The Work of Daniel Wells, Forty Years Late

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$$\langle f\rangle_{\mu,\Lambda} = Z^{-1}\langle f e^{-H}\rangle_{\mu,0}; \qquad Z = \langle e^{-H}\rangle_{\mu,0}$$

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$$\langle \sigma^B \rangle_{\mu,\Lambda} \leq \langle \sigma^B \rangle_{\nu,\Lambda}$$

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The left hand side is an Ising expectation and the right with the apriori measure of the $2D$ rotor with only couplings of the 1 components. So this was part of what seems to be an Ising domination result (the 2 indicates the Ising measure should really be $b_{1/\sqrt{2}}$).

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A *Ginibre system* is a triple $\langle X, \mu, \mathcal{F} \rangle$ of a compact Hausdorff space, X, a probability measure, μ, on X (with expectations $\langle \cdot \rangle_\mu$) and a class of continuous real valued functions $\mathcal{F} \subset C(X)$ that obeys:

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Note that

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We will see shortly that $(G2) \Rightarrow (G1)$

Extending Ginibre Systems

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It is trivial that $(G2)$ holds for sums and positive multiples of functions for which it holds, so it suffices to prove it holds for products. By induction, we need only handle products of two functions. We note that

$$fg \pm f'g' = \tfrac{1}{2}(f + f')(g \pm g') + \tfrac{1}{2}(f - f')(g \mp g')$$

Extending Ginibre Systems

What makes the notion so powerful is that there are three theorems for getting new Ginibre systems from old ones.

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which allows us to prove $(G2)$ for a single product when we have it for individual functions (and shows (G2)⇒(G1)).

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Ginibre Theorem 2 *Let* $\{\langle X_j, \mu_j, \mathcal{F}_j \rangle\}_{j=1}^n$ *be a family of Ginibre systems. Then* $\langle \times_{j=1}^n X_j, \otimes_{j=1}^n \mu_j, \cup_{j=1}^n \mathcal{F}_j \rangle$ *is also a Ginibre system*

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The proof is easy. The normalization is irrelevant and we expand the exponential $\exp(-H(x) - H(y))$.

Classical Ising System

Ginibre Theorem 4 *Let* X *be* $\mathbb{R}$ *or a compact subset of the form* $[-A, A]$ *and let* $d\mu$ *be a probability measure which is invariant under* $x \mapsto -x$ *and so that (only non-trivial in case* X *is not compact)* $\int x^{2n}\, d\mu(x) < \infty$ *for all* n*. Let* $\mathcal{F}$ *contain the single function,* $f(x) = x$*. Then* $\langle X, \mu, \mathcal{F} \rangle$ *is a Ginibre system.*

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Interchanging x and y implies the integral is zero if m is odd and $x \mapsto -x$ symmetry implies the integral is zero if $m+k$ is odd. Thus the only possible non-zero integrals are when m and k are even in which case the integrand is positive!

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$$-H = \sum_{A \subset \Lambda} J(A)\sigma^A \qquad \sigma^A = \prod_{j \in A} \sigma_j$$

with ANY (!!!) even apriori measure, one has positive expectations and positive correlations of the σ^A.

Final Ginibre Thoughts

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The first is to note that he proves that if $d\mu$ is a product of rotation invariant measures on circles, the set of functions $\cos(\sum_{j=1}^{n} m_j\theta_j)$ is a Ginibre system. This and some extensions are essentially half the correlation inequalities for plane rotors.

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Basic Definition

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$$\int \int (f_1(x) \pm f_1(y)) \ldots (f_n(x) \pm f_n(y)) d\mu(x) d\nu(y) \geq 0$$

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for all non-negative integers, n and m in which case we use the symbol $\triangleleft$ without being explicit about $\mathcal{F}$. Since the measures are even, one need only check this when $n+m$ is even. It is trivial if both are even, so we only need worry about the case that both are odd. Since the measures are different, we don't have the exchange symmetry that makes the integral vanish if both are odd but symmetry under $y \mapsto -y$ implies invariance under interchange of m and n, so we need only check for $m \geq n$. We'll see examples later.

