A CELEBRATION OF JÜRG AND TOM

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It is both a pleasure and an honor to write the introduction of this issue in honor of the (recent) sixtieth birthdays of Jürg Fröhlich and Tom Spencer and to be able to place their joint work in some perspective.

Tom and Jürg have about twenty-five joint papers, several with additional authors including two with me. I want to focus here on two sets of methods: infrared bounds and multiscale analysis, which are surely among the most significant developments in rigorous statistical physics in the last quarter of the last century.

Infrared bounds ([29, 30]), discovered in 1975 and proven using reflection positivity, provide upper bounds on the Fourier transform of the spin-spin correlation at nonzero momentum and force a macroscopic occupation of zero momentum at low temperature (aka Bose–Einstein condensation of spin waves). This implies long-range order, and so, a phase transition.

The method was used for quantum spin antiferromagnets by Dyson–Lieb–Simon [20, 21]. Remarkably, after more than thirty years, it remains the only method known to rigorously prove breaking of nonabelian symmetry—even for the abelian case, there is only one other approach to the short-range case using multiscale analysis (see below). For slow decay two-dimensional plane rotors, there are also results of Kunz–Pfister [49].

Among later applications of infrared bounds are Sokal's specific heat bounds [59], Aizenman's [1] and Fröhlich's [24] proofs of the triviality of ϕ^4 theories in five or more dimensions, the Aizenman–Fernández analysis of long-range models [4], Helffer's estimates of eigenvalue splitting for certain Schrödinger operators in the thermodynamic limit [43, 44], and the work of Biskup–Chayes on mean-field driven phase transitions [7, 8].

Reflection positivity was introduced in Euclidean field theory by Osterwalder–Schrader [54] and was a key element, albeit implicitly,

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in the work of FSS [29, 30]. This was a partial motivation for a series of works on chessboard estimates, many on phase transitions; for example, [9, 10, 11, 17, 22, 25, 26, 27, 50, 51, 52].

This material is further discussed in the review article of Shlosman [55] and in four books [42, 39, 23, 57].

We turn next to multiscale analysis. It is, of course, ancient wisdom to study infinite volume systems by approximating by finite volumes, say, cubes of length L. Before the Fröhlich–Spencer breakthrough, the sequence of L's one looks at were typically $L_{k+1} = L_k + 1$, or if one were especially brave, $L_{k+1} = 2L_k$. Multiscale analysis looks at $L_{k+1} = (L_k)^\alpha$ for some $\alpha > 1$.

At any level, there are typically good boxes where one gets estimates of a convenient form and bad boxes where estimates are much weaker. One decomposes L_{k+1} boxes into $(L_k)^{\nu(\alpha-1)}$ ($\nu =$ dimension) boxes of side L_k. If most of these smaller boxes are good, L_{k+1} is good. In this way, one inductively gets estimates on good boxes, proving that as $k \to \infty$, all boxes but a vanishingly small number are good.

What is particularly fascinating about the situation is that the underlying physics is either scaleless or has a single scale, but the mathematical machinery uses these multiple scales.

The breakthrough appeared first in 1981 in their proof of the Kosterlitz–Thouless transition [31, 32] and was later used by Jürg and Tom to analyze the phase transition in the one-dimensional Ising model with $1/r^2$ interaction energy [33], the deconfinement transition in four-dimensional $U(1)$ gauge theory [34], the phase transition in the three- or higher-dimensional plane rotor model (obtained already by FSS) [35], and localized states for certain quasi-periodic Schrödinger operators describing a particle hopping on the integers or moving on the real line [38]. Of course, these papers also used other clever ideas and techniques. Among precursors, one should certainly single out the paper of Glimm–Jaffe [41].

With Wayne [37], Jürg and Tom applied multiscale analysis to construct invariant tori in some Hamiltonian systems with infinitely many degrees of freedom; for later work in this direction, see [12, 13]. Except for localization, most of the results obtained by multiscale analysis have not been obtained by other methods. For localization, there is another approach found ten years later: the fractional moment method of Aizenman–Molchanov [5, 2, 3, 6].

In [36], Fröhlich and Spencer proved exponential decay of the Green's function in the spectrum for the Anderson model and left open the expected existence of point spectrum with exponentially decaying eigenfunctions. This was supplied by [28]. Shortly thereafter, the now standard way of going from Green's function decay to point spectrum was found by Simon–Wolff [56]. Further important localization criteria go under the names of dynamical localization and semi-uniform localization of eigenfunctions (SULE)—these can be tracked down through the reviews mentioned below.

Critical reworkings of multiscale analysis were developed by von Dreifus in his thesis [18] and with Klein [19]. Among extensions to settings beyond the lattice models that [36] consider, I would mention [14, 15, 16, 40, 46, 48, 53]; see the review articles mentioned next for more.

For book or book-length presentations of multiscale analysis for the Anderson model, see Stollmann [58] and Kirsch [45]. For a comprehensive review, see Klein [47].

Jürg and Tom: you have provided the seed-corn for a generation of mathematical physicists. So thanks and many happy returns.

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