ZEROS OF ORTHOGONAL POLYNOMIALS ON THE REAL LINE

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Abstract. Let $p_n(x)$ be the orthonormal polynomials associated to a measure $d\mu$ of compact support in $\mathbb{R}$. If $E \notin \operatorname{supp}(d\mu)$, we show there is a $\delta > 0$ so that for all n, either p_n or p_{n+1} has no zeros in $(E-\delta, E+\delta)$. If E is an isolated point of $\operatorname{supp}(\mu)$, we show there is a δ so that for all n, either p_n or p_{n+1} has at most one zero in $(E-\delta, E+\delta)$. We provide an example where the zeros of p_n are dense in a gap of $\operatorname{supp}(d\mu)$.

1. Introduction

Let $d\mu$ be a measure on $\mathbb{R}$ whose support is not a finite number of points and with $\int |x|^n \, d\mu(x) < \infty$ for all $n = 0, 1, 2, \dots$. The orthonormal polynomials $p_n(x; d\mu)$ or $p_n(x)$ are determined uniquely by

$$p_n(x) = \gamma_n x^n + \text{lower order} \qquad \gamma_n > 0 \tag{1.1}$$

$$\int p_n(x) p_m(x) \, d\mu(x) = \delta_{nm} \tag{1.2}$$

There are $a_n > 0$, $b_n \in \mathbb{R}$ for $n \geq 1$ so that

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_{n+1} p_n(x) + a_n p_n(x) \tag{1.3}$$

(many works use a_{n-1}, b_{n-1} where we use a_n, b_n).

In this paper, we will be interested in the zeros of $p_n(x; d\mu)$. The following results are classical (see, e.g., Freud's book [3]):

(1) The zeros of $p_n(x)$ are real and simple.

(2) If $(a,b) \cap \operatorname{supp}(d\mu) = \emptyset$, then if $a = -\infty$ or $b = +\infty$, p_n has no zeros in (a,b) and, in any event, (a,b) has at most one zero of $p_n(x)$.

(3) In the determinate case, if $x_0 \in \operatorname{supp}(d\mu)$ and $\delta > 0$, for all large n, $p_n(x)$ has a zero in $(x_0 - \delta, x_0 + \delta)$.

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Define

$$N_n(x_0,\delta) = \#\text{ of zeros of } p_n(x) \text{ in } (x_0-\delta, x_0+\delta)$$

Then (1)–(3) immediately imply:

(i) If x_0 is a non-isolated point of $\text{supp}(d\mu)$, then for any $\delta > 0$, $\lim_{n\to\infty} N_n(x_0,\delta) = \infty$.

(ii) If x_0 is an isolated point of $\text{supp}(d\mu)$ and $\delta = \text{dist}(x_0, \text{supp}(d\mu)\backslash\{x_0\})$, then $N_n(x_0,\delta)$ is never more than 2, and for all $\delta > 0$ and n large, $N_n(x_0,\delta) \geq 1$.

(iii) If $x_0 \notin \text{supp}(d\mu)$ and $\delta = \text{dist}(x_0, \text{supp}(d\mu))$, then $N_n(x_0,\delta)$ is never more than 1.

(i) is fairly complete, but (ii), (iii) leave open how often there is one vs. two points in case (ii) and zero vs. one in case (iii). One might guess that a zero near $x_0 \notin \text{supp}(d\mu)$ and two zeros near an isolated x_0 in $\text{supp}(d\mu)$ are not too common occurrences.

Example. If $d\mu$ is even about $x = 0$, then $p_n(-x) = (-1)^n p_n(x)$. Thus, if n is odd, $p_n(0) = 0$. So if $0 \notin \text{supp}(d\mu)$, we still have $N_n(0,\delta) = 1$ for all small δ and n odd. If zero is an isolated point of $d\mu$, p_n for n even has a zero at x_n near 0, but not equal to 0 (since zeros are simple), so also at $-x_n$, that is, $N_n(0,\delta) = 2$ for δ small and n even. So "not too common" can be as often as 50% of the time. Our goal here is to show this 50% is a maximal value.

It is surprising that there do not seem to be any results on these issues until a recent paper of Ambroladze [1], who proved

Theorem (Ambroladze [1]). *If* $\text{supp}(d\mu)$ *is bounded and* $x_0 \notin \text{supp}(d\mu)$*, then for some* $\delta > 0$*,* $\liminf_{n\to\infty} N_n(x_0,\delta) = 0$.

