ON THE DETERMINATION OF A POTENTIAL FROM THREE SPECTRA

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Dedicated to M.S. Birman on the occasion of his seventieth birthday

Abstract. We prove that under suitable circumstances, the spectra of a Schrödinger operator on the three intervals $[0,1]$, $[0,a]$, and $[a,1]$ for some $a \in (0,1)$ uniquely determine the potential q on $[0,1]$.

§1. Introduction

This is a paper in our series [2,4,5,6] on the use of Weyl-Titchmarsh m-function methods to obtain information on what spectral information uniquely determines the potential q in a one-dimensional Schrödinger operator $-\frac{d^2}{dx^2} + q$. Typical of our results is:

Theorem 1. *Fix $c, d \in \mathbb{R}$ with $c < d$ and $q \in L^1((c,d))$ real-valued. Let $S(c,d;q)$ denote the set of eigenvalues of $-\frac{d^2}{dx^2} + q$ on $L^2((c,d))$ with the boundary conditions $u(c) = u(d) = 0$. Suppose $q_1, q_2 \in L^1((0,1))$ are real-valued and there is some $a \in (0,1)$ so that*

(i) *$S(0,1;q_1) = S(0,1;q_2)$, $S(0,a;q_1) = S(0,a;q_2)$, and $S(a,1;q_1) = S(a,1;q_2)$.*
(ii) *The sets $S(0,1;q_1)$, $S(0,a;q_1)$, and $S(a,1;q_1)$ are pairwise disjoint.*

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Then $q_1 = q_2$ *a.e. on* $[0,1]$.

Our immediate motivation for this result is a recent preprint of Pivovarchik [13], who stated this result for $a = \frac{1}{2}$ without condition (ii). As we will see in Section 5, there are counterexamples if (ii) fails. After we pointed out the relevance of condition (ii) to Pivovarchik, he provided a corrected version. One of our goals here is to show that the methods of [2,5,6] provide a natural way to understand and extend this result.

There is a second motivation for this work. While not stated in this language, we actually considered a problem very close to a finite-difference analog of Theorem 1 in [5]. There we considered a tridiagonal Jacobi matrix in $\mathbb{C}^N$

$$A = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & a_{N-1} & b_N \end{pmatrix},$$

with $a_k > 0$, $k = 1, \dots, N-1$. Let $A^{[i,j]}$ be the submatrix of A obtained by keeping rows and columns $i, i+1, \dots, j-1, j$. In [5] we considered to what extent A is determined by $g(z,k)$, the kk matrix element of $(A-z)^{-1}$ (for all $z \in \mathbb{C} \backslash \operatorname{spec}(A)$). We found that generically there were $\binom{N-1}{k-1}$ possible A's consistent with a given $g(z,k)$.

The proof of this fact depends on the argument that looked at the eigenvalues of $A^{[1,k-1]}$ and $A^{[k+1,N]}$. The function $g(z,k)$ determined the union of these sets. Then $\binom{N-1}{k-1}$ possible values depended on the choice of which were actually eigenvalues of $A^{[1,k-1]}$ and which of $A^{[k+1,N]}$. If one a priori knows which are which (the hypothesis of Theorem 1), one has uniqueness.

The non-generic case in [5] occurs precisely when $A^{[1,k]}$ and $A^{[k+1,N]}$ share an eigenvalue, in which case there is a manifold of possible A's consistent with $g(z,k)$.

In a sense, Theorem 1 can be thought of as a continuum analog of a part of the result in [5].

We actually prove a more general result than Theorem 1. Let $h_c, h_d \in \mathbb{R} \cup \{\infty\}$. We let $H(c,d;h_c,h_d;q)$ be the operator $-\frac{d^2}{dx^2} + q$ on $L^2((c,d))$ with boundary conditions

$$u'(c) + h_c u(c) = 0, \quad u'(d) + h_d u(d) = 0,$$

where $h_{x_0} = \infty$ is a shorthand notation for the Dirichlet boundary condition at $x = x_0$ (i.e., $u(x_0) = 0$). Let $S(c,d;h_c,h_d;q)$ be the set of eigenvalues (i.e., the spectrum) of $H(c,d;h_c,h_d;q)$.

