The Work of Daniel Wells,
Forty Years Late

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Introduction

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I am writing a book for Cambridge Press entitled *Phase Transitions in the Theory of Lattice Gases*. It is in many ways the successor to my 1993 book *The Statistical Mechanics of Lattice Gases*, Vol. I, from Princeton University Press. That earlier book was mainly framework and largely left out all the most fun and beautiful elements of the theory:
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or more general over multiindices, i.e. assignments of an integer, $n_j \geq 0$ with then $\sigma^A = \prod_{j \in A} \sigma_j^{n_j}$ (and a finite sum or else $\ell^1$ condition). One then considers, the Gibbs state

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The left hand side is an Ising expectation and the right with the apriori measure of the \(2D\) rotor with only couplings of the 1 components. So this was part of what seems to be an Ising domination result (the 2 indicates the Ising measure should really be \( b_{1/\sqrt{2}} \)).
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The Rest of the Talk

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A \textit{Ginibre system} is a triple \( \langle X, \mu, \mathcal{F} \rangle \) of a compact Hausdorff space, \( X \), a probability measure, \( \mu \), on \( X \) (with expectations \( \langle \cdot \rangle_\mu \)) and a class of continuous real valued functions \( \mathcal{F} \subset C(X) \) that obeys:
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\begin{align*}
(G1) & \quad \forall f_1, \ldots, f_n \in \mathcal{F} \int_X f_1(x) \ldots f_n(x) \, d\mu(x) \geq 0 \\
(G2) & \quad \forall f_1, \ldots, f_n \in \mathcal{F} \int_{X \times X} \prod_{j=1}^n (f_j(x) \pm f_j(y)) \, d\mu(x)d\mu(y) \geq 0
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for all \( 2^n \) choices of the plus and minus sign.
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Note that

\[(G2) \Rightarrow 2\langle f \rangle_\mu = \int_X f(x) + f(y) \, d\mu(x) \, d\mu(y) \geq 0 \]

\[\int_{X \times X} (f(x) - f(y))(g(x) - g(y)) \, d\mu(x) \, d\mu(y)\]

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We will see shortly that \( (G2) \Rightarrow (G1) \)
Extending Ginibre Systems

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Given a family of functions, \( \mathcal{F} \subset C(X) \), we define the \textit{Ginibre cone}, \( \mathcal{C}(\mathcal{F}) \), as the set of linear combinations with non-negative coefficients of products of functions from \( \mathcal{F} \).
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It is trivial that $(G2)$ holds for sums and positive multiples of functions for which it holds, so it suffices to prove it holds for products. By induction, we need only handle products of two functions. We note that

$$f g \pm f' g' = \frac{1}{2}(f + f')(g \pm g') + \frac{1}{2}(f - f')(g \mp g')$$
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Given a family of functions, $\mathcal{F} \subset C(X)$, we define the Ginibre cone, $\mathcal{C}(\mathcal{F})$, as the set of linear combinations with non-negative coefficients of products of functions from $\mathcal{F}$.

**Ginibre Theorem 1** If a triple $\langle X, \mu, \mathcal{F} \rangle$ obeys $(G2)$, so does $\langle X, \mu, \mathcal{C}(\mathcal{F}) \rangle$.

It is trivial that $(G2)$ holds for sums and positive multiples of functions for which it holds, so it suffices to prove it holds for products. By induction, we need only handle products of two functions. We note that

$$fg \pm f'g' = \frac{1}{2}(f + f')(g \pm g') + \frac{1}{2}(f - f')(g \mp g')$$

which allows us to prove $(G2)$ for a single product when we have it for individual functions (and shows $(G2) \Rightarrow (G1)$).
The following is trivial

**Ginibre Theorem 2** Let \( \{ \langle X_j, \mu_j, \mathcal{F}_j \rangle \}_{j=1}^n \) be a family of Ginibre systems. Then \( \langle \times_{j=1}^n X_j, \otimes_{j=1}^n \mu_j, \cup_{j=1}^n \mathcal{F}_j \rangle \) is also a Ginibre system.
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$$\langle f \rangle_{\mu_H} = \frac{\langle fe^{-H} \rangle_{\mu}}{\langle e^{-H} \rangle_\mu}$$

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The proof is easy. The normalization is irrelevant and we expand the exponential \( \exp(-H(x) - H(y)) \).
Ginibre Theorem 4 Let $X$ be $\mathbb{R}$ or a compact subset of the form $[-A, A]$ and let $d\mu$ be a probability measure which is invariant under $x \mapsto -x$ and so that (only non-trivial in case $X$ is not compact) $\int x^{2n} d\mu(x) < \infty$ for all $n$. Let $\mathcal{F}$ contain the single function, $f(x) = x$. Then $\langle X, \mu, \mathcal{F} \rangle$ is a Ginibre system.
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Interchanging $x$ and $y$ implies the integral is zero if $m$ is odd and $x \mapsto -x$ symmetry implies the integral is zero if $m + k$ is odd. Thus the only possible non-zero integrals are when $m$ and $k$ are even in which case the integrand is positive!
A little thought shows that for Hamiltonians of the form

\[-H = \sum_{A \subseteq \Lambda} J(A)\sigma^A\]

