Periodic Jacobi Matrices on Trees

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Joint Work with Nir Avni (Northwestern) and Jonathan Breuer (HUJI)

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Work in Progress

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It is obvious that $\operatorname{spec}(H) = \operatorname{supp}(d\nu)$.

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2. **spec$(\mathbf{H})$ is a finite number of disjoint closed bands, at most p**. This goes back to the dawn of quantum mechanics.
3. ***(gap labelling)* k in any gap of the spectrum is a multiple of $1/\mathrm{p}$**. Again this goes back to the dawn of quantum mechanics although its important extension to the almost periodic case goes back to Johnson–Moser, Avron–Simon and Bellisard in the early 1980s.

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6. **For all $\mathbf{n}$, $\mathbf{G_n(z)}$ and $\mathbf{m_n^{\pm}(z)}$ have analytic continuations from $\mathbb{C}\backslash\mathbf{spec(H)}$ to a finitely sheeted Riemann surface with a discrete set of branch points.**

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7. **These functions are hyperelliptic and, in particular, have only square root branch points and the surface is two sheeted.**

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These results follow by writing down an explicit quadratic equation for the m-functions and analyzing it using, in part, the monotonicity of G in gaps and the fact that poles of m correspond to zeros of G.

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Both Borg (1946) and Hochstadt (1984) proved their results for Hill's equation (i.e. continuum Schrödinger operators) but it is known to hold for the Jacobi case.

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There are several beautiful underlying structures connected with these facts. One involves the Toda flow and gives the nested tori the structure of a completely integrable Hamiltonian system. Another views the isospectral torus as the Jacobian variety of hyperelliptic surface.

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While it is often expressed in terms of Floquet boundary conditions, it is better for our purposes to consider the group of symmetries $W_n = U^n$ where U is the symmetry $Uu_j = u_{j+p}$ so $W_n H = H W_n$.

17. **The representation of $\{\mathbf{W_n}\}_{\mathbf{n}\in\mathbb{Z}}$ acting on $\ell^{\mathbf{2}}(\mathbb{Z})$ is a direct integral of all the irreps of $\mathbb{Z}$, each with multiplicity $\mathbf{p}$**
18. **$\mathbf{H}$ is a direct integral of $\mathbf{p}\times\mathbf{p}$ matrices $\mathbf{H}(\theta)$; $\mathbf{e^{i\theta}}\in\partial\mathbb{D}$ so that**

$$\mathrm{spec}(\mathbf{H}) = \bigcup_{\mathbf{e^{i\theta}}\in\partial\mathbb{D}} \mathrm{spec}(\mathbf{H}(\theta))$$

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In the math physics literature, Δ arises as the trace of a transfer matrix while in the OP literature as a Chebyshev polynomial. This is a key tool in some proofs of the above results.

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We will most often consider regular graphs.

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If there are self–loops, one needs to modify this.

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Pick a base point, $j_0 \in \mathcal{T}$ and define the ball, Λ_r, as the set of all vertices in $\mathcal{T}$ with distance at most r from j_0. Because the number of boundary points in Λ_r is comparable to the total number of points in Λ_r, you **cannot** get $d\nu$ as a limit eigenvalue counting measures with free boundary conditions but

DOS

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$$k(E) = \mathrm{Tr}(P_{(-\infty,E)}(H))$$

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Because in this discrete case, the trace can be normalized, he gets a full gap labelling result although nothing is noted explicitly.

Projections in $\mathcal{C}(\mathcal{T})$

Theorem (Sunada) *For a period p periodic Jacobi matrix on a tree, $k(E)$ in any gap has a value which is a multiple of $1/p$. This implies the spectrum has at most p bands.*

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Given the above formula for k, this result is a corollary of

Theorem (Sunada) *If Ξ is a covering map from a tree to a graph with p vertices, then for any projection, $P \in \mathcal{C}(\mathcal{T}, \Xi)$, its normalized trace, $\mathrm{Tr}(P)$, is a multiple of $1/p$.*

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G and M

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$$G_j(z) = \langle \delta_j, (H - z)^{-1} \delta_j \rangle \qquad m_j^\alpha = \langle \delta_j, (H(\mathcal{T}_j^\alpha) - z)^{-1} \delta_j \rangle$$

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$$N = \begin{pmatrix} X & Z \\ Z^* & Y \end{pmatrix}$$

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$$L = \begin{pmatrix} \mathbf{1} & 0 \\ -Y^{-1}Z^* & \mathbf{1} \end{pmatrix} \text{ so } L^{-1} = \begin{pmatrix} \mathbf{1} & 0 \\ Y^{-1}Z^* & \mathbf{1} \end{pmatrix}$$