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(d) *If $\mu \lhd \nu$ with respect to a set of functions $\mathcal{F}$, then for every $f \in \mathcal{F}$, we have that*

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Corollary *If for $j = 1, \dots, n$, $\mu_j \triangleleft \nu_j$ for probability measures on spaces X_j with respect to sets of functions $\mathcal{F}_j$ on X_j, then if $-H \in \mathcal{C}(\cup_{j=1}^n \mathcal{F}_j)$ and if μ_H, ν_H are formed from the underlying product measures $\otimes_{j=1}^n \mu_j$ and $\otimes_{j=1}^n \nu_j$, then for all $F \in \mathcal{C}(\cup_{j=1}^n \mathcal{F}_j)$, one has that $\int f(x)\, d\mu_H(x) \le \int f(x)\, d\nu_H(x)$. In particular, if each $X_j = \mathbb{R}$, (so implicitly F_j is the single function σ_j) and if H has the general Ising form, then for all $A \subset 2^{\{1,\dots,n\}}$ one has that*

$$\langle \sigma^A \rangle_{\mu_H} \le \langle \sigma^A \rangle_{\nu_H}$$

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Since Ising domination is trivially transitive, for applications, this lack isn't so important.

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Big Theorem *Let* $d\mu$ *be an even probability measure on* $\mathbb{R}$ *with compact support that is not a point mass at* 0. *Then there are two strictly positive numbers* $T_-(\mu)$ *and* $T_+(\mu)$ *so that* $\mu \triangleleft b_S$ *if and only if* $S \geq T_+$ *and* $b_S \triangleleft \mu$ *if and only if* $S \leq T_-$. *Moreover*

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Big Theorem *Let $d\mu$ be an even probability measure on $\mathbb{R}$ with compact support that is not a point mass at 0. Then there are two strictly positive numbers $T_-(\mu)$ and $T_+(\mu)$ so that $\mu \lhd b_S$ if and only if $S \geq T_+$ and $b_S \lhd \mu$ if and only if $S \leq T_-$. Moreover*

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$$S \leq T_- \iff \forall_{n\in\mathbb{N}} \int_{\mathbb{R}} (x^2 - S^2)^n \, d\mu(x) \geq 0$$

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One consequence of the theorem is

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It is an interesting question when one has equality. Before leaving this theorem, I should mention I happened to look at a 1981 paper of Bricmont, Lebowitz and Pfister that includes in an appendix a proof (with attribution to Wells) of Wells result about the existence of $T_- > 0$.

Three Spin Values

For $0 \leq \lambda \leq 1$, consider the probability measure supported by the three points $\{0, \pm 1\}$ given by

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$$T_{-}(\lambda) = \begin{cases} \sqrt{\lambda}, & \text{if } \lambda \leq \frac{1}{2} \\ \sqrt{\frac{1}{2}}, & \text{if } \lambda \geq \frac{1}{2} \end{cases}$$

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So we see there are cases where $T_- = \langle x^2 \rangle^{1/2}$ and other cases where the inequality is strict. Note also that at $\lambda = 1/2$, the integral $\langle (x^2 - T_-^2)^{2m+1} \rangle_\lambda$ vanishes for all n, a sign that the distribution of $x^2 - T_-^2$ is symmetric about 0.

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It could even happen that there is Wells domination. It would even be interesting to know that $\tilde{\mu}_S$ Ising dominates normalized Lebesgue measure on $[-1, 1]$.

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While this notion is useful, it has one nearly fatal flaw (that comes from the strength of the conclusion - all of $\mathcal{M}$ rather than just linear functions) one has that

$$b_T \prec \mu \text{ for some } T > 0 \Rightarrow \mu(([0,T)) = 0$$

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Wells' Framework
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Examples
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The Statement

Recall the big theorem

Big Theorem *Let* $d\mu$ *be an even probability measure on* $\mathbb{R}$ *with compact support that is not a point mass at* 0. *Then there are two strictly positive numbers* $T_-(\mu)$ *and* $T_+(\mu)$ *so that* $\mu \triangleleft b_S$ *if and only if* $S \geq T_+$ *and* $b_S \triangleleft \mu$ *if and only if* $S \leq T_-$. *Moreover*

$$T_+ = \sup\{s \mid s \in \operatorname{supp}(\mu)\}$$

and

$$S \leq T_- \iff \forall_{n \in \mathbb{N}} \int_{\mathbb{R}} (x^2 - S^2)^n \, d\mu(x) \geq 0$$

The Proof: T_+

If $S \geq \sup\{s \mid s \in \operatorname{supp}(\mu)\}$, then, for the integrand to be positive, we need that $(S+y)^n(S-y)^m + (S+y)^m(S-y)^n \geq 0$ for all $y \geq 0$ in $\operatorname{supp}(\mu)$.

