Natural Boundaries and Spectral Theory

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Joint Work with Jonathan Breuer

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and with great thanks to Larry Zalcman for his many years of service

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Weierstrass and All That: Gap Theorems

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In general, one is interested in natural boundaries on arbitrary complex domains, but, in this talk I'll focus only on $\mathbb{D}$. Indeed, I will only consider a special situation which is plenty interesting.

We will only look at sequences $\{b_n\}_{n=0}^{\infty}$ and functions

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Fabry (1896) like Hadamard but only needed $n_j/j \to \infty$.

Random Power Series

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And in 1921, Hecke proved that if $\{x\}$ is the fractional part of a real number x, then

Theorem For any irrational q, we have that

$$f(z) = \sum_{n=0}^{\infty} \{nq\} z^n$$

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The fundamental objects are matrix elements of the resolvents

$$G(z) = \langle \delta_0, (H - z)^{-1} \delta_0 \rangle,$$

$$m_+(z) = \langle \delta_1, (H_+ - z)^{-1} \delta_1 \rangle, \quad m_-(z) = \langle \delta_0, (H_- - z)^{-1} \delta_0 \rangle$$

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Two decompositions of the spectrum concern us. Recall that any measure can be split into three parts: an absolutely continuous (a.c.) part f(x)dx, a pure point(p.p.) and a singular continuous (s.c.) part like the Cantor measure. By looking at supports of the parts of the spectral measure (for H, one needs to also look at a spectral measure associated to δ_1), the spectrum, $\sigma(H)$ has three parts, $\sigma_{ac}(H)$, $\sigma_{pp}(H)$, $\sigma_{sc}(H)$.

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To distinguish "normal" spectra from situations with some singular continuous or dense point spectra, the later has come to be called exotic spectra. A hallmark of such spectra is the absence of a.c. spectrum.

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Avron-Simon (1982) proved that the AMO for $\lambda > 2$ ($b_n = \lambda \cos(\pi\alpha n + \theta)$) has no a.c. spectrum if α is irrational and is purely singular continuous for α very well approximated by rationals (the spectral type is now totally understood depending on Diophantine properties of α).

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The random model and AMO are special cases of a class called ergodic Schrödinger operators where $\{\omega_n\}_{n=-\infty}^{\infty}$ is an ergodic stochastic process and $b_n^\omega = F(\omega_n)$. In this regard

Kotani(1983) (who did continuum models, discrete is Simon (1983)) proved that if an ergodic Schrodinger operator has any a.c. spectrum, then b is deterministic (i.e. each $b_n, n \geq 1$ is a function of $\{b_n\}_{n=-\infty}^{0}$). Random are not, almost periodic is deterministic. (for many years - Kotani-Last conjecture that the only ergodic Schrödinger operators with a.c. spectrum are almost periodic, counter-examples in Avila and Volberg-Yuditskii (2014)).

Kotani (1989) proved that an ergodic Jacobi matrix taking finitely many values either is periodic or has no a.c. spectrum.

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Remling (2011; preprint 2007) proved (this and much more as I'll discuss) that if a half-line operator has b_n taking only finitely many values and any a.c. spectrum, then b_n is eventually periodic.

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Most striking is this last. Perhaps, if I'd known of Szegő's theorem when Kotani did his work, I'd have had my aha moment then but I only learned of Szegő's theorem after Remling which was good because his ideas, which gave a general understanding of the lack of a.c. spectrum, were the key to what Breuer and I found.

The Remling Revolution: Right Limits

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Last-Simon proved that

$$\sigma_{ess}(H_+) \supseteq \bigcap_{H_r\in\mathcal{R}} \sigma(H_r) \qquad \sigma_{ac}(H_+) \subseteq \bigcup_{H_r\in\mathcal{R}} \sigma_{ac}(H_r)$$

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This turns out to be equivalent to the diagonal Green's function $G_{nn}(x+i0)$ being pure imaginary for all n. The name comes from the fact that in the case where b_n goes to zero rapidly as $n \to \pm\infty$ so that a scattering theory exists, the scattering theoretic reflection coefficient is zero for $x \in S$.

