x + \frac{b}{n})}{K_n(x, x)} \to \frac{\sin(\pi \rho(x)(a - b))}{\pi \rho(x)(a - b)}
\]

- In Freud for smooth weights on \([-1, 1]\)
- Using random matrix theory; proven using RH for analytic weights
- Lubinsky under great generality on \([-1, 1]\)
Notice \( \sin(\pi \rho (a - b)) = 0 \Leftrightarrow a - b = \frac{j}{\rho} \)

Universality \( \Rightarrow \) clock behavior, \( x^{(j+1)} - x^{(j)} \sim \frac{1}{n\rho} \)
Modified Universality

\[ \rho_n \equiv \frac{w(x)K_n(x, x)}{n} \]

"Usually" \( \rho_n \to \rho_\varepsilon(x) \) (MNT, Totik for regular; Simon that \( \frac{1}{n}K_n(x, x) d\mu \xrightarrow{w} d\rho \) (DOS))

Modified universality:

\[ \frac{K_n(x + \frac{a}{n\rho_n}, x + \frac{b}{n\rho_n})}{K_n(x, x)} \to \frac{\sin \pi(a - b)}{\pi(a - b)} \]

Notice if \( \rho_n \to \rho \),

Universality \( \iff \) Modified Universality

Indeed, many random matrix people and Lubinsky essentially write it in this form.
Modified Universality

By modifying Freud–Levin,

\[
\text{Modified Universality} \Rightarrow \text{Weak Clock}
\]

It could be true that always a.e. on the a.c. spectrum, modified universality holds. For now, all examples where modified universality is known, one has \( \rho_n \) converging (although not necessarily to \( \rho_e \)) and so universality.

In my Equilibrium Measure paper, I have an example where \( \rho_n \) does not converge on the a.c. spectrum.
**Theorem 1**

**Definition.** \( x_0 \) is a Lebesgue point for \( \mu \) if

1. \( w(x_0) > 0 \)
2. \( \frac{1}{2\delta} \int_{x_0-\delta}^{x_0+\delta} |w(x) - w(x_0)| \, dx \to 0 \)
3. \( \mu_s(x_0 - \delta, x_0 + \delta)/2\delta \to 0 \)

**Note.** For a.e. \( x_0 \in \Sigma_{ac}(d\mu) = \) essential support of a.c. spectrum is a Lebesgue point.

**Theorem 1.** Suppose for some \( x_0 \in \mathbb{R}, \ x_0 \) is a Lebesgue point for \( \mu \).

1. \( \forall \varepsilon, \exists C_{\varepsilon}, \forall R, \exists N \) so that \( \forall z \in \mathbb{C}, |z| < R, n > N, \)
   \[ |K_n(x_0 + \frac{z}{n}, x_0 + \frac{z}{n})| \leq C_{\varepsilon} \exp(\varepsilon|z|^2) \]

2. \( \forall A > 0, a \) real with \( |a| < A, \)
   \[ \sup_{|a| \leq A} \left| \frac{K_n(x_0 + \frac{a}{n}, x_0 + \frac{a}{n})}{K_n(x_0, x_0)} - 1 \right| \to 0 \]

Then modified universality holds at \( x_0 \).
Theorem 1

- If $C^\varepsilon \exp(\varepsilon |z|^2)$ is replaced by $C \exp(A|z|)$, this is a result of Lubinsky.

- This result is due to ALS.

- Our strategy is Lubinsky; our tactics are quite different.