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\[
\int \int (f_1(x) \pm f_1(y)) \cdots (f_n(x) \pm f_n(y)) d\mu(x) d\nu(y) \geq 0
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Basic Definition

We will be most interested in case $X = \mathbb{R}$, $\mu$ and $\nu$ are both even measures with all moments finite and $\mathcal{F}$ has the single function $f(x) = x$ in which case the condition takes the form

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (x + y)^n (x - y)^m \, d\mu(x) \, d\nu(y) \geq 0$$

for all non-negative integers, $n$ and $m$ in which case we use the symbol $\ll$ without being explicit about $\mathcal{F}$. Since the measures are even, one need only check this when $n + m$ is even. It is trivial if both are even, so we only need worry about the case that both are odd. Since the measures are different, we don’t have the exchange symmetry that makes the integral vanish if both are odd but symmetry under $y \mapsto -y$ implies invariance under interchange of $m$ and $n$, so we need only check for $m \geq n$. We’ll see examples later.
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(d) If $\mu \prec \nu$ with respect to a set of functions $\mathcal{F}$, then for every $f \in \mathcal{F}$, we have that

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\]

In particular, if each \( X_j = \mathbb{R} \), (so implicitly \( F_j \) is the single function \( \sigma_j \)) and if \( H \) has the general Ising form, then for all \( A \subset 2^{\{1, \ldots, n\}} \) one has that

\[
\langle \sigma^A \rangle_{\mu_H} \leq \langle \sigma^A \rangle_{\nu_H}
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Almost a Partial Order

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Since Ising domination is trivially transitive, for applications, this lack isn’t so important.
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**Big Theorem** Let \( d\mu \) be an even probability measure on \( \mathbb{R} \) with compact support that is not a point mass at 0. Then there are two strictly positive numbers \( T_-(\mu) \) and \( T_+(\mu) \) so that \( \mu \prec b_S \) if and only if \( S \geq T_+ \) and \( b_S \prec \mu \) if and only if \( S \leq T_- \). Moreover
We say an even probability measure is non-trivial if and only if it is not a unit mass at 0. The following theorem says that any non-trivial measure of compact support is Ising dominated by a scaling of any other such measure and gives quantitative optimal bounds when one of the measures is the Bernoulli measure.

**Big Theorem** Let $d\mu$ be an even probability measure on $\mathbb{R}$ with compact support that is not a point mass at 0. Then there are two strictly positive numbers $T_-(\mu)$ and $T_+(\mu)$ so that $\mu \prec b_S$ if and only if $S \geq T_+$ and $b_S \prec \mu$ if and only if $S \leq T_-$. Moreover

$$T_+ = \sup\{s \mid s \in \text{supp}(\mu)\}$$
Statement of the Theorem

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\]

and

\[
S \leq T_- \iff \forall n \in \mathbb{N} \int_{\mathbb{R}} (x^2 - S^2)^n d\mu(x) \geq 0
\]
What is $T$?

The proof is not hard but I will defer it and include it if there is time.
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One consequence of the theorem is

$$T_- \leq \left( \int_{\mathbb{R}} x^2 \, d\mu(x) \right)^{1/2}$$

It is an interesting question when one has equality.
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One consequence of the theorem is

$$T_- \leq \left( \int_{\mathbb{R}} x^2 \, d\mu(x) \right)^{1/2}$$

It is an interesting question when one has equality. Before leaving this theorem, I should mention I happened to look at a 1981 paper of Bricmont, Lebowitz and Pfister that includes in an appendix a proof (with attribution to Wells) of Wells result about the existence of $T_- > 0$. 
Three Spin Values

For $0 \leq \lambda \leq 1$, consider the probability measure supported by the three points $\{0, \pm 1\}$ given by

$$d\mu_\lambda = \frac{\lambda}{2} (\delta_1 + \delta_{-1}) + (1 - \lambda)\delta_0$$
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$$\iff \frac{1 - T^2}{T^2} \geq \left( \frac{1 - \lambda}{\lambda} \right)^{1/2m+1}$$
If $\lambda \leq 1/2$, then $(1 - \lambda)/\lambda \geq 1$ and the maximum on the right side of the last formula occurs for $m = 0$. 