We will prove

Theorem 2. *Fix* $a \in (0,1)$ *and* $h_0, h_1, h_a \in \mathbb{R} \cup \{\infty\}$. *Suppose* $q_1, q_2 \in L^1((0,1))$ *are real-valued and*

(i) $S(0,1;h_0,h_1;q_1) = S(0,1;h_0,h_1;q_2)$, $S(0,a;h_0,h_a;q_1) = S(0,a;h_0,h_a;q_2)$, *and* $S(a,1;h_a,h_1;q_1) = S(a,1;h_a,h_1;q_2)$.

(ii) *The sets* $S(0,1;h_0,h_1;q_1)$, $S(0,a;h_0,h_a;q_k)$, *and* $S(a,1;h_a,h_1;q_k)$ *are pairwise disjoint.*

Then $q_1 = q_2$ *a.e. on* $[0,1]$.

Remark. The proof actually shows that not only is q determined by $S(0,1)$, $S(0,a;h_a)$, and $S(a,1;h_a)$, but so are h_0 and h_1.

The structure of this paper is as follows: In Section 2, we prove several results which illustrate when Green's functions are determined by zeros, poles, and residues. In Section 3, we prove Theorem 2 when $h_a = \infty$ (including Theorem 1); and in Section 4, we prove Theorem 2 when $|h_a| < \infty$. In Section 5, we discuss the case where condition (ii) fails. In Section 6, we consider some cases where q is defined on all of $\mathbb{R}$.

It is a great pleasure to dedicate this paper as a seventieth birthday present to M.S. Birman, whose work has long inspired us. In our use of Green's functions and analytic function theory, the reader will see echoes of his influence.

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§2. Some Uniqueness Theorems of Meromorphic Herglotz Functions

One could prove the basic result of this paper using the theorems in [2,6] on the determination of an entire function by its values on a set of suitable density. Instead we will use some alternative theorems that allow ready extension to q's on all of $\mathbb{R}$, a typical one being

Theorem 2.1. *Let* $0 < w_1 < z_1 < w_2 < z_2 < \cdots$ *with* $\lim_{n\to\infty} w_n = \infty$. *Then*

$$g(z) = \lim_{n\to\infty} \prod_{j=1}^{n} (1 - z/z_j) \Big/ \prod_{j=1}^{n} (1 - z/w_j)$$

exists for any z *in* $\mathbb{C}\backslash\{w_j\}_{j=1}^{\infty}$, *with convergence uniform on compact subsets of* $\mathbb{C}\backslash\{w_j\}_{j=1}^{\infty}$. $g(z)$ *is a meromorphic function with*

$$\frac{\mathrm{Im}\,(g(z))}{\mathrm{Im}\,(z)} > 0 \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R} \tag{2.1}$$

and hence a Herglotz function. Moreover, any meromorphic function $f(z)$ *satisfying* (2.1) *with zeros precisely at* $\{z_j\}_{j=1}^{\infty}$ *and poles precisely at* $\{w_j\}_{j=1}^{\infty}$ *is a positive multiple of* $g(z)$.

Remarks. 1. Theorems of this genre can be found in Levin [7].

2. This is a variant of the standard theorem on the convergence of alternating series.

3. One can easily accommodate situations where there are also zeros and poles alternating towards $-\infty$.

4. Any meromorphic Herglotz function (i.e., any meromorphic function satisfying (2.1)) can be seen to satisfy $f'(z) > 0$ away from its polar singularities, so its zeros and poles are simple, its zeros and poles alternate, and residues at poles are negative. Thus Theorem 2.1 describes all meromorphic Herglotz functions which are positive on $(-\infty, w_1)$ for some $w_1 > 0$.