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Thus if H is reflectionless one can go from $\{b_n\}_{n=-\infty}^{0}$ to m_- to m_+ to $\{b_n\}_{n=1}^{\infty}$ so the sequence is deterministic. Indeed, Kotani proved his result on stochastic Schrödinger with a.c. spectrum being deterministic by proving H reflectionless on the a.c. spectrum.

The Remling Revolution: His Big Theorem

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His proof is not simple (the exposition in one of my books is 10 dense pages) and no one has found another proof! Fortunately, his result is suggestive to our needs and the proof of the complex variables result doesn't use any ideas from his proof.

Definitions

Definition We say that power series $f(z) = \sum_{n=0}^{\infty} b_n z^n$ with $\sup_n |b_n| < \infty$ has a *strong natural boundary* on $\partial\mathbb{D}$ if and only if, for every interval $I \subset \partial\mathbb{D}$ the quantity below is infinite

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Big Theorem of Breuer-Simon

Theorem. *Let* $f(z) = \sum_{n=0}^{\infty} b_n z^n$ *be a power series with* b_n *bounded. Suppose that* $I \subset \partial\mathbb{D}$ *is an open interval so that the* ($\sup$ *of the) integral in (2) is finite. Then every right limit of* $\{b_n\}_{n=0}^{\infty}$ *is reflectionless on* I.

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Thus, if for any I, we can find a right limit which is not reflectionless on I, then f has a strong natural boundary on $\partial\mathbb{D}$.

Here is a general gap result

Theorem. *Suppose* $\{b_n\}_{n=0}^{\infty}$ *is a bounded sequence so that there exists, some* $C, D > 0$ *and* $n_j \to \infty$ *so that for all* $k < 0$

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After we completed our work, we discovered a remarkable 1949 paper of Agmon (alas little quoted; only 10 MathSciNet citations) that discussed gap theorems by a method not unrelated to ours, of course without the Remling theory analogy.

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Natural Boundaries

Spectral Theory

Main Result

Consequences

M. Riesz' Lemma and The Proof

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The proof is easy. There is a right limit c_n with $c_0 \neq 0$ and with $c_n \leq Ce^{-D|n|}$ for all $n < 0$.

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His result is more general in that he considers cases where b_n is unbounded or $b_n \to 0$ and he can renormalize but less general in that he only gets (for our situation) that f is unbounded in any sector rather than a strong NB. Except for this last, the above theorem is his!!! He didn't try to apply his method to anything but gap theorems.

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Stochastic Power Series

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Determinism Principle If a bounded sequence $\{b_n\}_{n=0}^{\infty}$ has two different right limits $\{c_n\}_{n=-\infty}^{\infty}$ and $\{d_n\}_{n=-\infty}^{\infty}$ so that $c_n = d_n$ for all $n < 0$ but $c_0 \neq d_0$, then $\sum_{n=0}^{\infty} b_n z^n$ has a strong natural boundary on $\partial\mathbb{D}$.

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This immediately implies that any non-deterministic stochastic power series has a strong natural boundary on $\partial\mathbb{D}$ (e.g. Markov processes which are new unless independent) and recovers Steinhaus and Paley-Zygmund.

New Genericity Results

Motivated by my results on Baire generic singular continuous spectrum, Breuer and I used our result to prove

Theorem. *Let $\Omega \subset \mathbb{C}$ be a compact set with more than one point. Let Ω^∞ be the countable product of copies of Ω in the weak topology. Then $\{b \in \Omega^\infty \mid \sum_{n=0}^\infty b_n z^n$ has a strong natural boundary on $\partial\mathbb{D}\}$ is a dense G_δ in Ω^∞.*

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There is a very interesting open question. In spectral theory, Baire generic potentials lead to singular continuous spectrum while random potentials to dense point spectrum. Is there a difference between the natural boundaries for the Baire generic and random power series?

Szegő's Theorem

The key is

Boas' Lemma If $\{b_n\}_{n=0}^{\infty}$ is a sequence taking finitely many values which is not eventually periodic, then there exist right limits $\{c_n\}_{n=-\infty}^{\infty}$ and $\{d_n\}_{n=-\infty}^{\infty}$ so that $c_n = d_n$ for all $n \leq 0$ and $c_1 \neq d_1$.