Note also that at $\lambda = 1/2$, the integral $\langle (x^2 - T^2 - m + 1)^2 \rangle_\lambda$ vanishes for all $n$, a sign that the distribution of $x^2 - T^2$ is symmetric about 0.
If $\lambda \leq 1/2$, then $(1 - \lambda)/\lambda \geq 1$ and the maximum on the right side of the last formula occurs for $m = 0$ while, if $\lambda \geq 1/2$, then $(1 - \lambda)/\lambda \leq 1$ and we get the maximum as $m \to \infty$. 


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$$T_-(\lambda) = \begin{cases} \sqrt{\lambda}, & \text{if } \lambda \leq \frac{1}{2} \\ \sqrt{\frac{1}{2}}, & \text{if } \lambda \geq \frac{1}{2} \end{cases}$$
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So we see there are cases where $T_- = \langle x^2 \rangle^{1/2}$ and other cases where the inequality is strict. Note also that at $\lambda = 1/2$, the integral $\langle (x^2 - T_-^2)^{2m+1} \rangle_\lambda$ vanishes for all $n$, a sign that the distribution of $x^2 - T_-^2$ is symmetric about 0.
Spin $S$

For each value of $S = 1/2, 1, 3/2, \ldots$, consider the measure $d\tilde{\mu}_S$ which takes $2S + 1$ values equally spaced between $-1$ and $1$, each with weight $1/(2S + 1)$.
Spin $S$

For each value of $S = 1/2, 1, 3/2, \ldots$, consider the measure $d\tilde{\mu}_S$ which takes $2S + 1$ values equally spaced between $-1$ and $1$, each with weight $1/(2S + 1)$. We have just seen that for $S = 1$ ($\lambda = 2/3$ in the above example), one has that $T_- = \sqrt{\frac{1}{2}} < \sqrt{\frac{2}{3}} = \left(\int_{\mathbb{R}} x^2 d\tilde{\mu}_{S=1}(x)\right)^{1/2}$.
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I have used Mathematica to compute $\langle (x^2 - a_S)^{2n+1} \rangle_S$ where $a_S = \left(\int_{\mathbb{R}} x^2 \, d\tilde{\mu}_S(x)\right)$ for $S = 3/2, 2, 5/2$ and $m = 1, 2, \ldots, 5$ and found them all positive which leads to a natural conjecture which I state as an open question.
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**Question 3** Prove for spin $S$ that $T_\frac{2}{2}^2 \geq 1/3$. 
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The only result I know on Ising domination lower bounds on spin $S'$ by $b_T$ for general $S$ is Griffiths (by clever choice of analog spin 1/2 systems) has $T^2 = 1/4$ so I am especially interested in these two questions.
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**Question 4** Prove for spin $S$ that $\tilde{\mu}_S$ Ising dominates $\tilde{\mu}_{S+1/2}$. 
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**Question 4** Prove for spin $S$ that $\tilde{\mu}_S$ Ising dominates $\tilde{\mu}_{S+1/2}$.

It could even happen that there is Wells domination. It would even be interesting to know that $\tilde{\mu}_S$ Ising dominates normalized Lebesgue measure on $[-1, 1]$. 
Most of this talk is about work of Ginibre, Wells (and van Beijeren-Sylvester). I turn next to what may be my only new result on this subject.
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$$d\mu_D(x) = \left[ \frac{\Gamma\left(\frac{D}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{D-1}{2}\right)} \right] (1 - x^2)^{\frac{1}{2}(D-3)} \chi_{[-1,1]}(x) dx$$

This is the distribution of $x_1$ is one looks at a $D$-component unit vector distributed with the rotation invariant measure on $S^{D-1}$.

Since with respect to this measure all $x_j$ have the same distribution and $\sum_{j=1}^D x_j^2 = 1$, we clearly have that

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Totally Anisotropic D-vector model

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The result for \( D = 2 \) is especially easy because
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$$\langle (x^2 - 1/2)^{2m+1} \rangle_{D=2} = 0 \text{ since it is equivalent to } \langle (2x^2 - 1)^{2m+1} \rangle_{D=2} = \langle (x_1^2 - x_2^2)^{2m+1} \rangle_{\text{rotor}} = 0 \text{ by } x_1 \leftrightarrow x_2.$$
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I note that this result for \( D = 2 \) is precisely the result that Aizenman and I say is in Wells mystery preprint. He may have the general \( D \) result there but since \( D = 2 \) is much easier, maybe not.
van Beijeren-Sylvester order

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$$\forall f \in \mathcal{M}_+ \frac{\int fg \, d\hat{\mu}}{\int g \, d\hat{\mu}} \leq \frac{\int fg \, d\hat{\nu}}{\int g \, d\hat{\nu}}$$
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While this notion is useful, it has one nearly fatal flaw
We then write $\mu \prec \nu$ say that $\nu$ \textit{van Beijeren-Sylvester dominates} $\mu$. The first says that $\frac{\hat{\mu}([x,\infty))}{\hat{\nu}([x,\infty))}$ is monotone decreasing as $x$ increases (when we can take the ratio, i.e. so long as $\hat{\nu}([y,\infty)) \neq 0$). And these in turn imply even more than Ising domination of $\mu$ by $\nu$ - it is true for Hamiltonians built by more than products of $\sigma$ - products of any elements of $\mathcal{M}$.