Proof. Let $g_N(z) = \prod_{j=1}^{N}(1 - z/z_j)/\prod_{j=1}^{N+1}(1 - z/w_j)$. Then g_N has simple poles at $w_1, w_2, \dots, w_{N+1}$ and because of the alternating nature of the z_j's and w_j's, each residue is negative. Since $g_N(z) \to 0$ as $|z| \to \infty$, it follows that $g_N(z) = \sum_{j=1}^{N+1} \frac{\alpha_j^{(N)}}{w_j - z}$ with $\alpha_j^{(N)} > 0$, $j = 1, \dots, N+1$.

Thus, each g_N is a Herglotz function and so g_N maps $\mathbb{C}\backslash[0,\infty)$ to $\mathbb{C}\backslash(-\infty,0]$. Let H be a biholomorphic map of $\mathbb{C}\backslash(-\infty,0]$ to the open unit disk (e.g., $H(w) = \frac{\sqrt{w}-1}{\sqrt{w}+1}$). By applying the Vitali convergence theorem (see, e.g., [15], Ch. 5) to $H \circ g_N$, we see it suffices to show $g_N(x)$ converges for each $x \in (-\infty, 0)$ to conclude that $g_N(z)$ converges as $N \to \infty$ for $z \in \mathbb{C}\backslash(0,\infty)$.

Since $w_j < z_j$, we have $(1 - x/z_j)/(1 - x/w_j) < 1$, and since $w_{j+1} > z_j$, we have $(1 - x/z_j)/(1 - x/w_{j+1}) > 1$ assuming $x < 0$. Thus $g_1(x) < g_2(x) < \cdots < g_N(x) < g_{N+1}(x) < 1$, so $\lim_{N\to\infty} g_N(x)$ exists for $x < 0$.

Once we have convergence on $\mathbb{C}\backslash(0,\infty)$, it is easy to extend the argument to $\mathbb{C}\backslash\{w_j\}_{j=1}^{\infty}$.

Finally, let $f(z)$ be a Herglotz function with the stated zeros and poles. Then $f(z)/g(z)$ is an entire non-vanishing function, and on $\mathbb{C}\backslash[0,\infty)$, $|\mathrm{Im}\,(\ln(f(z)/g(z)))| \leq 2\pi$ since $|\mathrm{Im}\,(\ln(f(z)))| \leq \pi$ and $|\mathrm{Im}\,(\ln(g(z)))| \leq \pi$ on $\mathbb{C} \setminus [0,\infty)$. It follows that $f(z)/g(z)$ is constant. $\square$

In exactly the same way one infers

Theorem 2.2. *Let* $0 < z_1 < w_1 < z_2 < w_2 < \cdots$ *with* $\lim_{n\to\infty} w_n = \infty$. *Then*

$$g(z) = \lim_{n\to\infty} \prod_{j=1}^{n}(1 - z/z_j) \Big/ \prod_{j=1}^{n}(1 - z/w_j)$$

exists for any z *in* $\mathbb{C}\backslash\{w_j\}_{j=1}^{\infty}$ *with convergence uniform on compact subsets of* $\mathbb{C}\backslash\{w_j\}_{j=1}^{\infty}$. $g(z)$ *is a meromorphic function with* $\frac{\mathrm{Im}\,(g(z))}{\mathrm{Im}\,(z)} < 0$ *for* $z \in \mathbb{C} \setminus \mathbb{R}$.

Moreover, any meromorphic function $f(z)$ *satisfying* (2.1) *with zeros precisely at* $\{z_j\}_{j=1}^{\infty}$ *and poles precisely at* $\{w_j\}_{j=1}^{\infty}$ *is a negative multiple of* $g(z)$.

We also have theorems on asymptotics, poles, and residues determining a meromorphic Herglotz function.