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Obviously this lemma and what I called the Determinism Principle immediately imply a strong version of Szegő's theorem on natural boundaries in that one can conclude that the functions have a strong natural boundary. That result is actually also due to Boas.

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Kotani essentially rediscovered Boas' Lemma (and Remling improved Kotani's variant) with a proof which I think is not as intuitive as Boas' proof.

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Recall Hecke's example, that for all irrational q, the function $f(z) = \sum_{n=0}^{\infty} \{nq\} z^n$ has a natural boundary.

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M. Reisz' Lemma

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Theorem. *Suppose that* $\{b_n\}_{n=0}^{\infty}$ *is a bounded sequence and the function,* f*, defined by the associated Taylor series has an analytic continuation to a neighborhood of* $\mathbb{D} \cup S$ *where*

$$S = \{re^{i\theta} \mid 0 < r \leq R, \alpha \leq \theta \leq \beta\}$$

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respectively on $\mathbb{D} \setminus \{0\} \cup S$ for the first equation (by analytic continuation to S), uniformly on compacts of $\mathbb{D}$ for the second and uniformly on compacts of $\mathbb{C} \cup \{\infty\} \setminus \overline{\mathbb{D}}$ for the last.

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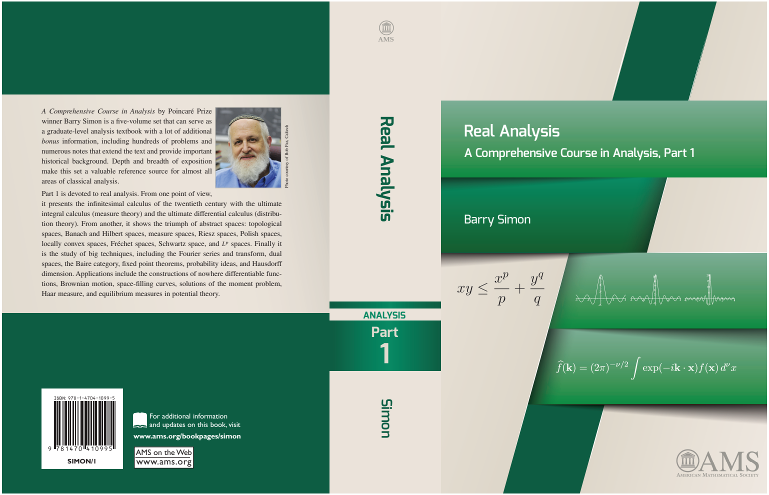

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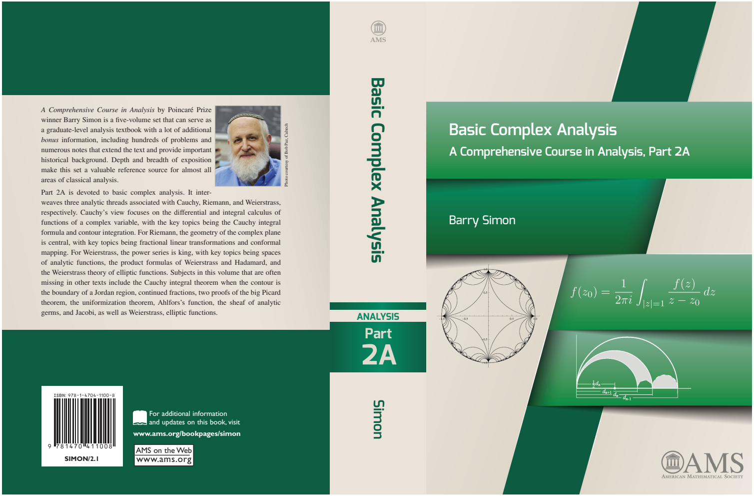

