Universality and Clock Behavior for Ergodic Jacobi Matrices

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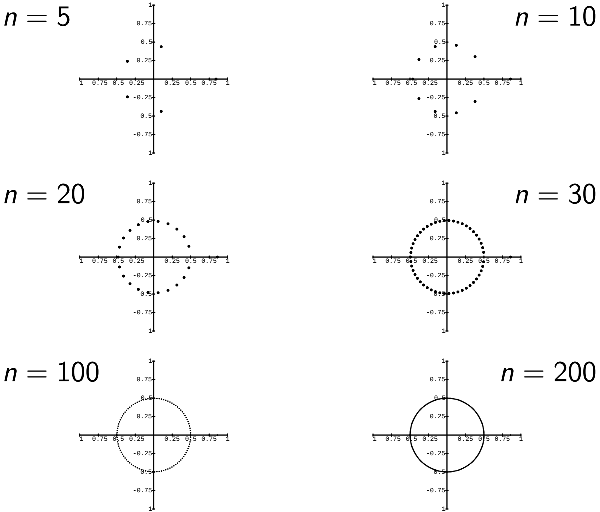
Clock Behavior

The zeros of OPs are of intrinsic interest; also Gauss quadrature (Riemann integrals and zeros of Legendre) and eigenvalues in a box

Clock Behavior

Consider the zeros of OPUC with $\alpha_n = (\frac{1}{2})^{n+1}$. From my book; check Mhaskar–Saff; clock name.

Clock Behavior



Non-Circular Clock

Density of states; $d\nu_n = \frac{1}{n}$ counting measure of zeros

$$d\nu_n \to d\rho = \rho(x)\, dx \qquad \text{typically}$$

Often, $d\rho$ is potential theoretic equilibrium measure

$$x_n(x), \text{ zeros with } \ldots < x_{-1} < x \leq x_0 < x_1 < \ldots$$

Clock. $n(x_{j+1} - x_j) \to \frac{1}{\rho(x)}$

Weak Clock. $\frac{x_{j-1} - x_j}{x_1 - x_0} \to 1$ (does not require DOS to exist)

On $[-1, 1]$, $d\rho_{\mathfrak{e}}(x) = \frac{1}{\pi\sqrt{1-x^2}}\, dx$

History of Clock Behavior

- Jacobi polynomials, classical (Szegő's book)
- Erdös–Turán, 1940: on $[-1,1]$ regular a.c. weight
- Freud, 1971: universality and clock
- Simon, 2005: series of papers
- Last–Simon, 2006: bounded variation perturbation of free $\Rightarrow$ clock
- Lubinsky, 2007: first universality (Lubinsky inequality), approach of $[-1,1]$
- Levin–Lubinsky, 2007: rediscovered Freud and applied
- Simon & Totik, 2007: first Lubinsky approach, general $\mathfrak{e}$
- Lubinsky, 2008: second approach to universality
- Avila–Last–Simon, 2008: used second Lubinsky approach for ergodic Jacobi matrices

CD Kernel

$d\mu(x) = w(x)\,dx + d\mu_{\rm s}(x)$ measure "on" $\mathfrak{e} \subset \mathbb{R}$

$p_n(x)$ orthonormal polynomials

$K_n(x,y) = \sum_{j=0}^n \overline{p_j(x)}\,p_j(y)$ CD kernel

CD formula:

$$K_n(x,y) = \frac{a_{n+1}(\,\overline{p_{n+1}(x)}\,p_n(y) - p_{n+1}(y)\,\overline{p_n(x)}\,)}{\bar{x} - y}$$

Universality

$$\frac{K_n(x+\frac{a}{n}, x+\frac{b}{n})}{K_n(x,x)} \to \frac{\sin(\pi\rho(x)(a-b))}{\pi\rho(x)(a-b)}$$

- In Freud for smooth weights on $[-1,1]$
- Using random matrix theory; proven using RH for analytic weights
- Lubinsky under great generality on $[-1,1]$

