RANK ONE PERTURBATIONS AT INFINITE COUPLING

F. Gesztesy[1] and B. Simon[2]

Abstract. We discuss rank one perturbations $A_\alpha = A+\alpha(\varphi,\cdot)\varphi, \alpha \in \mathbb{R}, A \geq 0$ self-adjoint. Let $d\mu_\alpha(x)$ be the spectral measure defined by $(\varphi,(A_\alpha - z)^{-1}\varphi) = \int d\mu_\alpha(x)/(x-z)$. We prove there is a measure $d\rho_\infty$ which is the weak limit of $(1+\alpha^2)\,d\mu_\alpha(x)$ as $\alpha \to \infty$. If φ is cyclic for A, then A_∞, the strong resolvent limit of A_α, is unitarily equivalent to multiplication by x on $L^2(\mathbb{R}, d\rho_\infty)$. This generalizes results known for boundary condition dependence of Sturm-Liouville operators on half-lines to the abstract rank one case.

§1. Introduction

This paper is a contribution to the theory of rank one perturbations which in its natural format involves a self-adjoint operator, $A \geq 0$ in a complex separable Hilbert space $\mathcal{H}$, and a vector, $\varphi \in \mathcal{H}_{-1}(A)$, with $\mathcal{H}_s(A)$ the scale of spaces associated to A. Then $q_\varphi(\psi,\eta) = (\psi,\varphi)(\varphi,\eta)$ defines a quadratic form on $\mathcal{H}_{+1}(A)$ with q_φ a form-bounded perturbation of A with relative bound zero. Accordingly, $A_\alpha \equiv A + \alpha(\varphi,\cdot)\varphi, \alpha \in \mathbb{R}$ defines a self-adjoint operator with $\mathcal{H}_s(A_\alpha) = \mathcal{H}_s(A)$ for $|s| \leq 1$.

We will suppose that φ is cyclic for A in which case it is easy to see that φ is also cyclic for each A_α. If $d\mu_\alpha$ is the spectral measure for φ associated to A_α, then A_α is unitarily equivalent to multiplication by x on $L^2(\mathbb{R}, d\mu_\alpha)$. Define

$$F_\alpha(z) = \int_{\mathbb{R}} \frac{d\mu_\alpha(x)}{x-z}$$

where $\varphi \in \mathcal{H}_{-1}(A_\alpha)$ implies that

$$\int_{\mathbb{R}} \frac{d\mu_\alpha(x)}{|x|+1} < \infty$$

[1]Department of Mathematics, University of Missouri, Columbia, MO 65211. E-mail: mathfg@mizzou1.missouri.edu

[2]Division of Physics, Mathematics and Astronomy, California Institute of Technology, 253-37, Pasadena, CA 91125. This material is based upon work supported by the National Science Foundation under Grant No. DMS-9101715. The Government has certain rights in this material.

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so that the integral defining F converges. One has the basic formula (with $F(z) \equiv F_{\alpha=0}(z)$)

$$F_\alpha(z) = \frac{F(z)}{1+\alpha F(z)}. \tag{1}$$

We are interested here in the case $\alpha = \infty$. By the monotone convergence theorem for forms ([3,6]), we have that $\operatorname{s-lim}_{\alpha\to\infty}(A_\alpha - z)^{-1}$ exists (the existence also follows from the explicit formula for $(A_\alpha - z)^{-1}$, eq. (6) below) and can be described as follows. Let

$$\mathcal{H}_{+1}(A_\infty) = \{\psi \in \mathcal{H}_{+1} \mid (\varphi, \psi) = 0\}$$

and $\mathcal{H}(A_\infty) = \overline{\mathcal{H}_{+1}(A_\infty)}$. This is all of $\mathcal{H}$ if $\varphi \notin \mathcal{H}$ and a codimension one subspace if $\varphi \in \mathcal{H}$. Let A_∞ be the self-adjoint operator on $\mathcal{H}(A_\infty)$ defined by the closed quadratic form $\psi, \eta \mapsto (\psi, A\eta)$ on $\mathcal{H}_{+1}(A_\infty)$. If $\mathcal{H}(A_\infty) \neq \mathcal{H}$, extend $(A_\infty - z)^{-1}$ to all of $\mathcal{H}$ by setting it zero on $\mathcal{H}(A_\infty)^\perp$. Then $\operatorname{s-lim}(A_\alpha - z)^{-1} = (A_\infty - z)^{-1}$.