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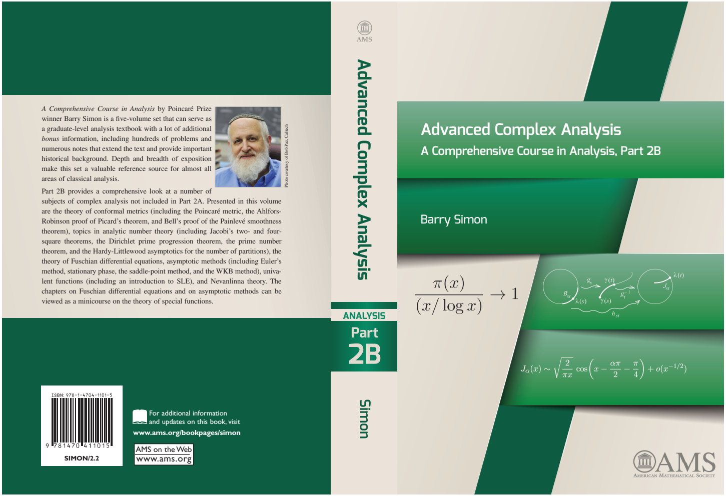

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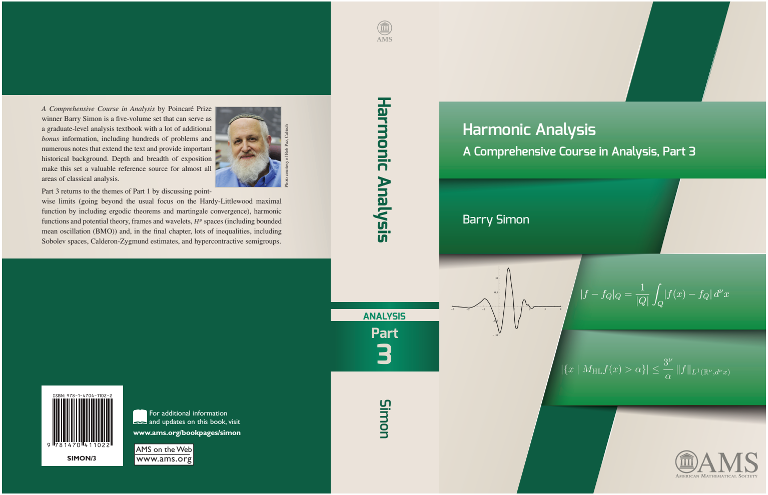

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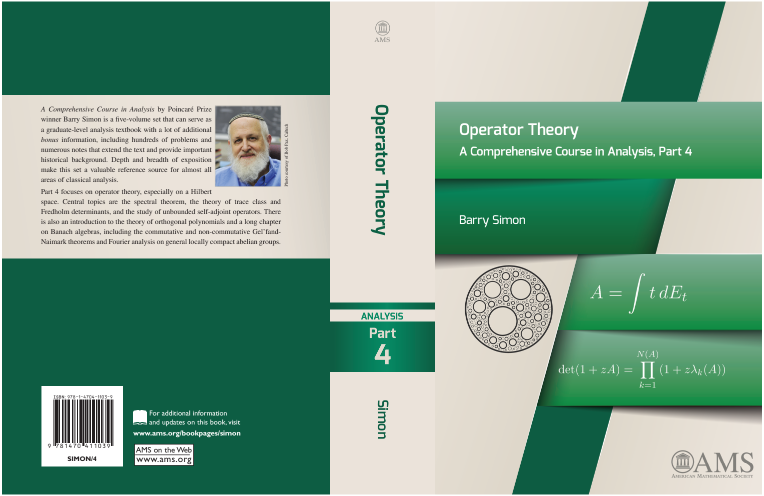

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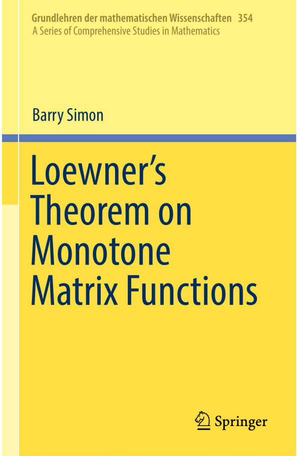

And tada, the latest book