Universality $\Rightarrow$ Clock (Freud–Levin)

Notice $\sin(\pi\rho(a-b)) = 0 \Leftrightarrow a - b = \frac{j}{\rho}$

Universality $\Rightarrow$ clock behavior, $x^{(j+1)} - x^{(j)} \sim \frac{1}{n\rho}$

Modified Universality

$$\rho_n \equiv \frac{w(x)K_n(x,x)}{n}$$

"Usually" $\rho_n \to \rho_{\mathfrak{e}}(x)$ (MNT, Totik for regular; Simon that $\frac{1}{n}K_n(x,x)\,d\mu \xrightarrow{w} d\rho$ (DOS))
Modified universality:

$$\frac{K_n(x+\frac{a}{n\rho_n}\,, x+\frac{b}{n\rho_n})}{K_n(x,x)} \to \frac{\sin\pi(a-b)}{\pi(a-b)}$$

Notice if $\rho_n \to \rho$,

$$\text{Universality} \;\Leftrightarrow\; \text{Modified Universality}$$

Indeed, many random matrix people and Lubinsky essentially write it in this form.

Modified Universality

By modifying Freud–Levin,

$$\text{Modified Universality} \;\Rightarrow\; \text{Weak Clock}$$

It could be true that always a.e. on the a.c. spectrum, modified universality holds. For now, all examples where modified universality is known, one has ρ_n converging (although not necessarily to $\rho_{\mathfrak{e}}$) and so universality.

In my Equilibrium Measure paper, I have an example where ρ_n does not converge on the a.c. spectrum.

Theorem 1

Definition. x_0 is a Lebesgue point for μ if

(a) $w(x_0) > 0$

(b) $\frac{1}{2\delta}\int_{x_0-\delta}^{x_0+\delta} |w(x) - w(x_0)|\, dx \to 0$

(c) $\mu_{\mathrm{s}}(x_0-\delta, x_0+\delta)/2\delta \to 0$

Note. For a.e. $x_0 \in \Sigma_{\mathrm{ac}}(d\mu)$ = essential support of a.c. spectrum is a Lebesgue point.

Theorem 1. *Suppose for some $x_0 \in \mathbb{R}$, x_0 is a Lebesgue point for μ.*

(1) $\forall\varepsilon$, $\exists C_\varepsilon$, $\forall R$, $\exists N$ *so that* $\forall z \in \mathbb{C}$, $|z| < R$, $n > N$,

$$|K_n(x_0 + \tfrac{z}{n}, x_0 + \tfrac{z}{n})| \le C_\varepsilon \exp(\varepsilon|z|^2)$$

(2) $\forall A > 0$, *a real with* $|a| < A$,

$$\sup_{|a|\le A}\left|\frac{K_n(x_0 + \frac{a}{n}, x_0 + \frac{a}{n})}{K_n(x_0, x_0)} - 1\right| \to 0$$

Then modified universality holds at x_0.

Theorem 1

- If $C_\varepsilon \exp(\varepsilon|z|^2)$ is replaced by $C \exp(A|z|)$, this is a result of Lubinsky.
- This result is due to ALS.
- Our strategy is Lubinsky; our tactics are quite different.
- In all examples where this is known to hold, stronger $C \exp(A|z|)$ bound holds.

Theorem 2

Recall $\rho_n(x_0) = \frac{1}{n} w(x_0) K_n(x_0, x_0)$. Consider the condition

$$\lim_{n\to\infty} \rho_n(x_0) = \rho_\infty(x_0) > 0 \tag{3}$$

Theorem 2. $\Sigma \subset \Sigma_{\mathrm{ac}}$. *Then*

$$(3) \text{ for a.e. } x_0 \in \Sigma + (1) \Rightarrow (2) \text{ for a.e. } x_0 \in \Sigma$$

Corollary. $(1) + (3)$ *for a.e.* $x_0 \in \Sigma_{\mathrm{ac}} \Rightarrow$ *modified universality; indeed, universality with* $\rho_{\mathfrak{e}}$ *replaced by* ρ_∞*, and so clock behavior.*

Theorem 3

Recall the transfer matrix given by

$$T_n(z) = \begin{pmatrix} p_n(z) & q_n(z) \\ a_n p_{n-1}(z) & a_n q_{n-1}(z) \end{pmatrix}$$

where q_n are second kind polynomials.