Theorem 2.3. *Let $f_1(z), f_2(z)$ be two meromorphic Herglotz functions with identical sets of poles and residues, respectively. If*

$$f_1(ix) - f_2(ix) \to 0 \text{ as } x \to \infty, \tag{2.2}$$

then $f_1 = f_2$.

Proof. By the Herglotz representation theorem, if $f(z)$ is a meromorphic Herglotz function with poles at $\{w_j\}_{j=1}^{\infty}$ in $\mathbb{R}$ and residues $-\alpha_k < 0$ at $z = w_k$, then for some constants $A \geq 0$ and $B \in \mathbb{R}$,

$$f(z) = Az + B + \sum_{j=1}^{\infty} \alpha_j \left[\frac{1}{w_j - z} - \frac{w_j}{1 + w_j^2} \right],$$

where the sum is absolutely convergent since $\sum_{j=1}^{\infty} \frac{\alpha_j}{1+w_j^2} < \infty$.

Thus $f_1(z) - f_2(z) = \tilde{A}z - \tilde{B}$ for some $\tilde{A}, \tilde{B} \in \mathbb{R}$, and therefore, (2.2) implies $\tilde{A} = \tilde{B} = 0$. $\square$

In applications, either $f_1(ix)$ and $f_2(ix)$ are both $o(1)$ as $x \to \infty$ or else, $f_1(ix)$ and $f_2(ix)$ are both $\sqrt{ix} + o(1)$ as $x \to \infty$.

§3. The Case of a Dirichlet Boundary Condition $h_a = \infty$

We want to prove Theorem 2 when $h_a = \infty$. If $h_0 < \infty$, let $u_-(z, x; q)$ solve $-u'' + qu = zu$ with boundary conditions $u_-(z, 0; q) = 1$, $u_-'(z, 0; q) = -h_0$. If $h_0 = \infty$, let it satisfy $u_-(z, 0; q) = 0$, $u_-'(z, 0; q) = 1$. As is well known (see, e.g., [11], Ch. 1), u_- is an entire function of z. Similarly, u_+ satisfies the h_1 boundary condition at 1.

Let

$$W(z; q) = u_-'(z, x; q) u_+(z, x; q) - u_-(z, x; q) u_+'(z, x; q),$$

which is independent of x. The zeros of W are precisely the points w_i of $S(0, 1; h_0, h_1; q)$, that is, the eigenvalues of $H := H(0, 1; h_0, h_1; q)$.

Fix $a \in (0, 1)$ and q. Let $g(z) = G(z, a, a)$ be the Green's function of H in $L^2((0, 1))$ at (a, a), that is, the integral kernel of $(H - z)^{-1}$ at (a, a). (We also use the notation $g(z; q)$ for $g(z)$ whenever the dependence of $g(z)$ on q needs to be underscored.) Then, by a standard formula for the Green's function of H,

$$g(z; q) = \frac{u_-(z, a; q) u_+(z, a; q)}{W(z; q)}. \tag{3.1}$$

The zeros of $u_+(z, a; q)$ are precisely the points of $S(a, 1; h_a = \infty, h_1; q)$ and the zeros of $u_-(z, a; q)$ are precisely the points of $S(0, a; h_0, h_a = \infty; q)$. The hypothesis (ii) on disjointness of the S sets in Theorem 2 says that the poles of $g(z)$ are precisely the points of $S(0,1)$, and the zeros, the points of $S(0,a)\cup S(a,1)$. (If the sets are not disjoint, there are cancellations between zeros and poles.)

By Theorem 2.1 (adding a constant to q if need be, we can assume all poles and zeros are positive), the zeros and poles of $g(z)$ and the known asymptotics $g(-\kappa^2;q)=(2\kappa)^{-1}(1+o(1))$ as $\kappa\to\infty$ determine g, that is, $g(z;q_1)=g(z;q_2)$.