Thus we can use $N_n(x_0,\delta)$ to distinguish when $x_0 \in \text{supp}(d\mu)$. Our goal in this paper is to prove

Theorem 1. *Let* $d = \text{dist}(x_0, \text{supp}(d\mu)) > 0$*. Let* $\delta_n = d^2/(d + \sqrt{2}\, a_{n+1})$ (*where* a_n *is the recursion coefficient given by* (1.3)). *Then either* p_n *or* p_{n+1} (*or both*) *has no zeros in* $(x_0-\delta_n, x_0+\delta_n)$*. In particular, if* $a_\infty = \sup_n a_n < \infty$ *and* $d_\infty = d^2/(d+\sqrt{2}\, a_\infty)$*, then* $(x_0-\delta_\infty, x_0+\delta_\infty)$ *does not have zeros of* p_j *for two successive values of* j.

Theorem 2. *Let* x_0 *be an isolated point of* $\text{supp}(d\mu)$*. Then there exists a* $d_0 > 0$*, so that if* $\delta_n = d_0^2/(d_0 + \sqrt{2}\, a_{n+1})$*, then at least one of* p_n *and* p_{n+1} *has no zeros or one zero in* $(x_0-\delta_n, x_0+\delta_n)$*. In particular, if* $a_\infty = \sup_n a_n < \infty$ *and* $\delta_\infty = d_0^2/(d_0 + \sqrt{2}\, a_\infty)$*, then for all large* n*, either* $N_n(x_0,\delta_\infty) = 1$ *or* $N_{n+1}(x_0,\delta_\infty) = 1$.

We will prove Theorem 1 in Section 2 and Theorem 2 in Section 3. In Section 4, we present an example of a set of polynomials whose zeros are dense in a gap of the spectrum.

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2. Points Outside the Support of $d\mu$

We arrived at the following lemma by trying to abstract the essence of Ambroladze's argument [1]; it holds for orthogonal polynomials on the complex plane. Let $d\mu$ be a measure on $\mathbb{C}$ with finite moments and infinite support, and let $p_n(z; d\mu)$ be the orthonormal polynomials. Define the reproducing kernel

$$K_n(z,w) = \sum_{j=0}^{n} p_n(z)\,\overline{p_n(w)} \tag{2.1}$$

so in $L^2(\mathbb{C}, d\mu)$, for any polynomial π of degree n or less,

$$\int K_n(z,w)\pi(w)\,d\mu(w) = \pi(z) \tag{2.2}$$

Lemma 2.1. *Suppose $z_0 \in \mathbb{C}$, $p_j(w) = 0$ for some $j \leq n+1$. Then*

$$|z_0 - w| \geq \frac{|p_j(z_0)|}{K_n(z_0,z_0)^{1/2}} \operatorname{dist}(w, \operatorname{supp}(d\mu)) \tag{2.3}$$

Proof. Let $q(z) = p_j(z)/(z-w)$, which has $\deg(q) \leq n$. Thus, by (2.2), $\langle K(\,\cdot\,, z_0), q(\,\cdot\,)\rangle = q(z_0)$ so, by the Schwarz inequality,

$$\frac{|p_j(z_0)|}{|z_0 - w|} \leq \|q\|\,\|K(\,\cdot\,, z_0)\|$$

By (2.2), $\|K(\,\cdot\,, z_0)\| = K(z_0,z_0)^{1/2}$ and clearly, $\|q\| \leq \operatorname{dist}(w, \operatorname{supp}(d\mu))^{-1}\|p_j\| = \operatorname{dist}(w, \operatorname{supp}(d\mu))^{-1}$. This yields (2.3).

