THE GOLINSKII-IBRAGIMOV METHOD AND A THEOREM OF DAMANIK AND KILLIP

BARRY SIMON

Abstract. In 1971, Golinskii and Ibragimov proved that if the Verblunsky coefficients, $\{\alpha_n\}_{n=0}^\infty$, of a measure $d\mu$ on $\partial\mathbb{D}$ obey $\sum_{n=0}^\infty n|\alpha_n|^2 < \infty$, then the singular part, $d\mu_{\rm s}$, of $d\mu$ vanishes. We show how to use extensions of their ideas to discuss various cases where $\sum_{n=0}^N n|\alpha_n|^2$ diverges logarithmically. As an application, we provide an alternative to a part of the proof of a recent theorem of Damanik and Killip.

1. Introduction

Let $\partial\mathbb{D}$ be the unit circle $\{z \mid |z| = 1\}$ in $\mathbb{C}$ and $\mathbb{D}$ the open disc, $\{z \mid |z| < 1\}$. Let μ be a probability measure on $\partial\mathbb{D}$ which is not supported on a finite number of points. Then using the Gram-Schmidt procedure, we can define monic orthogonal polynomials on the unit circle (OPUC) $\Phi_n(z; d\mu)$ and normalized polynomials $\varphi_n(z; d\mu)$. These obey the Szegő recursion formulae [8, 9]:

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z) \tag{1.1}$$

where

$$\Phi_n^*(z) = z^n \, \overline{\Phi_n(\tfrac{1}{\bar{z}})} \tag{1.2}$$

The parameters α_n are called the Verblunsky coefficients of $d\mu$ (also called Schur, Szegő, Geronimus, or reflection parameters or coefficients). They lie in $\mathbb{D}$ and any $\alpha \in \times_{n=0}^\infty \mathbb{D}$ is the Verblunsky coefficient of a unique measure [2, 8]. In this note, we are mainly interested in the spectral problem of going from information on the Verblunsky coefficients to information on the measure.

Our starting point is a method from a lovely 1971 paper of B. Golinskii and I. Ibragimov [4], who used this method to prove:

Theorem 1 (Golinskii-Ibragimov [4]). *If* $\sum_{n=0}^\infty n|\alpha_n|^2 < \infty$*, then* $d\mu_{\rm s} = 0$.

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[1] Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125. E-mail: bsimon@caltech.edu.

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Here we make a canonical decomposition:

$$d\mu = w(\theta)\,\frac{d\theta}{2\pi} + d\mu_{\rm s} \tag{1.3}$$

where $d\mu_{\rm s}$ is singular with respect to $d\theta$. We are interested in α's that obey a weaker condition:

$$\sum_{n=0}^{N} n|\alpha_n|^2 \leq A\log N + C \tag{1.4}$$

with A, C constant. Think of $\alpha_n = \sqrt{A}/n$ as a prototypical example. First, we will prove

Theorem 2. *If* (1.4) *holds with* $A < \frac{1}{4}$, *then* $d\mu_{\rm s} = 0$.

The GI method most directly only gets $A < \frac{1}{16}$, but by replacing their L^1 methods by an L^2 method, we will bring things up to $A < \frac{1}{4}$.

Theorem 2 is almost optimal in that there are examples with $A > \frac{1}{4}$ but $|A - \frac{1}{4}|$ arbitrarily small, where $d\mu$ has an eigenvalue. However, we can do better if we assume the α's are real:

Theorem 3. *If all* α_n *are real* (*equivalently,* $d\mu$ *is invariant under complex conjugation*) *and* (1.4) *holds with* $A < \frac{1}{2}$, *then* $d\mu_{\rm s}$ *can only consist of possible pure points at* $z = +1$ *or* $z = -1$. *If* $A = \frac{1}{4}$, $d\mu_{\rm s} = 0$.

We will use an idea motivated by Damanik-Killip [1] to prove Theorem 3. In this regard, we will prove the following special-looking result which, as we will explain, is related to [1]:

Theorem 4. *Let all* α_n *be real and obey*

(i)
$$\sum_{j=0}^{N} j|\alpha_{2j}|^2 \leq A\log N + C \tag{1.5}$$

(ii) $|\alpha_{2j-1}| \leq |\alpha_{2j+1}|$ *for* $j = 1, 2, \dots$ *with either all* $\alpha_{2j-1} \geq 0$ *or all* $\alpha_{2j-1} \leq 0$.