By (1), $d\mu_\alpha(x) \to 0$ weakly as $\alpha \to \infty$, so we do not have any obvious spectral measure of A_∞. Our main goal here is to prove that $(1+\alpha^2)\, d\mu_\alpha$ does have a weak limit as $\alpha \to \infty$ which is the spectral measure for a vector $\eta \in \mathcal{H}_{-2}(A_\infty)$. Explicitly, define

$$d\rho_\alpha(x) = (1+\alpha^2)\, d\mu_\alpha(x). \tag{2}$$

Then we will prove that

Theorem 1. *There exists a vector, $\eta \in \mathcal{H}_{-2}(A_\infty)$, cyclic for A_∞ so that if $d\rho_\infty(x)$ is the spectral measure for η with respect to A_∞, then*

$$\int_{\mathbb{R}} f(x)\, d\rho_\alpha(x) \to \int_{\mathbb{R}} f(x)\, d\rho_\infty(x) \tag{3}$$

for all continuous functions, f, of compact support.

Note that since $\eta \in \mathcal{H}_{-2}(A_\infty)$,

$$\int_{\mathbb{R}} \frac{d\rho_\infty(x)}{(|x|+1)^2} < \infty. \tag{4}$$

It may be that (4) fails if $(|x|+1)^{-2}$ is replaced by $(|x|+1)^{-1}$. We will see explicit examples in §5 where the integral diverges for $(|x|+1)^{-2+\epsilon}$. The proof will show that (3) holds if $f(x) = (|x|+1)^{-\alpha}$ with $\alpha > 2$. There will be examples when it fails if $\alpha = 2$.

Another major result we'll prove is that

$$d\rho_\infty(x) = \lim_{\epsilon\downarrow 0} \pi^{-1}\big[\operatorname{Im}((-F(x+i\epsilon))^{-1})\, dx\big].$$

The abstract theory appears in §2. We discuss boundary condition dependence of Schrödinger operators on the half-line in §3. In that case, $d\rho_\alpha$ is the Weyl spectral measure and $d\rho_\infty$ is the Dirichlet spectral measure. In §4, we consider the case when A is bounded. In §5, we discuss a further example.

§2. The Main Results

We begin by recalling some of the standard formulae for rank one perturbations [7]:

$$F_\alpha(z) = F(z)/[1+\alpha F(z)],$$

$$(A_\alpha - z)^{-1}\varphi = (1+\alpha F(z))^{-1}(A-z)^{-1}\varphi, \tag{5}$$

$$(A_\alpha - z)^{-1} = (A-z)^{-1} - \frac{\alpha}{1+\alpha F(z)}\,((A-\bar z)^{-1}\varphi,\cdot)(A-z)^{-1}\varphi, \tag{6}$$

$$\mathrm{Tr}[(A-z)^{-1} - (A_\alpha - z)^{-1}] = \int_{E_\alpha}^{\infty} (\lambda - z)^{-2}\xi_\alpha(\lambda)\,d\lambda, \quad E_\alpha = \min(0, \inf\,\mathrm{spec}(A_\alpha)),$$

where ξ_α is the Krein spectral shift [4] given by

$$\xi_\alpha(x) = \frac{1}{\pi}\,\mathrm{Arg}(1+\alpha F(x+i0)). \tag{7}$$

For $\alpha > 0$ we have $\mathrm{Arg}(\cdot) \in [0,\pi]$ and hence $0 \le \xi_\alpha \le 1$ in this case.

If $\|\varphi\| = \infty$, let $P = 0$, and if $\|\varphi\| < \infty$, let P be the projection onto $\{c\varphi \mid c \in \mathbb{C}\}$. Thus, $\mathcal{H}(A_\infty) = \mathrm{Ran}(1-P)$.