Schur Complements

A simple calculation shows that

$$L^* N L = \begin{pmatrix} S & 0 \\ 0 & Y \end{pmatrix} \tag{4.1}$$

so

$$\begin{aligned} N^{-1} &= L \begin{pmatrix} S^{-1} & 0 \\ 0 & Y^{-1} \end{pmatrix} L^* \\ &= \begin{pmatrix} S^{-1} & -S^{-1} Z Y^{-1} \\ -Y^{-1} Z^* S^{-1} & Y^{-1} + Y^{-1} Z^* S^{-1} Z Y^{-1} \end{pmatrix} \end{aligned}$$

which proves Banachiewicz' formula $(N^{-1})_{11} = S^{-1}$

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Similarly, if $\beta = (rj)$ is an edge in $\mathcal{T}$, we have that

$$m_j^\beta(z) = \frac{1}{-z + b_j - \sum_{\alpha=(jk);\, k\neq r} a_\alpha^2 m_k^\alpha(z)}$$

Note that if e is the number of edges in the underlying graph, $\mathcal{G}$, then there are $2e$ m–functions.

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Formulae for G and M

If you compare the two equations for G and m, they differ in a single term, so if $\beta = (rj)$ is an edge in $\mathcal{T}$, we have that

$$G_j(z) = \frac{1}{\left[m_j^{\beta}(z)\right]^{-1} - a_{\beta}^2 m_r^{\beta}(z)}$$

an analog of a well known formula from the 1D case.

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the famed Kesten–McKay distribution, which arose first in random graph models.

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If $b \neq 0$, there is a single gap which is always open.

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Problem 1 *Find a simpler proof of the above bound state theorem of Aomoto.*

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I want to explain our more explicit form of Green's function regularity.

A Theorem on Algebraic Varieties

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(3) The map π *is finite on* $\mathcal{V} \setminus F$*; indeed (Bezout)* $\#\pi^{-1}(z) \leq \prod_{j=1}^{\ell} \deg P_j$.

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Corollary *Periodic Jacobi matrices on trees have no singular continuous spectrum.*

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Problem 3 *Prove a Hochstadt type theorem for general periodic trees with this improved definition of period.*

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We at least know the set is non-empty, for if all b are different and $\sum a < \min_{i\neq j} |b_i - b_j|$, then all gaps are open.

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Problem 4 *Find an effective description of gap edges.*

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Problem 5 *Classify the possible DOS allowed for a given tree, period and set.*

IsoDOS sets

The 1D Case
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Gap Labelling
Equations for M and G
Examples
Aomoto Results
Many Conjectures and Questions
Final Remarks

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Problem 7 *Is there an natural flow on the IsoDOS set?*

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Problem 8 *Explore what connection there is between non-physical sheet poles of an m_j^β and physical sheet poles of the other rooted trees. Resolve the notion that there are d rooted trees and, we suspect, only two branches.*

Relevant Representations

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Problem 11 *Determine if the direct integral decomposition is of any use in spectral analysis. In particular, do gap edges have to do with particular irreps?*

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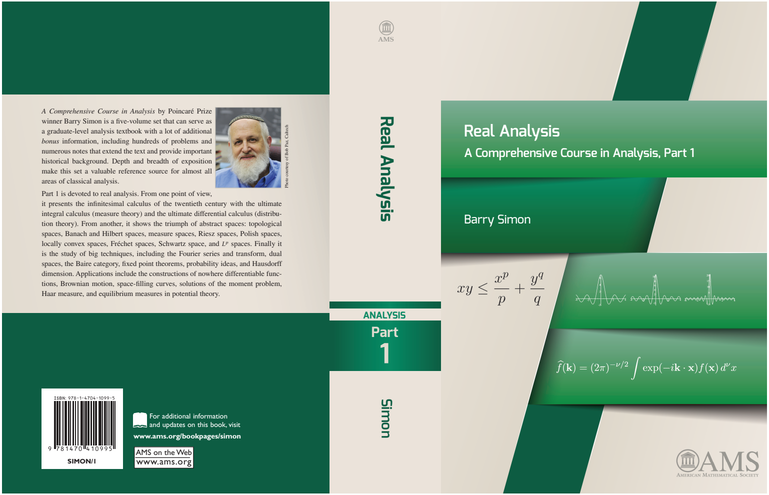