The following only holds in the real case:

Lemma 2.2. *For any $x \in \mathbb{R}$ and n,*

$$K_n(x,x)\operatorname{dist}(x, \operatorname{supp}(d\mu))^2 \leq a_{n+1}^2[p_{n+1}^2(x) + p_n^2(x)] \tag{2.4}$$

Proof. The Christoffel-Darboux formula [3] says

$$K_n(x,y) = a_{n+1}\left[\frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{x-y}\right]$$

so since $\langle p_j, p_k\rangle = \delta_{jk}$,

$$\|(x - \,\cdot\,)K_n(x,\,\cdot\,)\|^2 = |a_{n+1}|^2[p_{n+1}^2(x) + p_n^2(x)] \tag{2.5}$$

Clearly,

$$\|(x - \cdot\,)K_n(x,\,\cdot\,)\|^2 \geq \operatorname{dist}(x, \operatorname{supp}(d\mu))^2 \|K_n(x,\,\cdot\,)\|^2 \tag{2.6}$$

and, as above, $\|K_n(x,\,\cdot\,)\|^2 = K_n(x,x)$, which yields (2.4).

Remark. An alternate way of seeing (2.5) is to let ψ be the trial vector $(p_0(x), \dots, p_n(x), 0, 0, \dots)$ and note that in terms of the standard Jacobi matrix $((J - x)\psi)_j = 0$ unless $j = n, n+1$, in which case the values are $-a_{n+1}p_{n+1}(x)$ and $a_{n+1}p_n(x)$. (2.6) is then just $\|(J-x)\psi\| \geq \operatorname{dist}(x, \operatorname{supp}(d\mu))\|\psi\|$.

Proof of Theorem 1. By (2.4), we have that

$$K_n(x_0, x_0)\operatorname{dist}(x_0, \operatorname{supp}(d\mu))^2 \leq 2a_{n+1}^2 p_{n+1}^2(x_0) \tag{2.7}$$

and/or

$$K_n(x_0, x_0)\operatorname{dist}(x_0, \operatorname{supp}(d\mu))^2 \leq 2a_{n+1}^2 p_n^2(x_0) \tag{2.8}$$

Suppose (2.7) holds. Then, by (2.3), if w is a zero of $p_{n+1}(x)$ and if $d = \operatorname{dist}(x_0, \operatorname{supp}(d\mu))$,

$$\begin{aligned} |x_0 - w| &\geq \frac{1}{\sqrt{2}}\,\frac{1}{a_{n+1}}\, d\operatorname{dist}(w, \operatorname{supp}(d\mu)) \\ &\geq \frac{1}{\sqrt{2}}\,\frac{1}{a_{n+1}}\, d(d - |w - x_0|) \end{aligned}$$

which leads directly to $|x_0 - w| \geq d^2/(d + a_{n+1}\sqrt{2})$.

Remark. There is also a Christoffel-Darboux result for polynomials on the unit circle $\partial D = \{z \mid |z| = 1\}$ in $\mathbb{C}$. This leads to the following: If $d\mu$ is a measure on ∂D and $z_0 \in \partial D$ has $d = \operatorname{dist}(z_0, \operatorname{supp}(d\mu)) > 0$, then the circle of radius $d^2/(2 + d)$ has no zeros of the orthogonal polynomials. L. Golinskii has pointed out that the theorem of Fejér [2] that the zeros lie in the convex hull of $\operatorname{supp}(d\mu)$ implies there are no zeros in the circle of radius $d^2/2$ — which is a stronger result, so we do not provide the details.

3. Isolated Points of the Support of $d\mu$

To prove Theorem 2, we will make use of the second kind polynomials [3, 5] associated to $d\mu$ and $\{p_n\}$. This is a second family of polynomials, q_n defined by recursion coefficients, $\tilde{a}_n$, $\tilde{b}_n$ with

$$\tilde{a}_n = a_{n+1} \qquad \tilde{b}_n = b_{n+1} \tag{3.1}$$

They have the following two critical properties:

Proposition 3.1. (i) *The zeros of* p_{n+1} *and* q_n *interlace. In particular, between any two zeros of* p_{n+1} *is a zero of* q_n.
(ii) *If* x_0 *is an isolated point of* $d\mu$ *and* $d\nu$ *is a suitable measure with respect to which the q's are orthogonal, then* $x_0 \notin \text{supp}(d\nu)$.