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On the other hand, if $\mu \triangleleft b_S$, we have that $\int x^{2n}\, d\mu(x) \leq S^{2N}$, so, taking $2N$th roots and then $N \to \infty$, we see that $S \geq \sup\{s \mid s \in \text{supp}(\mu)\}$ which proves the formula for T_+.

The Proof: Preliminary Lemma

Lemma Let μ be a positive measure on an interval $I \subset \mathbb{R}$ (ether open or closed at each endpoint). Let $f, g \in L^2(d\mu)$ and suppose that g is monotone increasing on I and there is $c \in I$ so that $f(x) \leq 0$ (resp $f(x) \geq 0$) if $x \leq c$ (resp $x \geq c$). Then

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Proof The function $f(x)[g(x) - g(c)]$ is positive so its integral is positive which is the claim.

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$$\frac{1}{2} \int (x^2 - S^2)^n \left[(x + S)^{m-n} + (x - S)^{m-n} \right] \, d\mu(x)$$

$$= \int (x^2 - S^2)^n \left[(x + S)^{m-n} + (x - S)^{m-n} \right] \, d\tilde{\mu}(x)$$

The Proof: Reduction of Lower Bound to $m = n$

By the binomial theorem, the polynomial $Q_{2k}(y) = (y+S)^{2k} + (y-S)^{2k}$ only has even degree terms with only positive coefficients so the function in $[\cdot]$ in the last equation is monotone on $I = [0, \infty)$. Applying the lemma with $c = S$, we see that

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$$\int_{\mathbb{R}} \int_{\mathbb{R}} (x+y)^n (x-y)^m d\mu(x) d\nu(y) \geq (2S)^{m-n} \int_{\mathbb{R}} (x^2 - S^2)^n \, d\mu(x$$

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Thus, we have shown that

$$b_S \lhd \mu \iff \forall_{n \text{ odd}} \int_{\mathbb{R}} (x^2 - S^2)^n \, d\mu(x) \geq 0$$

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$$\int (x^2 - b^2)^{2k+1} d\mu(x) \geq -(b^2)^{2k+1} + 2(a^2 - b^2)^{2k+1} \mu([a, \infty))$$

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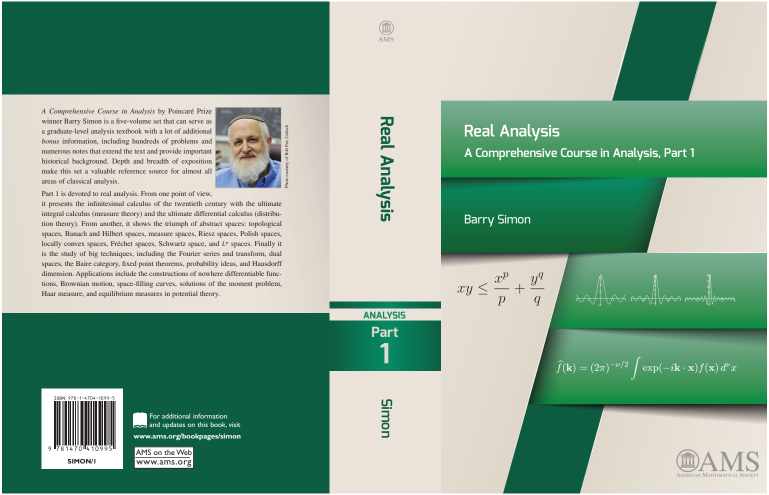

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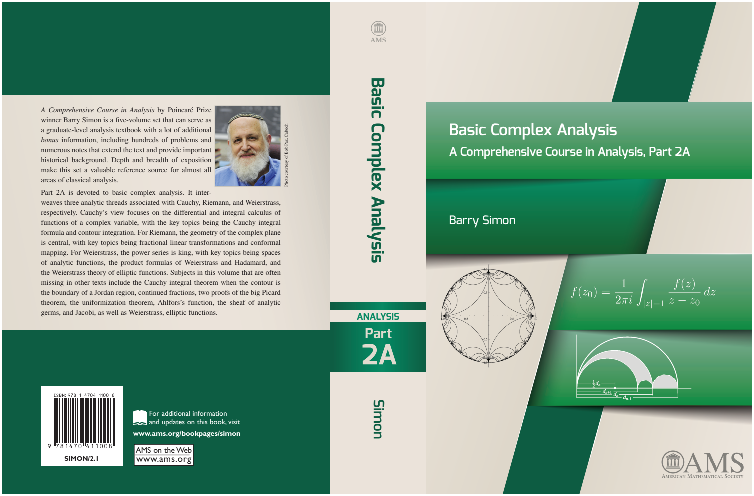