- In all examples where this is known to hold, stronger $C \exp(A|z|)$ bound holds.
Recall $\rho_n(x_0) = \frac{1}{n} w(x_0) K_n(x_0, x_0)$. Consider the condition

$$\lim_{n \to \infty} \rho_n(x_0) = \rho_{\infty}(x_0) > 0$$

(3)

Theorem 2. $\Sigma \subset \Sigma_{ac}$. Then

(3) for a.e. $x_0 \in \Sigma + (1) \Rightarrow (2)$ for a.e. $x_0 \in \Sigma$

Corollary. (1) + (3) for a.e. $x_0 \in \Sigma_{ac} \Rightarrow \text{modified universality}; \text{ indeed, universality with } \rho_e \text{ replaced by } \rho_{\infty}, \text{ and so clock behavior}$.
Recall the transfer matrix given by

\[ T_n(z) = \begin{pmatrix} p_n(z) & q_n(z) \\ a_n p_{n-1}(z) & a_n q_{n-1}(z) \end{pmatrix} \]

where \( q_n \) are second kind polynomials.
Theorem 3. If

\[ \sup_n \frac{1}{n+1} \sum_{j=0}^{n} \| T_j(x_0) \|^2 = C < \infty \]

and

\[ \inf_n a_n = \alpha_- > 0 \]

then for all \( z \in \mathbb{C}, n, \)

\[ \frac{1}{n+1} \sum_{j=0}^{n} \| T_j(x_0 + \frac{z}{n+1}) \|^2 \leq C \exp(2C\alpha_-^{-1}|z|) \]
Ergodic Jacobi Matrices

$(\Omega, d\eta)$ probability space; $S: \Omega \to \Omega$ invertible ergodic transformation. $B$, bounded real function on $\Omega$, $A$ bounded positive function with $A^{-1}$ bounded. $J_\omega$ has Jacobi parameters $a_n(\omega) \equiv A(S^n\omega)$, $b_n(\omega) \equiv B(S^n\omega)$. We are interested in cases where $\Sigma_{ac}(d\mu_\omega)$ is nonempty.

Example is almost Mathieu; $a_n \equiv 1$, $b_n = \lambda \cos(\pi \alpha n + \theta)$ ($\theta = \omega$). $\lambda < 2$, $\alpha$ irrational. Pure a.c. spectrum for a.e. $\theta$. Spectrum is Cantor set.
Theorem 4. For a.e. $\omega$ and a.e. $x_0 \in \Sigma_{ac}$, we have

$$\lim_{n \to \infty} \frac{1}{n} K_n(x_0, x_0)$$

exists and is strictly positive.
Theorem 5. Let $\rho_\infty(x)$ be the density of the a.c. part of the DOS. Then the limit in Theorem 4 is $\rho_\infty(x_0)/w(x_0, \omega)$.

This is the Nevai–Freud vision for this case. Analog of results of MNT and Totik.

**Corollary.** *Universality* (with $\rho_\infty$ given by the a.c. part of the DOS) and clock behavior a.e. on $\Sigma_{ac}$.

We can’t give complete proofs here—we will settle for sketch, sometimes of weaker or partial variants.
Reduction to Bounds

Lemma 1. \( f \) supported on \([-L, L]\) & \( \|f\|_{L^2} \leq 1 \) & \( \int f = (2L)^{1/2} \Rightarrow f = (2L)^{-1/2} \chi_{[-L,L]} \)

Proof. \( \int |f - \chi_{[-L,L]}(2L)^{-1/2}|^2 \, dx = 0 \), aka equality in Schwarz inequality

\( \square \)
Lemma 2. \[ \int_{-\infty}^{\infty} |f(x)|^2 \, dx \leq 1 \text{ and } \hat{f} \text{ supported on } [-\pi, \pi] \text{ and } f(0) = 1 \Rightarrow f(x) = \frac{\sin(\pi x)}{\pi x} \]

**Proof.** Apply Lemma 1 to \( \hat{f} \) with \( L = \pi \).

Lemma 3. \( f \) entire & \[ \int_{-\infty}^{\infty} |f(x)|^2 \, dx \leq 1 \text{ and } f(0) = 1 \text{ and } |f(z)| \leq C_\varepsilon e^{(\pi + \varepsilon)|\operatorname{Im} z|} \Rightarrow f(z) = \frac{\sin(\pi z)}{\pi z} \]