Next we use the m-functions $m_\pm$ defined by $m_\pm(z;q)=\pm u'_\pm(z,a;q)/u_\pm(z,a;q)$. By (3.1),

$$g(z;q)=-\frac{1}{[m_+(z;q)+m_-(z;q)]}\,. \tag{3.2}$$

Moreover, the poles of m_+ (resp. m_-) are precisely the points λ of $S(a,1;h_a=\infty,h_1;q)$ (resp. $S(0,a;h_0,h_a=\infty;q)$). And the residues of the poles are determined by g. Explicitly, if λ_0 is a pole of m_+, by hypothesis (ii) in Theorem 2, it is not a pole of m_-, and so its residue is $-1/\frac{\partial g}{\partial z}\Big|_{z=\lambda_0}$.

By Theorem 2.2 and the asymptotics $m_\pm(-\kappa^2;q)=-\kappa+o(1)$ as $\kappa\to\infty$, the poles and residues determine $m_\pm$; that is, $m_\pm(z;q_1)=m_\pm(z;q_2)$.

Finally, the uniqueness result of Borg [1] and Marchenko [12] guarantees that $m_\pm(z;q)$ uniquely determine g on $[0,a]$ and $[a,1]$, so $q_1=q_2$ a.e. on $[0,1]$.

§4. The Case $h_a\in\mathbb{R}$

The changes in the proof when $|h_a|<\infty$ are minimal. Define $u_\pm$ as in the last section, but instead of (3.1), define

$$g(z;q)=\frac{[u'_-(z,a;q)+h_a u_-(z,a;q)][u'_+(z,a;q)+h_a u_+(z,a;q)]}{W(z;q)}\,. \tag{4.1}$$

Since $W=(u'_-+h_a u_-)u_+-u_-(u'_++h_a u_+)$, (3.2) becomes

$$g(z;q)=\frac{1}{\frac{1}{m_+(z;q)+h_a}+\frac{1}{m_-(z;q)-h_a}}\,. \tag{4.2}$$

The spectra determine the zeros and poles of g which, together with the asymptotics $g(-\kappa^2;q)=-\frac{1}{2}\kappa(1+o(1))$ as $\kappa\to\infty$, determine g by Theorem 2.1 or 2.2.

By hypothesis (ii) of Theorem 2, the poles of $(m_\pm\pm h_a)^{-1}$ are distinct and so their residues are determined by (4.2) and the knowledge of g. The poles and residues of $-(m_\pm\pm h_a)^{-1}$ and the fact that $|m_\pm(ix)|\to\infty$ as $x\to\infty$ determine $(m_\pm\pm h_a)^{-1}$ by Theorem 2.3. The Borg-Marchenko uniqueness theorem then completes the proof.

§5. Examples of Non-Uniqueness

Our goal here is to show that if condition (ii) fails, then the uniqueness result in Theorem 2 can also fail. We will take an extreme case where $S(0,\frac{1}{2})=S(\frac{1}{2},1)$ for simplicity; but we have no doubt that a single point in common suffices to construct counterexamples to the extension of Theorem 2 with (ii) absent. We note that $S(0,1) \cap S(0,\frac{1}{2}) = S(0,1) \cap S(\frac{1}{2},1) = S(0,\frac{1}{2}) \cap S(\frac{1}{2},1)$ so that if two S's fail to be disjoint, each pair has non-zero intersection.

To begin we note

Lemma 5.1. *Let f be a continuous map of $Q := [0,1] \times [0,1]$ to the unit circle. Then, there exists a pair of points $p_0, p_1 \in Q$ with $p_0 \neq p_1$ and $f(p_0) = f(p_1)$.*