Proposition 2. *There exists $\eta \in \mathcal{H}_{-2}(A_\infty)$ so that for all $z \in \mathbb{C}$:*

$$(A_\infty - z)^{-1}\eta = \lim_{\alpha\to\infty} \alpha(1-P)(A_\alpha - z)^{-1}\varphi. \tag{8}$$

If φ is cyclic for A, then η is cyclic for A_∞.

Proof. By (5), the limit on the right side of (8) exists, call it $\psi(z)$, and is given by

$$\psi(z) = F(z)^{-1}(1-P)(A-z)^{-1}\varphi. \tag{9}$$

We have that

$$(A_\alpha - z)^{-1}\varphi - (A_\alpha - w)^{-1}\varphi = (z-w)(A_\alpha - z)^{-1}(A_\alpha - w)^{-1}\varphi. \tag{10}$$

Multiply by α, take $\alpha \to \infty$, and note that if $\|\varphi\| < \infty$, then $P(A_\infty - w)^{-1}\varphi = 0$. We conclude that

$$\psi(z) - \psi(w) = (z-w)(A_\infty - z)^{-1}\psi(w)$$

or

$$\psi(z) = [1 + (z-w)(A_\infty - z)^{-1}]\psi(w). \tag{11}$$

Note that $\psi(z) \in \mathcal{H}(A_\infty)$ (because of the $1-P$) so we can define $\eta(z) \equiv (A_\infty - z)\psi(z)$ in $\mathcal{H}_{-2}(A_\infty)$. (11) precisely says that $\eta(z) = \eta(w)$, that is, it is independent of z; call it η. Cyclicity follows from (9) since if $\{(A-z)^{-1}\varphi\}$ is total in $\mathcal{H}$, then clearly $\{(1-P)(A-z)^{-1}\varphi\}$ is total in $(1-P)\mathcal{H} = \mathcal{H}(A_\infty)$.

Remark. In §4, we'll prove that when A is bounded, then $\eta = -(1-P)A\varphi$.

Theorem 3. *Let $d\rho_\infty$ be the spectral measure for η. Then*

$$\int_{\mathbb{R}} \frac{d\rho_\infty(x)}{(x-z)^2} = F(z)^{-2}\,\frac{dF}{dz} - \frac{1}{\|\varphi\|^2}. \tag{12}$$

Proof. For simplicity, suppose z is real and negative. By definition of η:

$$\begin{aligned}\int_{\mathbb{R}} \frac{d\rho_\infty(x)}{(x-z)^2} &\equiv (\eta, (A_\infty - z)^{-2}\eta) \\ &= \big(\varphi, (A-z)^{-1}(1-P)(A-z)^{-1}\varphi\big)/F(z)^2 \\ &= \Big[(\varphi, (A-z)^{-2}\varphi) - \frac{1}{\|\varphi\|^2}\,\langle\varphi, (A-z)^{-1}\varphi\rangle^2\Big]/F(z)^2\end{aligned}$$

since $P = \|\varphi\|^{-2}(\varphi,\cdot)\varphi$. But this is precisely the right side of (12). (12) for general z follows by analyticity.

Recall that $d\rho_\alpha$ is defined by (2). Then

Theorem 4. (i)

$$\lim_{\alpha\to\infty} \int_{\mathbb{R}} \frac{d\rho_\alpha(x)}{(x-z)^2} = \frac{1}{\|\varphi\|^2} + \int_{\mathbb{R}} \frac{d\rho_\infty(x)}{(x-z)^2}.$$

(ii)

$$\lim_{\alpha\to\infty} \int_{\mathbb{R}} \frac{d\rho_\alpha(x)}{(x-z)^3} = \int_{\mathbb{R}} \frac{d\rho_\infty(x)}{(x-z)^3}.$$

(iii) *For any continuous f of compact support*

$$\lim_{\alpha\to\infty} \int_{\mathbb{R}} f(x)\,d\rho_\alpha(x) = \int_{\mathbb{R}} f(x)\,d\rho_\infty(x).$$

Proof. (ii) implies (iii) by a Stone-Weierstrass type argument. (i) implies (ii) by using the fact that both sides are analytic in z on $\mathbb{C}\backslash\mathbb{R}$ so their derivatives in z converge. To prove (i), use Theorem 3 and the calculation

$$\begin{aligned}\int_{\mathbb{R}} \frac{d\rho_\alpha(x)}{(x-z)^2} &= (1+\alpha^2)(\varphi, (A_\alpha - z)^{-2}\varphi) \\ &= \frac{(1+\alpha^2)}{(1+\alpha F)^2}\,(\varphi, (A-z)^{-2}\varphi) \\ &= \frac{1+\alpha^2}{(1+\alpha F)^2}\,\frac{dF}{dz}.\end{aligned} \tag{13}$$

(13) follows from (5).