**Proof.** Paley–Wiener + Lemma 2
**Theorem.** \( f \) entire with \( f \) real on \( \mathbb{R} \) and

1. \( \int_{-\infty}^{\infty} |f(x)|^2 \, dx \leq 1 \)
2. \( f(0) = 1, \, |f(x)| \leq 1 \) \((x \text{ real})\)
3. \( f(z) = 0 \) only on real axis and zeros
   \[ \ldots z_{-1} < 0 < z_1 < \ldots \] with \( z_0 = 0 \) obey
   \[ |z_j - z_k| \geq |j - k| - 1 \]
4. \( |f(z)| \leq C \varepsilon e^{\varepsilon |z|^2} \)

Then \( f(z) = \sin(\pi z) / \pi z \).
**Proof.** (When (4) is replaced by $|f(z)| \leq Ce^{A|z|}$; general needs Phragmén–Lindelőf)

Hadamard $\Rightarrow f(z) = e^{Bz} \prod_{j \neq 0}(1 - \frac{z}{z_j})e^{z/z_j}$; $B$ real

$z = iy$; $|e^{Bz}| = 1$, $|1 - \frac{iy}{z_j}| = 1 + \frac{y^2}{z_j}$ $\Rightarrow |f(iy)| \leq \prod_{j \neq 0}(1 + \frac{y^2}{z_j}) \leq C(1 + y^4) \sinh(\pi y)/\pi y$

$g(z) = e^{i(\pi+\varepsilon)z}f(z)$ bounded on $\arg z = 0, \frac{\pi}{2}, \pi$

Plus Phragmén–Lindelőf $\Rightarrow$ bounded on $\mathbb{C}_+$

So $|f(z)| \leq C\varepsilon e^{(\pi+\varepsilon)|\text{Im } z|}$

Now use lemma.
Consider a fixed and look at limit points, \( f(z) \), of \( K_n(x_0 + \frac{a}{n}, x_0 + \frac{a+z}{n})/K_n(x_0 + \frac{a}{n}, x_0 + \frac{a}{n}) \) at points \( x_0 \) where \( \frac{1}{n}K_n(x_0, x_0) \) has a pointwise limit.

By bounds (condition 1) and compactness (Montel), such limits exist and obey \( f(0) = 1 \). By Cauchy–Schwarz and condition 2, \( |f(x)| \leq 1 \).
Completion of Proof of Theorem 1

From
\[ \int K_n(x, y)K_n(y, w) \, d\mu(y) = K_n(x, w) \]

and \( d\mu \geq w \, dx \) and Lebesgue point \( \int |f(x)|^2 \, dx \leq 1 \)

By condition 1, \( |f(z)| \leq C_\varepsilon \exp(\varepsilon |z|^2) \)

By Markov–Stieltjes inequality (extended form of Freud) for \( j < k \),

\[ x_j(x_0) - x_k(x_0) \geq \sum_{\ell=j+1}^{k-1} \frac{1}{K_n(x_\ell, x_\ell)} \]

implies, using condition 2,

\[ |z_j - z_k| \geq |j - k| - 1 \]

Now use analytic function result.
Preliminaries

By Egorov, for any $\varepsilon$, there exists a compact set, $K$, so that $Q_n(x) = \frac{1}{n}K_n(x, x)$ converges uniformly on $K$ to a limit $Q_\infty(x)$ and $|\Sigma \setminus K| \leq \varepsilon$.

By Lebesgue, for almost every $x_0 \in K$,

$$\frac{n}{2}|K \cap (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})| \to 1$$

We will prove:

$$G_n(a) \equiv \frac{K_n(x_0 + \frac{a}{n}, x_0 + \frac{a}{n})}{K_n(x_0, x_0)} \to 1$$

for such $x_0$. 
Controlling $G_n(a)$

By Cauchy–Schwarz and bound on $\frac{1}{n}K_n(x_0 + \frac{z}{n}, x_0 + \frac{z}{n})$, we get bound on $\frac{1}{n}K_n(x_0 + \frac{z}{n}, x_0 + \frac{w}{n})$.