Theorem 3

Theorem 3. *If*

$$\sup_n \frac{1}{n+1} \sum_{j=0}^{n} \| T_j(x_0) \|^2 = C < \infty$$

and

$$\inf_n a_n = \alpha_- > 0$$

then for all $z \in \mathbb{C}, n$*,*

$$\frac{1}{n+1} \sum_{j=0}^{n} \| T_j(x_0 + \tfrac{z}{n+1}) \|^2 \leq C \exp(2C\alpha_-^{-1}|z|)$$

Ergodic Jacobi Matrices

$(\Omega, d\eta)$ probability space; $S\colon \Omega \to \Omega$ invertible ergodic transformation. B, bounded real function on Ω, A bounded positive function with A^{-1} bounded. J_ω has Jacobi parameters $a_n(\omega) \equiv A(S^n\omega)$, $b_n(\omega) \equiv B(S^n\omega)$. We are interested in cases where $\Sigma_{\rm ac}(d\mu_\omega)$ is nonempty.

Example is almost Mathieu; $a_n \equiv 1$, $b_n = \lambda \cos(\pi\alpha n + \theta)$ $(\theta = \omega)$. $\lambda < 2$, α irrational. Pure a.c. spectrum for a.e. θ. Spectrum is Cantor set.

Theorem 4

Theorem 4. *For a.e. ω and a.e. $x_0 \in \Sigma_{\mathrm{ac}}$, we have*

$$\lim_{n\to\infty} \frac{1}{n} K_n(x_0, x_0)$$

exists and is strictly positive.

Theorem 5

Theorem 5. *Let $\rho_\infty(x)$ be the density of the a.c. part of the DOS. Then the limit in Theorem 4 is $\rho_\infty(x_0)/w(x_0,\omega)$.*

This is the Nevai–Freud vision for this case. Analog of results of MNT and Totik.

Corollary. *Universality (with ρ_∞ given by the a.c. part of the DOS) and clock behavior a.e. on Σ_{ac}.*

We can't give complete proofs here—we will settle for sketch, sometimes of weaker or partial variants.

Reduction to Bounds

Lemma 1. *f supported on $[-L, L]$ & $\|f\|_{L^2} \leq 1$ & $\int f = (2L)^{1/2} \Rightarrow f = (2L)^{-1/2}\chi_{[-L,L]}$*

Proof. $\int |f - \chi_{[-L,L]}(2L)^{-1/2}|^2 \, dx = 0$, aka equality in Schwarz inequality □

Reduction to Bounds

Lemma 2. *$\int_{-\infty}^{\infty} |f(x)|^2\,dx \le 1$ & $\widehat{f}$ supported on $[-\pi,\pi]$ & $f(0)=1 \Rightarrow f(x) = \sin(\pi x)/\pi x$*

Proof. Apply Lemma 1 to $\widehat{f}$ with $L=\pi$. □

Lemma 3. *f entire & $\int_{-\infty}^{\infty} |f(x)|^2\,dx \le 1$ & $f(0)=1$ & $|f(z)| \le C_\varepsilon e^{(\pi+\varepsilon)|\operatorname{Im} z|} \Rightarrow f(z) = \sin(\pi z)/\pi z$*

Proof. Paley–Wiener + Lemma 2 □

Reduction to Analytic Function Bounds + Zeros

Theorem. *f entire with f real on $\mathbb{R}$ and*

(1) $\int_{-\infty}^{\infty} |f(x)|^2 \, dx \leq 1$

(2) $f(0) = 1$, $|f(x)| \leq 1 \quad (x \text{ real})$

(3) $f(z) = 0$ *only on real axis and zeros* $\dots z_{-1} < 0 < z_1 < \dots$ *with* $z_0 = 0$ *obey*

$$|z_j - z_k| \geq |j - k| - 1$$

(4) $|f(z)| \leq C_\varepsilon e^{\varepsilon |z|^2}$

Then $f(z) = \sin(\pi z)/\pi z$.