Theorem 5.

$$d\rho_\infty(x) = \pi^{-1} \lim_{\epsilon \downarrow 0} \operatorname{Im}\biggl[-\frac{1}{F(x+i\epsilon)}\biggr] dx.$$

Proof. We start with (12) and integrate, noting that $F'/F^2 = \frac{d}{dz}(-1/F)$ to get

$$\int_{\mathbb{R}} d\rho_\infty(x)\biggl(\frac{1}{x-z} - \frac{1}{x+1}\biggr) = -\frac{1}{F(z)} + \frac{1}{F(-1)} - (z+1)\,\frac{1}{\|\varphi\|^2}.$$

The theorem then follows by the standard relations between a measure and the boundary values of its Borel transform.

§3. Variation of Boundary Condition

As an example of the general theory, we consider the case of boundary conditions variation for Schrödinger operators on $L^2(0,\infty)$. The formulae that result are well-known (see, e.g., [1,2,5,8]). The point is that they fit into a more general framework. Let V be continuous and bounded below on $[0,\infty)$. Let H_θ be the operator on $L^2([0,\infty), dx)$ formally given by $-\frac{d^2}{dx^2} + V(x)$ with $u(0)\cos\theta + u'(\theta)\sin\theta = 0$ boundary conditions. One defines the Weyl m-function, $m_\theta(z)$, and Weyl spectral measure, $d\rho_\theta(x)$, so that for $\theta \neq 0$:

$$m_\theta(z) = \cot(\theta) + \int_{\mathbb{R}} \frac{d\rho_\theta(x)}{x-z} \tag{14}$$

and $d\rho_\theta \to d\rho_{\theta=0}$ as $\theta \downarrow 0$. Moreover,

$$m_{\theta=0}(z) = -1/m_{\theta=\pi/2}(z). \tag{15}$$

For $\theta \neq 0$, the Green's function, $G_\theta(0,0;z)$ is related to $m_\theta(z)$ by

$$G_\theta(0,0;z) = \sin^2(\theta)[-\cot\theta + m_\theta(z)]. \tag{16}$$

This fits into the general framework by taking $A = H_{\theta=\pi/2}$ and $\varphi = \delta_0$, the delta function at 0. Then for $\theta \neq 0$,

$$H_\theta = A - \cot(\theta)(\varphi,\cdot)\varphi$$

and $F_{-\cot(\theta)}(z) = G_\theta(0,0;z)$. By (14) and (16), $d\rho_\theta$ is just $(1+\alpha^2)\,d\mu_\alpha$ where $\alpha = -\cot\theta$ and $\lim_{\theta\to 0} d\rho_\theta = d\rho_0$ is just what we found in the last section. (15) is just Theorem 5.

We want to identify the vector η. Let $\psi_+(x,z)$ be the solution of $(-\frac{d^2}{dx^2} + V(x) - z)\psi = 0$ which is L^2 at infinity normalized any way that is convenient. Then from the Wronskian formula for $G_{\theta=\pi/2}$, we get

$$((A-z)^{-1}\varphi)(x) = \frac{\psi_+(x,z)}{\psi'_+(0,z)}$$

and

$$F(z) = -\psi_+(0,z)/\psi'_+(0,z).$$

It follows that

$$F(z)^{-1}(A-z)^{-1}\varphi = \psi_+(x,z)/\psi_+(0,z)$$

which, by the Wronskian formula for $G_{\theta=0}$, is just

$$(A_\infty - z)^{-1}\delta'(x),$$

that is, η is δ' (note that $P=0$ in this case) and $d\rho_\infty$ is the spectral function for the vector δ'.