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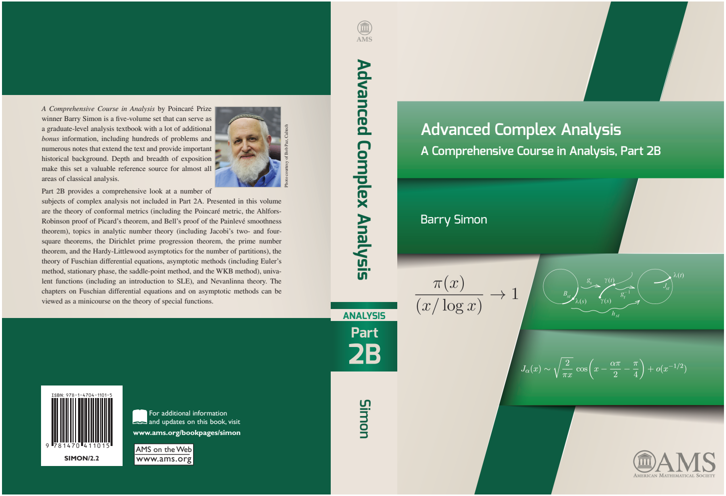

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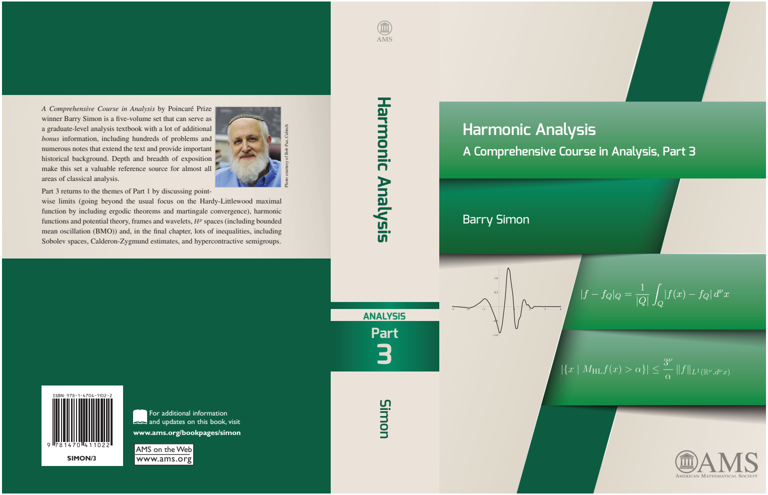

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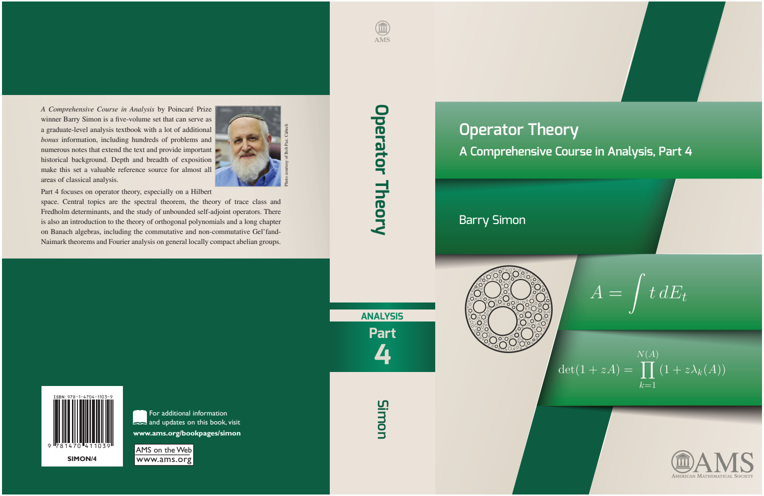

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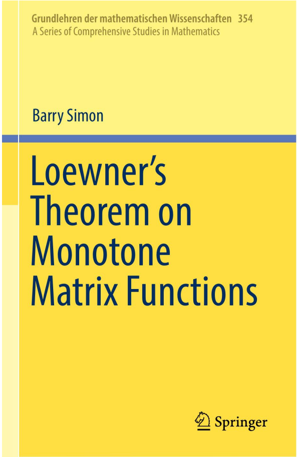

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