These are well known. (i) follows from the fact that the zeros of p_{n+1} are eigenvalues of the matrix

$$J_{ij}^{(n+1)} = b_i \delta_{ij} + a_i \delta_{i\,i+1} + a_{i-1}\delta_{i\,i-1} \qquad 1 \leq i, j \leq n+1$$

and the zeros of q_n are the eigenvalues of

$$\tilde{J}_{ij}^{(n)} = \tilde{b}_i \delta_{ij} + \tilde{a}_i \delta_{i\,i+1} + \tilde{a}_{i-1}\delta_{i\,i-1} \qquad 1 \leq i, j \leq n$$

which is the matrix $J_{ij}^{(n+1)}$ with the top row and left column removed. (ii) follows because of the relation that ν obeys for all $z \in \mathbb{C}\setminus\mathbb{R}$ [5]:

$$\int \frac{d\nu(x)}{x-z} = a_1^{-2}\left[b_1 - z - \left(\int \frac{d\mu(x)}{x-z} \right)^{-1} \right] \tag{3.2}$$

(if the moment problem is indeterminate, this is one possible ν). Isolated points of $d\mu$ are poles of $\int d\mu(x)/(x-z)$ so $\int d\nu(x)/(x-z)$ is regular there.

Proof of Theorem 2. Let $d_0 = \text{dist}(x_0, \text{supp}(d\nu)) > 0$ by (ii) of Proposition 3.1. By Theorem 1 and (3.1), either q_{n-1} or q_n has no zeros in $(x_0 - \delta_n, x_0 + \delta_n)$. By the intertwining result (Proposition 3.1(i)), either p_n or p_{n+1} cannot have two zeros in this interval.

Remark. If $b \in \text{supp}(d\mu)$ is such that $|x_0 - b| = \text{dist}(x_0, \text{supp}(d\mu))$ and $\int d\mu(y)/|y-b| = \infty$, then $d\nu$ has an isolated point in between x_0 and b, and so $d_0 < \text{dist}(x_0, \text{supp}(d\mu))$.

4. An Example of Dense Zeros in the Gap

Nevai raised the issue of whether as n varies, the single possible zero of p_n in a gap (a, b) of $\text{supp}(d\mu)$ can yield all of (a, b) as limit points, or if the situation of a single (or finite number of) limit point as in the example in Section 1 is the only possibility. In this section, we describe an explicit bounded Jacobi matrix so that $\text{supp}(d\mu) = [-5, -1] \cup [1, 5]$ but the set $\{x \in (-1, 1) \mid p_n(x) = 0 \text{ for some } n\}$ is dense in $[-1, 1]$.

Let $\{\beta_j\}_{j=1}^{\infty}$ be the sequence

$$\beta_1, \beta_2, \ldots = 0, -\tfrac{1}{2}, 0, \tfrac{1}{2}, -\tfrac{3}{4}, -\tfrac{1}{2}, -\tfrac{1}{4}, 0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, -\tfrac{7}{8}, \ldots$$

which goes through *all* dyadic rationals in $(-1,1)$ with denominator 2^k successively for $k = 1,2,3,\dots$ with each $j/2^k$ "covered" multiple times. Let L be the Jacobi matrix with

$$a_{2n-1} = 3, \qquad a_{2n} = 1, \qquad n = 1,2,\dots \tag{4.1}$$

$$b_k = \beta_n \qquad \text{if } 2n^2 \le k < 2(n+1)^2 \tag{4.2}$$

$$b_1 = \beta_1$$

We claim that

(1) $\operatorname{supp}(d\mu) = [-5,-1] \cup [1,5]$

(2) There is an x_n with $|x_n - \beta_n| \le 2\ 3^{-2n}$ so that

$$P_{2(n+1)^2-1}(x_n) = 0 \tag{4.3}$$

This provides the claimed example.

Remarks. 1. By adjusting a_1 and a_2 (but keeping $a_{2n+1} = a_1$; $a_{2n} = a_2$), we can replace $[-5,-1]\cup[1,5]$ by $[-3-\varepsilon,-1]\cup[1,3+\varepsilon]$, but our method seems to require bands larger than the size of the gap.

2. One can replace (4.2) by $b_k = \beta_n$ for $\ell_n \le k < \ell_{n+1}$ so long as $\ell_{n+1} - \ell_n \to \infty$.

3. We believe that the measure associated to L is purely singular. It is perhaps true that the phenomenon of zeros dense in a gap requires purely singular spectral measure.