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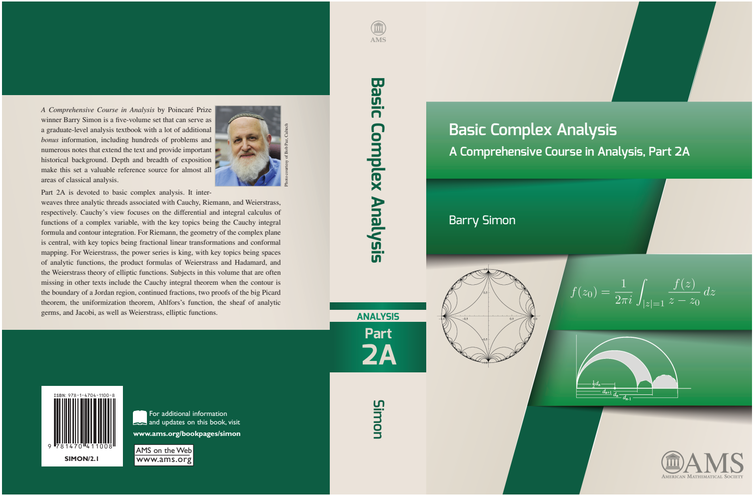

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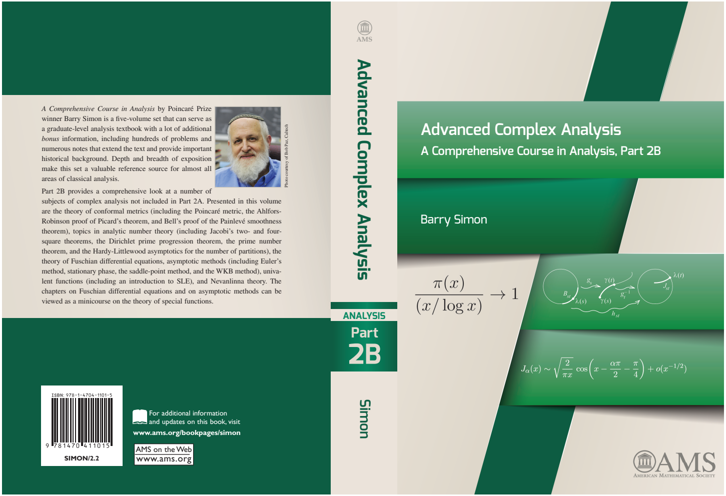

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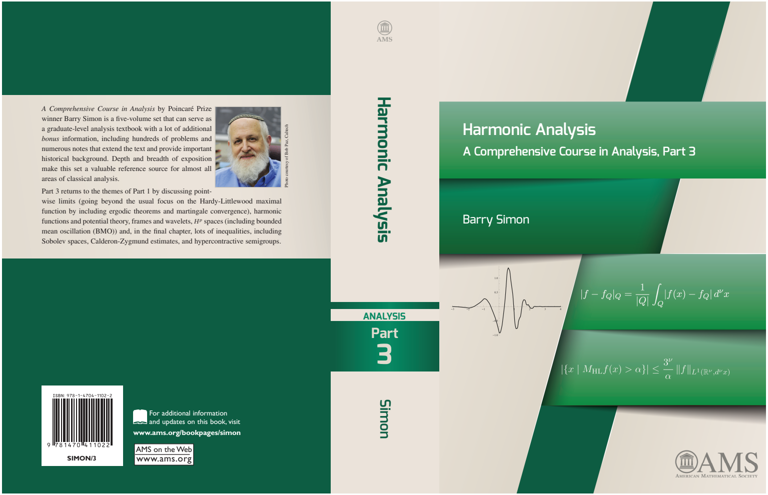

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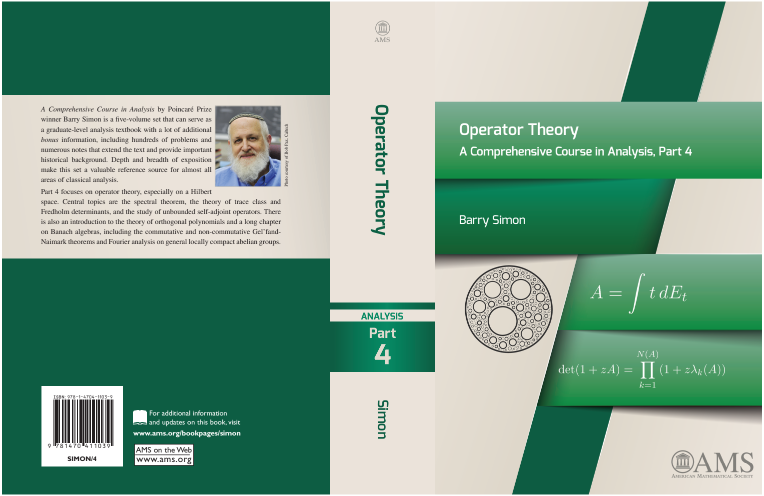

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