This plus analyticity proves an $n$-independent bound, even if $x_0 + \frac{a}{n} \notin K$, uniformly in $|a|, |b| \leq A$,

$$|Q_n(x_0 + \frac{a}{n}) - Q_n(x_0 + \frac{b}{n})| \leq C|a - b|$$

By uniform convergence of $Q_n$, we get first continuity of $Q_\infty$, and then uniformly in $|a| \leq A$,

$$x_0 + \frac{a}{n} \in K \Rightarrow |Q_n(x_0 + \frac{a}{n}) - Q_n(x_0)| = o(1)$$

By Lebesgue density, for any $a, n$, $\exists b_n$ so $|a - b_n| \to 0$ and $x_0 + \frac{b_n}{n} \in K$.

This completes the proof of Theorem 2.
Proof of Theorem 3
(following Avila–Krikorian)

\[ T_n = A_n \ldots A_1 \]

\[ \tilde{T}_n \equiv T_n(x_0 + \frac{z}{n+1}) = \tilde{A}_n \ldots \tilde{A}_1 \]

\[ \|\tilde{A}_j - A_j\| \leq \frac{\alpha_{-1}^{-1}|z|}{n+1} \]

\[ T_j^{-1}\tilde{T}_j = (T_j^{-1}\tilde{A}_j T_{j-1})(T_{j-1}^{-1}\tilde{A}_{j-1} T_{j-2}) \ldots \]

\[ = (1 + B_j)(1 + B_{j-1}) \ldots \]

\[ B_k = T_k^{-1}(\tilde{A}_k - A_k) T_{k-1} \]
Proof of Theorem 3
(following Avila–Krikorian)

\[ \| \tilde{T}_j \| \leq \| T_j \| \exp \left( \frac{|z| \alpha^{-1}}{n + 1} \sum_{k=1}^{j} \| T_k \| \| T_{k-1} \| \right) \]

\[ \frac{1}{n+1} \sum_{j=0}^{n} \| \tilde{T}_j \|^2 \leq \left( \frac{1}{n+1} \sum_{j=0}^{n} \| T_j \|^2 \right) \exp \left( |z| \alpha^{-1} \frac{1}{n+1} \sum_{j=0}^{n} \| T_j \|^2 \right) \]
Deift–Simon (1983) showed a.e. \( x \) in the a.c. spectrum of two-sided ergodic Jacobi matrices, there are solutions \( u_n^±(x, \omega) \) obeying

\[
 a_n(\omega)u_{n+1}^±(x, \omega) + (b_n(\omega) - x)u_n^±(x, \omega) + a_{n-1}(\omega)u_{n-1}^±(x, \omega) = 0
\]

and

- (i) \( u_n^+ = u_n^- \)
- (ii) \( a_n(u_{n+1}^- u_n^+ - u_n^- u_{n+1}^-) = i \)
- (iii) \( |u_{n+1}^±(x, \omega)| = |u_n^±(x, S\omega)| \)
- (iv) \( \int |u_0^±(x, \omega)|^2 \, d\eta(\omega) < \infty \)

**Note.** It is known that the phase of \( u \) might not be “almost periodic.”
Deift–Simon Solutions

(iii) \( u_{n+j}^\pm(x, \omega) = e^{\pm i \theta_n(x, \omega)} u_j^\pm(x, S^n \omega) \)

If \( T_n \) is the transfer matrix,

\[
T_n(x, \omega) \begin{pmatrix} u_1^\pm \\ a_0 u_0^\pm \end{pmatrix} = \begin{pmatrix} u_{n+1}^\pm \\ a_n u_n^\pm \end{pmatrix}
\]

so if

\[
B_n(x, \omega) = (-i)^{1/2} \begin{pmatrix} u_1^+ \\ a_0 u_0^+ \\ u_1^- \\ a_0 u_0^- \end{pmatrix}
\]