We note that it is well known that $\int_0^\mu d\rho_\infty(x) \sim C\mu^{3/2}$ as $\mu \to \infty$ so that $\int_0^\infty \frac{d\rho_\infty(\lambda)}{(1+|\lambda|)^k} < \infty$ if and only if $k > \frac{3}{2}$. In particular, $\eta \notin \mathcal{H}_{-1}(A_\infty)$.

§4. Bounded Operators

One gets insight into the general theory by considering the case where A is bounded. Since $\|\varphi\|$ is then finite, we'll suppose $\|\varphi\| = 1$. We'll also get a better understanding of the $\frac{1}{\|\varphi\|^2}$ term in Theorem 4(i). We first note:

Theorem 6. *If A is bounded and $\|\varphi\| = 1$, then $\eta = -(1-P)A\varphi$.*

Proof. If A is bounded, then A_∞ is just $(1-P)A(1-P)$. Thus

$$\begin{aligned}\eta &= F(z)^{-1}(1-P)(A-z)(1-P)(A-z)^{-1}\varphi \\ &= F(z)^{-1}(1-P)(A-z)(A-z)^{-1}\varphi - F(z)^{-1}[(1-P)(A-z)\varphi]F(z).\end{aligned}$$

The first term is zero since $(1-P)\varphi = 0$. The second is $-(1-P)A\varphi$ since $(1-P)z\varphi = 0$.

Since $\|\varphi\| = 1$, $(\varphi, \cdot)\varphi$ is just a projection, P. Instead of $A + \alpha P$, look at

$$P + \alpha^{-1}A = B_\alpha.$$

P has an isolated simple eigenvalue at 1 with eigenvector φ. Thus by regular perturbation theory [3], B_α has the eigenvalue at $1 + (\varphi, A\varphi)\alpha^{-1} + O(\alpha^{-2})$ with eigenvector

$$\psi_\alpha = \varphi + \alpha^{-1}(1-P)A\varphi + O(\alpha^{-2}) = \varphi + \alpha^{-1}\eta + O(\alpha^{-2}).$$

The first order term is standard perturbation theory where the reduced resolvent $(H_0 - E)^{-1}(1-P)$ is just $-(1-P)$ since H_0 is P is 0 on $\operatorname{Ran}(1-P)$.

Thus, with respect to $A + \alpha P = \alpha B_\alpha$, the measure $(1+\alpha^2)\, d\mu_\alpha$ has a pole of weight $(1+\alpha^2)$ at $E_\alpha = \alpha + (\varphi, A\varphi) + O(\alpha^{-1})$ plus the spectral measure of η for the operator A_∞ plus an error of order α^{-1}. If $\nu > 2$, the pole at E_α makes no asymptotic contribution to $\int_{\mathbb{R}} \frac{d\rho_\alpha(x)}{|x-z|^\nu}$ as $\alpha \to \infty$ but for $\nu = 2$, it makes a contribution of $(1+\alpha^2)/E_\alpha^2 \to 1 = 1/\|\varphi\|^2$.

§5. A Further Example

Let $0 < \gamma < 1$. Let $d\mu_0(x) = \pi^{-1}|x|^{-\gamma}\sin(\pi\gamma)\,dx$ on $[0,\infty)$. Let A be multiplication by x on $L^2([0,\infty), d\mu_0(x))$ and $\varphi \equiv 1$. Then $\int_0^\infty \frac{d\mu_0(x)}{|x|+1} < \infty$ so $\varphi \in \mathcal{H}_{-1}(A_\infty)$.

$$F(z) = \int_0^\infty \frac{d\mu_0(x)}{x-z} = (-z)^{-\gamma}$$

(the easiest way to see this is to compute the imaginary part of $(-z)^{-\gamma}$ for $z = x + i\epsilon$ with $\epsilon \to 0$). Then, by Theorem 5,

$$d\rho_\infty(x) = \pi^{-1}|x|^{\gamma}\sin(\pi\gamma)\,dx.$$

It follows that $\int_0^\infty d\rho_\infty(x)/(|x|+1)^k < \infty$ only if $k > 1+\gamma$. Thus, we cannot conclude in general that $\int_0^\infty d\rho_\infty(x)/(|x|+1)^k < \infty$ for any $k < 2$.

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