Reduction to Analytic Function Bounds + Zeros

Proof. (When (4) is replaced by $|f(z)| \leq Ce^{A|z|}$; general needs Phragmén–Lindelöf)

Hadamard $\Rightarrow f(z) = e^{Bz} \prod_{j \neq 0}(1 - \frac{z}{z_j})e^{z/z_j}$; B real

$z = iy$; $|e^{Bz}| = 1$, $|1 - \frac{iy}{z_j}| = 1 + \frac{y^2}{z_j} \Rightarrow |f(iy)| \leq \prod_{j \neq 0}(1 + \frac{y^2}{z_j^2}) \leq C(1 + y^4)\sinh(\pi y)/\pi y$

$g(z) = e^{i(\pi+\varepsilon)z} f(z)$ bounded on $\arg z = 0, \frac{\pi}{2}, \pi$

Plus Phragmén–Lindelöf $\Rightarrow$ bounded on $\mathbb{C}_+$

So $|f(z)| \leq C_\varepsilon e^{(\pi+\varepsilon)|\mathrm{Im}\, z|}$

Now use lemma. $\square$

Completion of Proof of Theorem 1

Consider a fixed and look at limit points, $f(z)$, of $K_n(x_0 + \frac{a}{n}, x_0 + \frac{a+z}{n})/K_n(x_0 + \frac{a}{n}, x_0 + \frac{a}{n})$ at points x_0 where $\frac{1}{n}K_n(x_0, x_0)$ has a pointwise limit.

By bounds (condition 1) and compactness (Montel), such limits exist and obey $f(0) = 1$. By Cauchy–Schwarz and condition 2, $|f(x)| \leq 1$.

Completion of Proof of Theorem 1

From

$$\int K_n(x,y)K_n(y,w)\,d\mu(y) = K_n(x,w)$$

and $d\mu \geq w\,dx$ and Lebesgue point $\int |f(x)|^2\,dx \leq 1$

By condition 1, $|f(z)| \leq C_\varepsilon \exp(\varepsilon |z|^2)$

By Markov–Stieltjes inequality (extended form of Freud) for $j < k$,

$$x_j(x_0) - x_k(x_0) \geq \sum_{\ell=j+1}^{k-1} \frac{1}{K_n(x_\ell, x_\ell)}$$

implies, using condition 2,

$$|z_j - z_k| \geq |j-k| - 1$$

Now use analytic function result.

Preliminaries

By Egorov, for any ε, there exists a compact set, K, so that $Q_n(x) = \frac{1}{n} K_n(x,x)$ converges uniformly on K to a limit $Q_\infty(x)$ and $|\Sigma \setminus K| \leq \varepsilon$.

By Lebesgue, for almost every $x_0 \in K$,

$$\frac{n}{2} |K \cap (x_0 - \tfrac{1}{n}, x_0 + \tfrac{1}{n})| \to 1$$

We will prove:

$$G_n(a) \equiv \frac{K_n(x_0 + \frac{a}{n}, x_0 + \frac{a}{n})}{K_n(x_0, x_0)} \to 1$$

for such x_0.

Controlling $G_n(a)$

By Cauchy–Schwarz and bound on $\frac{1}{n}K_n(x_0 + \frac{z}{n}, x_0 + \frac{z}{n})$, we get bound on $\frac{1}{n}K_n(x_0 + \frac{z}{n}, x_0 + \frac{w}{n})$.