\((\det B_n) = 1\)

then

\[
T_n(x, \omega) = B(x, S^n \omega) \begin{pmatrix} e^{i \theta_n} & 0 \\ 0 & e^{-i \theta_n} \end{pmatrix} B(x, \omega)^{-1}
\]
A Hilfssatz

**Theorem.** \(D = 2 \times 2\) diagonal unitary \(A, B : \Omega \rightarrow SL(2, \mathbb{C})\) so \(B(S\omega)^{-1}A(\omega)B(\omega) \in D\) and \(\|B(\omega)\| \in L^2(d\eta)\).

\(q : SL(2, \mathbb{C}) \rightarrow \mathbb{R}\) obeys \(q(C) \leq (\text{const})\|C\|^2\).

\(T_n(\omega) = A(S^{n-1}\omega) \ldots A(\omega)\). Then for a.e. \(\omega\),

\[
\frac{1}{n} \sum_{k=0}^{n} q(T_k(\omega))
\]

converges to a finite limit.

Taking \(A, B\) as above and using that \(p\) is the one-one matrix element of \(T\) yields Theorem 4.
An approximation argument using $\|B\| \in L^2$ allows one to suppose $q$ has compact support in $\mathbb{SL}(2, \mathbb{C})$. Let $F : \Omega \times \mathbb{SL}(2, \mathbb{C}) \to \Omega \times \mathbb{SL}(2, \mathbb{C})$ by $F(\omega, C) = (S\omega, A(\omega)C)$, so if $\hat{q}(\omega, C) = q(C)$, then

$$\frac{1}{n} \sum_{k=0}^{n-1} \hat{q}(F^k(\omega, C)) = \frac{1}{n} \sum_{k=0}^{n} q(T_k(\omega)C)$$

A simple construction finds an $F$-invariant probability measure $\nu$ on $\Omega \times \mathbb{SL}(2, \mathbb{C})$ whose projection on $\Omega$ is $\eta$ and whose $\mathbb{SL}(2, \mathbb{C})$ fibers are equivalent to Haar measure for a.e. $\eta$.

That plus ergodic theorem (doesn’t need $F$ ergodic!) implies convergence of sums for a.e. $(\omega, C)$. $q$ compact support plus continuous implies enough equicontinuity to get convergence for a.e. $\omega$ and all $C$. So take $C = 1$. 

Proof of Hilfssatz
By Theorems 1–4,

$$\rho_L(x_0, \omega) = \lim_{n \to \infty} \frac{1}{n} w(x_0, \omega) K_n(x_0, x_0)$$

is asymptotic $O(1/n)$ zero spacing (microlocal DOS) while $\rho_\infty(x_0)$ is a local DOS.

A result of Kotani plus explicit formula for Deift–Simon, $u_n$, shows that

$$\frac{1}{2\pi} \rho_\infty(x_0) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} |u_j(x_0, \omega)|^2$$

(ergodic average since $|u_n(x, \omega)| = |u_0(x, S^n \omega)|$)
Since

\[ p_n(x_0, \omega) = \frac{\text{Im } u_{n+1}(x_0, \omega)}{\text{Im } u_1(x_0, \omega)} \]

and

\[ |\text{Im } u_1(x_0, \omega)|^2 = \pi w(x_0, \omega) \]

We have that

\[ \frac{1}{\pi} \rho_L(x_0, \omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} |\text{Im } u_j(x_0, \omega)|^2 \]
Microlocal DOS = Local DOS

Thus, \( \rho_{\infty} = \rho_L \) is equivalent to

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \text{Re}([u_j(x_0, \omega)]^2) = 0
\]

By zero interlacing, \( \rho_L(x_0, S\omega) = \rho_L(x_0, \omega) \) implies \( \rho_L \) is \( \omega \)-independent. This, combined with tracking phases, shows \( \text{Av}(u_j^2) = 0. \)