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We then write $\mu < \nu$ say that $\nu$ van Beijeren-Sylvester dominates $\mu$. The first says that $\frac{\hat{\mu}([x, \infty))}{\hat{\nu}([x, \infty))}$ is monotone decreasing as $x$ increases (when we can take the ratio, i.e. so long as $\hat{\nu}([y, \infty)) \neq 0$). And these in turn imply even more than Ising domination of $\mu$ by $\nu$ - it is true for Hamiltonians built by more than products of $\sigma$ - products of any elements of $\mathcal{M}$.

While this notion is useful, it has one nearly fatal flaw (that comes from the strength of the conclusion - all of $\mathcal{M}$ rather than just linear functions) one has that

$$b_T < \mu \text{ for some } T > 0 \Rightarrow \mu([[0, T)]) = 0$$
The Open Questions

To summarize

- Question 1: Is Wells relation transitive among all even measures on $\mathbb{R}$? How about among all measures on a general topological space if $F$ is rich enough?
- Question 2: Prove for spin $S \geq \frac{3}{2}$ that $T^2 - a^S = \alpha$.
- Question 3: Prove for spin $S$ that $T^2 - a^S \geq \frac{1}{3}$.
- Question 4: Prove for spin $S$ that $\tilde{\mu}^S_{\text{Ising}}$ dominates $\tilde{\mu}^S_{\text{Ising} + \frac{1}{2}}$. 

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$$T_+ = \sup\{s \mid s \in \text{supp}(\mu)\}$$

and

$$S \leq T_- \iff \forall n \in \mathbb{N} \int_{\mathbb{R}} (x^2 - S^2)^n d\mu(x) \geq 0$$
If \( S \geq \sup \{ s \mid s \in \text{supp}(\mu) \} \), then, for the integrand to be positive, we need that
\[
(S + y)^n(S - y)^m + (S + y)^m(S - y)^n \geq 0
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for all \( y \geq 0 \) in \( \text{supp}(\mu) \).
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On the other hand, if $\mu \prec b_S$, we have that

$$\int x^{2n} \, d\mu(x) \leq S^{2N}$$

so, taking $2N$th roots and then $N \to \infty$, we see that $S \geq \sup\{s \mid s \in \text{supp}(\mu)\}$ which proves the formula for $T_+$. 
Lemma Let $\mu$ be a positive measure on an interval $I \subset \mathbb{R}$ (either open or closed at each endpoint). Let $f, g \in L^2(d\mu)$ and suppose that $g$ is monotone increasing on $I$ and there is $c \in I$ so that $f(x) \leq 0$ (resp $f(x) \geq 0$) if $x \leq c$ (resp $x \geq c$). Then
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$$\int f(x)g(x) \, d\mu(x) \geq g(c) \int f(x) \, d\mu(x)$$

**Proof** The function $f(x)[g(x) - g(c)]$ is positive so its integral is positive which is the claim.
The Proof: Reduction of Lower Bound to 

\[ m = n \]

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we see that the integral in question is

$$\frac{1}{2} \int (x^2 - S^2)^n \left[ (x + S)^{m-n} + (x - S)^{m-n} \right] \, d\mu(x)$$

$$= \int (x^2 - S^2)^n \left[ (x + S)^{m-n} + (x - S)^{m-n} \right] \, d\tilde{\mu}(x)$$
By the binomial theorem, the polynomial
\[ Q_{2k}(y) = (y + S)^{2k} + (y - S)^{2k} \]
only has even degree terms with only positive coefficients so the function in \([ \cdot ]\) in the last equation is monotone on \(I = [0, \infty)\). Applying the lemma with \(c = S\), we see that
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Thus, we have shown that

$$b_S \prec \mu \iff \forall n \text{ odd } \int_{\mathbb{R}} (x^2 - S^2)^n d\mu(x) \geq 0$$
The Proof: $T_+ > 0$

First, pick $a > 0$ so that $\mu([a, \infty)) > 0$. 
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$$\int (x^2 - b^2)^{2k+1} d\mu(x) \geq -(b^2)^{2k+1} + 2(a^2 - b^2)^{2k+1} \mu([a, \infty))$$
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by the choice of $b$. Thus $T_\geq b > 0$. 
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\[
f(z_0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z - z_0} \, dz
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