This plus analyticity proves an n-independent bound, even if $x_0 + \frac{a}{n} \notin K$, uniformly in $|a|, |b| \leq A$,

$$|Q_n(x_0 + \tfrac{a}{n}) - Q_n(x_0 + \tfrac{b}{n})| \leq C|a - b|$$

By uniform convergence of Q_n, we get first continuity of Q_∞, and then uniformly in $|a| \leq A$,

$$x_0 + \tfrac{a}{n} \in K \Rightarrow |Q_n(x_0 + \tfrac{a}{n}) - Q_n(x_0)| = o(1)$$

By Lebesgue density, for any a, n, $\exists b_n$ so $|a - b_n| \to 0$ and $x_0 + \frac{b_n}{n} \in K$.

This completes the proof of Theorem 2.

Proof of Theorem 3 (following Avila–Krikorian)

$$T_n = A_n \dots A_1$$
$$\tilde{T}_n \equiv T_n(x_0 + \tfrac{z}{n+1}) = \tilde{A}_n \dots \tilde{A}_1$$
$$\|\tilde{A}_j - A_j\| \le \tfrac{\alpha_1^{-1}|z|}{n+1}$$

$$T_j^{-1}\tilde{T}_j = (T_j^{-1}\tilde{A}_j T_{j-1})(T_{j-1}^{-1}\tilde{A}_{j-1}T_{j-2})\dots$$
$$= (1 + B_j)(1 + B_{j-1})\dots$$
$$B_k = T_k^{-1}(\tilde{A}_k - A_k)T_{k-1}$$

Proof of Theorem 3 (following Avila–Krikorian)

$$\|\tilde{T}_j\| \leq \|T_j\| \exp\left(\frac{|z|\alpha_-^{-1}}{n+1} \sum_{k=1}^{j} \|T_k\| \, \|T_{k-1}\|\right)$$

$$\frac{1}{n+1} \sum_{j=0}^{n} \|\tilde{T}_j\|^2 \leq \left(\frac{1}{n+1} \sum_{j=0}^{n} \|T_j\|^2\right) \exp\left(|z|\alpha_-^{-1} \frac{1}{n+1} \sum_{j=0}^{n} \|T_j\|^2\right)$$

Deift–Simon Solutions

Deift–Simon (1983) showed a.e. x in the a.c. spectrum of two-sided ergodic Jacobi matrices, there are solutions $u_n^\pm(x,\omega)$ obeying

$$a_n(\omega)u_{n+1}^\pm(x,\omega)+(b_n(\omega)-x)u_n^\pm(x,\omega)+a_{n-1}(\omega)u_{n-1}^\pm(x,\omega)=0$$

and

(i) $\overline{u_n^+} = u_n^-$

(ii) $a_n(u_{n+1}^+ u_n^- - u_n^+ u_{n+1}^-) = i$

(iii) $|u_{n+1}^\pm(x,\omega)| = |u_n^\pm(x,S\omega)|$

(iv) $\int |u_0^\pm(x,\omega)|^2 \, d\eta(\omega) < \infty$

Note. It is known that the phase of u might not be “almost periodic.”

Deift–Simon Solutions

(iii) $\Rightarrow u^{\pm}_{n+j}(x,\omega) = e^{\pm i\theta_n(x,\omega)} u^{\pm}_j(x, S^n\omega)$

If T_n is the transfer matrix,

$$T_n(x,\omega)\begin{pmatrix} u_1^{\pm} \\ a_0 u_0^{\pm} \end{pmatrix} = \begin{pmatrix} u_{n+1}^{\pm} \\ a_n u_n^{\pm} \end{pmatrix}$$

so if

$$B_n(x,\omega) = (-i)^{1/2}\begin{pmatrix} u_1^{+} & u_1^{-} \\ a_0 u_0^{+} & a_0 u_0^{-} \end{pmatrix}$$

$$(\det B_n) = 1$$

then

$$T_n(x,\omega) = B(x, S^n\omega)\begin{pmatrix} e^{i\theta_n} & 0 \\ 0 & e^{-i\theta_n} \end{pmatrix} B(x,\omega)^{-1}$$