Proof. If $f(0,0) = f(1,1)$, we have the required points. If not, reparametrize the circle so that $f(0,0) = 1$, $f(1,1) = -1$. Consider the images $f(\gamma_j(t))$, $t \in [0,1]$, $j = 0,1,2$ of the three curves $\gamma_0, \gamma_1, \gamma_2$ given by $\gamma_j(t) = (t, t + (j-1)\pi^{-1}\sin(\pi t))$, $t \in [0,1]$, $j = 0,1,2$. If two of these images contain the point $(0,-1)$ on the unit circle, then that value is taken twice. If at most one of these images contains $(0,-1)$, then by the intermediate value theorem, two images must contain $(0,1)$. $\square$

As explained in [6], by results of Levitan [8], [9], Ch. 3 and Levitan-Gasymov [10], one can prove

Proposition 5.2. *Suppose that $x_0 < y_0 < x_1 < y_1 < \cdots$ so that for n sufficiently large, $x_n = [(2n)\pi]^2$, $y_n = [(2n+1)\pi]^2$. Then there exists a unique h_1 and a C^∞-function q on $[\frac{1}{2},1]$ so that*

$$-\frac{d^2}{dx^2} + q \text{ in } L^2((\tfrac{1}{2},1)); \quad u'(\tfrac{1}{2}) = 0, \quad u'(1) + h_1 u(1) = 0$$

has eigenvalues $\{x_n\}_{n=0}^\infty$ and

$$-\frac{d^2}{dx^2} + q \text{ in } L^2((\tfrac{1}{2},1)]; \quad u(\tfrac{1}{2}) = 0, \quad u'(1) + h_1 u(1) = 0$$

has eigenvalues $\{y_n\}_{n=0}^\infty$. Moreover, if a finite subset of x's and y's is varied, h_1 varies continuously as a function of these numbers.

Consider now fixing $y_n = [(2n+1)\pi]^2$ for all $n \in \mathbb{N}_0$ $(= \mathbb{N} \cup \{0\})$ and $x_n = [(2n)\pi]^2$ for $n \geq 2$ and varying (x_0, x_1) in $[0,1] \times [20,21]$. By Lemma 5.1 and Proposition 5.2, we can find $(x_0^{(0)}, x_1^{(0)}) \neq (x_0^{(1)}, x_1^{(1)})$ so that the corresponding values of h_1 are equal. Set $\tilde{q}_0, \tilde{q}_1$ as the corresponding q's and h as the common value of h_1. Let q_1, q_2 be defined on $[0,1]$ by

$$\begin{aligned} q_1(x) &= \tilde{q}_0(1-x), \qquad && 0 \leq x \leq \tfrac{1}{2}, \\ &= \tilde{q}_1(x), && \tfrac{1}{2} \leq x \leq 1, \\ q_2(x) &= q_0(1-x). \end{aligned}$$

Then $q_1 \neq q_2$ but $S(0,\frac{1}{2}; h_0 = -h, h_{\frac{1}{2}} = \infty; q_1) = S(0,\frac{1}{2}; h_0 = -h, h_{\frac{1}{2}} = \infty; q_2) = S(\frac{1}{2},1; h_{\frac{1}{2}} = \infty, h_1 = h; q_1) = S(\frac{1}{2},1; h_{\frac{1}{2}} = \infty, h_1 = h; q_2) = \{((2n+1)\pi)^2\}_{n\in\mathbb{N}}$, and by reflection symmetry:

$$S(0,1; h_0 = -h, h_1 = h; q_1) = S(0,1; h_0 = -h, h_1 = h; q_2).$$

Since $q_1 \neq q_2$, this provides the required counterexample. (There is no particular significance in our choice of $x_1 \in [20, 21]$. Any interval of length one contained in $(y_0, y_1) = (\pi^2, 9\pi^2)$ would be admissible.)

As in the finite-difference case [5], we believe an analysis of the situation where $S(0, \frac{1}{2}) \cap S(\frac{1}{2}, 1)$ has k-points will yield k-parameter sets of q's (as long as we are allowed to vary h_0, h_1 as well as q) consistent with the given sets of eigenvalues.