To prove the claims, we let L_0 be the Jacobi matrix with a's given by (4.1) but $b_n = 0$, and L_∞ the period two, doubly infinite matrix on $\mathbb{Z}$ which equals L_0 when restricted to $\mathbb{Z}^+$. By the general theory of periodic Schrödinger operators [4], the spectrum of L_∞ is the two bands where $|\Delta(x)| \le 2$ where Δ is the discriminant, that is, the trace of the two-step transfer matrix. If a_1, a_2 are the two values of a (so $a_1 = 3$, $a_2 = 1$ in our example), a simple calculation shows that

$$\Delta(x) = \frac{1}{a_1 a_2}\,(x^2 - (a_1^2 + a_2^2))$$

so $\Delta(x) = \pm 2$ occurs at $x = \pm|a_1 \pm a_2|$. Thus

$$\operatorname{spec}(L_\infty) = [-4,-2] \cup [2,4] \tag{4.4}$$

The orthonormal polynomials $p_n^{(0)}$ for L_0 at $x = 0$ obey the recursion relation

$$p_{2n+2}^{(0)}(0) = -3p_{2n}^{(0)}(0)$$

so we have

$$p_{2n+1}^{(0)} = 0 \qquad p_{2n}^{(0)}(0) = (-3)^n \tag{4.5}$$

By the general theory of restricting periodic operators to the half-line, $\operatorname{spec}(L_0)$ is $\operatorname{spec}(L_\infty)$ plus a possible single eigenvalue in the gap

$(-2,2)$. Since there is a symmetry, the only possible eigenvalue is at $x=0$, but (4.5) says that 0 is not an eigenvalue since $\sum_{j=0}^{\infty}|P_j(0)|^2 = \infty$. Thus $\operatorname{spec}(L_0) = [-4,-2]\cup[2,4]$ also. $L-L_0$ is a diagonal matrix, so it is easy to see $\|L-L_0\| = \sup_j|\beta_j| = 1$. Thus $\operatorname{spec}(L) \subset \cup_{x\in[-1,1]} x + \operatorname{spec}(L_0) = [-5,-1]\cup[1,5]$. On the other hand, since the b's are equal to β_j on arbitrary long runs, a Weyl vector argument shows that

$$\operatorname{spec}(L) \supset \overline{\cup_j \beta_j + \operatorname{spec}(L_0)} = [-5,-1]\cup[1,5]$$

so claim 1 is proven.

Let $L_{n;F}$ be the $n\times n$ matrix obtained by taking the first n rows and columns of L. Then the zeros of $p_n(x)$ are precisely the eigenvalues of $L_{n;F}$ (see [5, Proposition 5.6]). Let φ_j be the j component vector with $(P_0^{(0)}(0), P_1^{(0)}(0), \dots, P_{j-1}^{(0)}(0))$. Then if j is odd so $P_j^{(0)}(0) = 0$, and we have $L_{0;j,F}\varphi_j = 0$. Thus, if $j = 2(n+1)^2 - 1$,

$$[(L_{j;F} - \beta_n)\varphi_j]_k = (b_k - \beta_n)\varphi_{j,k} \tag{4.6}$$

If $2n^2 \le k \le 2(n+1)^2 - 1$, the right-hand side is zero and its absolute value is always less than $2|\varphi_{j,k}|$. Thus

$$\begin{aligned} \frac{\|(L_{j;F} - \beta_n)\varphi_j\|^2}{\|\varphi_j\|^2} &\le \frac{4\sum_{k=0}^{n^2-1} 3^{2k}}{\sum_{k=0}^{(n+1)^2-1} 3^{2k}} \\ &\le 4\ 3^{-4n} \end{aligned}$$

by a simple estimate. Thus $L_{j;F}$ has an eigenvalue within $2\ 3^{-2n}$ of β_n, proving claim 2.

This completes the example.

References

[1] A. Ambroladze, *On exceptional sets of asymptotic relations for general orthogonal polynomials*, J. Approx. Theory **82** (1995), 257–273.

[2] L. Fejér, *Über die Lage der Nullstellen von Polynomen, die aus Minimumforderungen gewisser Art entspringen*, Math. Ann. **85** (1922), 41–48.

[3] G. Freud, *Orthogonal Polynomials*, Pergamon Press, Oxford-New York, 1971.

[4] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV: Analysis of Operators*, Academic Press, New York, 1978.

[5] B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Adv. in Math. **137** (1998), 82–203.