A Hilfssatz

Theorem. $\mathcal{D} = 2 \times 2$ *diagonal unitary* $A, B \colon \Omega \to \mathbb{SL}(2, \mathbb{C})$ *so* $B(S\omega)^{-1} A(\omega) B(\omega) \in \mathcal{D}$ *and* $\|B(\omega)\| \in L^2(d\eta)$. $q \colon \mathbb{SL}(2, \mathbb{C}) \to \mathbb{R}$ *obeys* $q(C) \leq (const) \|C\|^2$. $T_n(\omega) = A(S^{n-1}\omega) \dots A(\omega)$. *Then for a.e.* ω,

$$\frac{1}{n} \sum_{k=0}^{n} q(T_k(\omega))$$

converges to a finite limit.

Taking A, B as above and using that p is the one-one matrix element of T yields Theorem 4.

Proof of Hilfssatz

Clock Behavior
Universality
Main Theorems
Proof of Theorem 1
The Lubinsky Wiggle (Proof of Theorem 2)
Exponential Bounds (Proof of Theorem 3)
Diagonal Limits (Proof of Theorem 4)
Equality of Local and Microlocal DOS (Proof of Theorem 5)

An approximation argument using $\|B\| \in L^2$ allows one to suppose q has compact support in $\mathbb{SL}(2,\mathbb{C})$.
Let $F\colon \Omega \times \mathbb{SL}(2,\mathbb{C}) \to \Omega \times \mathbb{SL}(2,\mathbb{C})$ by $F(\omega, C) = (S\omega, A(\omega)C)$, so if $\widehat{q}(\omega, C) = q(C)$, then

$$\frac{1}{n}\sum_{k-0}^{n-1} \widehat{q}(F^k(\omega, C)) = \frac{1}{n}\sum_{k=0}^{n} q(T_k(\omega)C)$$

A simple construction finds an F-invariant probability measure ν on $\Omega \times \mathbb{SL}(2,\mathbb{C})$ whose projection on Ω is η and whose $\mathbb{SL}(2,\mathbb{C})$ fibers are equivalent to Haar measure for a.e. η.

That plus ergodic theorem (doesn't need F ergodic!) implies convergence of sums for a.e. (ω, C). q compact support plus continuous implies enough equicontinuity to get convergence for a.e. ω and all C. So take $C = 1$.

Microlocal DOS = Local DOS

By Theorems 1–4,

$$\rho_L(x_0,\omega) = \lim_{n\to\infty} \frac{1}{n}\, w(x_0,\omega) K_n(x_0,x_0)$$

is asymptotic $O(1/n)$ zero spacing (microlocal DOS) while $\rho_\infty(x_0)$ is a local DOS.

A result of Kotani plus explicit formula for Deift–Simon, u_n, shows that

$$\frac{1}{2\pi}\, \rho_\infty(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |u_j(x_0,\omega)|^2$$

(ergodic average since $|u_n(x,\omega)| = |u_0(x,S^n\omega)|$)

Microlocal DOS = Local DOS

Since

$$p_n(x_0,\omega) = \frac{\operatorname{Im} u_{n+1}(x_0,\omega)}{\operatorname{Im} u_1(x_0,\omega)}$$

and

$$|\operatorname{Im} u_1(x_0,\omega)|^2 = \pi w(x_0,\omega)$$

We have that

$$\frac{1}{\pi}\rho_L(x_0,\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} |\operatorname{Im} u_j(x_0,\omega)|^2$$

Microlocal DOS = Local DOS

Thus, $\rho_\infty = \rho_L$ is equivalent to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathrm{Re}([u_j(x_0,\omega)]^2) = 0$$

By zero interlacing, $\rho_L(x_0, S\omega) = \rho_L(x_0,\omega)$ implies ρ_L is ω-independent. This, combined with tracking phases, shows $\mathrm{Av}(u_j^2) = 0$.