§6. The Whole Line Case

In this section, we will extend Theorem 2 to the situation where $[0, 1]$ is replaced by $\mathbb{R}$, but the spectrum of the corresponding Schrödinger operator H in $L^2(\mathbb{R})$ is purely discrete and bounded from below. Typical situations are, for instance, $q \in L^1_{\text{loc}}(\mathbb{R})$ real-valued with $q(x) \to \infty$ as $|x| \to \infty$ or, $q \in L^1_{\text{loc}}(\mathbb{R})$ real-valued, q bounded from below, and $\lim_{x\to\pm\infty} \int_x^{x+a} dy\, q(y) = \infty$ for any $a > 0$ (cf. [11], Sect. 4.1). In this case, the maximal operator H in $L^2(\mathbb{R})$ associated with the differential expression $-\frac{d^2}{dx^2} + q$ on $\mathbb{R}$ (with domain $\mathcal{D}(H) = \{f \in L^2(\mathbb{R}) \mid f, f'$ locally absolutely continuous on $\mathbb{R}; (-f'' + qf) \in L^2(\mathbb{R})\}$) is self-adjoint. In [6] our extensions required a hypothesis on q that $q(x) \geq C|x|^{2+\varepsilon} + 1$ for some $C, \varepsilon > 0$. This was because we used results on densities of zeros. Here, because we rely on Theorems 2.1, 2.2, we note that the following result holds by the identical proof to Theorem 2:

Theorem 3. *Suppose $q \in L^1_{\text{loc}}(\mathbb{R})$ is real-valued and H in $L^2(\mathbb{R})$ is bounded from below with purely discrete spectrum $S(-\infty, \infty; q)$. Let $S(-\infty, 0; h_0; q)$ denote the spectrum of the corresponding (maximally defined) operator in $L^2((-\infty, 0))$ with $u'(0) + h_0 u(0) = 0$ boundary conditions, and similarly for $S(0, \infty; h_0; q)$. Suppose that q_1, q_2 are given and we have a fixed $h_0 \in \mathbb{R} \cup \{0\}$ so that*

- (i) *$S(-\infty, \infty; q_1) = S(-\infty, \infty; q_2)$, $S(-\infty, 0; h_0; q_1) = S(-\infty, 0; h_0; q_2)$, and $S(0, \infty; h_0; q_1) = S(0, \infty; h_0; q_2)$*
- (ii) *The sets $S(-\infty, \infty; q_1)$, $S(-\infty, 0; h_0; q_1)$, and $S(0, \infty; h_0; q_1)$ are pairwise disjoint.*

Then $q_1 = q_2$ a.e. on $\mathbb{R}$.

As noted in Remark 2 following Theorem 2.1, this result extends to Schrödinger operators H with purely discrete spectra accumulating at $+\infty$ and $-\infty$. In particular, it extends to cases where H is in the limit circle case at $+\infty$ and/or $-\infty$ as long as the corresponding (separated) boundary condition at $+\infty$ and/or $-\infty$ is kept fixed for all three operators on $\mathbb{R}$, $(-\infty, 0)$, and $(0, \infty)$.

The reader might want to contrast Theorem 3 with Corollary 3.4 in [3], where we obtained uniqueness of q from three (discrete) spectra of operator realizations of $-\frac{d^2}{dx^2} + q$ on $\mathbb{R}$. There one of the three spectra is $S(-\infty, \infty; q)$ as above in Theorem 3; the other two, $S(-\infty, \infty; \beta_j, q)$, $j = 1, 2$, are associated with $-\frac{d^2}{dx^2} + q$ on $\mathbb{R}$ and the boundary conditions $\lim_{\varepsilon\downarrow 0}[u'(\pm\varepsilon) + \beta_j u(\pm\varepsilon)] = 0$, where $\beta_j \in \mathbb{R} \cup \{\infty\}$, $j = 1, 2$, $\beta_1 \neq \beta_2$, $(\beta_1, \beta_2) \neq (0, \infty)$, $(\infty, 0)$.

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