(iii) $\sum_{j=1}^{\infty} |\alpha_{2j-1}|^2 < \infty$

If $A < \frac{1}{2}$, $d\mu_{\rm s}$ *consists only of possible pure points at* $z = \pm 1$ *or* $z = \pm i$. *Moreover, if* $A = \frac{1}{4}$, *the only possible pure points are at* $z = \pm 1$, *and if* $A = \frac{1}{4}$ *and* $\alpha_{2j-1} \leq 0$, *then* $d\mu_{\rm s} = 0$.

This theorem is custom-made to provide part of a proof of the following recent striking result of Damanik-Killip [1]:

Theorem 5 (Damanik-Killip [1])**.** *Let* H *be a half-line Schrödinger operator on* $\ell^2(\mathbb{Z}_+)$,

$$(Hu)_n = u_{n+1} + u_{n-1} + v_n u_n$$

(u_0 *interpreted as* 0). *Suppose* $\operatorname{spec}(H) = [-2, 2]$. *Then* H *has purely a.c. spectrum, that is,* $\sigma_{\rm sc} = \sigma_{\rm pp} = \emptyset$.

Their proof has the following steps (they use γ_n for our α_n):

(i) Following Szegő [9], map H to an associated measure on $\partial\mathbb{D}$ by using $z = e^{i\theta} \mapsto E = 2\cos\theta$ to pull back the spectral measure $d\rho$ on $[-2, 2]$ for H to a measure μ on $\partial\mathbb{D}$. Note (following Geronimus) that the Verblunsky coefficients for $d\mu$ obey:

$$v_{n+1} = (1 - \alpha_{2n-1})\alpha_{2n} - (1 + \alpha_{2n-1})\alpha_{2n-2} \tag{1.6}$$

$$1 = (1 - \alpha_{2n-1})(1 - \alpha_{2n}^2)(1 + \alpha_{2n+1}) \tag{1.7}$$

(for general Jacobi matrices, the left side of (1.7) is a_{n+1}^2; the initial conditions are $\alpha_{-1} = -1$).

(ii) Analyze (1.7) with α_n real and $|\alpha_n| < 1$ to conclude:
  (a) [Their Lemma 4.1] $\alpha_{2n-1} \leq \alpha_{2n+1} \leq 0$
  (b) [Their Lemma 4.2] $|\alpha_{2n+1}| \leq \frac{1}{n+2}$ so $\sum_{n=1}^\infty |\alpha_{2n-1}|^2 < \infty$.
  (c) [Their Proposition 4.5] $\sum_{j=0}^N (j+1)\alpha_j^2 \leq \frac14 \log N + C$

(iii) Translate information on the α_n's to information on the v_n's.

(iv) Prove that $-2, 2, 0$ are not eigenvalues of H.

(v) Prove that solutions of

$$u_{n+1} + u_{n-1} + v_n u_n = Eu \tag{1.8}$$

with $E \in (-2, 0) \cup (0, 2)$ have $|u_n| \leq cn^\eta$ for any $\eta > \frac{1}{2\sqrt{2}}$.

(vi) Prove that the set of E in $(-2, 2)$ for which (1.8) has unbounded solutions has Hausdorff dimension 0.

(vii) Use (v), (vi), Hausdorff dimension, and the Jitomirskaya-Last inequalities [6] to show that $d\mu$ has no singular part on $(-2, 0) \cup (0, 2)$.

Given step (ii), one can use Theorem 4 to replace steps (iii)–(vii). For Theorem 4 says $d\mu$ is purely a.c. and the pull back then implies $d\rho$ is. Since Theorem 4 depends on ideas closely related to steps (iv) and (v), what we are really doing is using an appeal to the GI method to replace steps (vi) and (viii) and, in particular, the use of Hausdorff dimension and the Jitomirskaya-Last inequalities. These steps follow ideas of Remling [7] so, in essence, where [1] extends [7], we extend [4]. It is pointless to argue which approach is "simpler" (since some of their techniques have appeared extensively in the Schrödinger operator literature), but we believe it useful to have the alternate approach.

In Section 2, we discuss the ideas of Golinskii-Ibragimov and, in particular, prove Theorem 2. In Section 3, we use Prüfer variables for OPUC to prove Theorem 3. In Section 4, we prove Theorem 4.

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2. The GI Method

Golinskii-Ibragimov make the assumption that

$$\sum_{n=0}^{\infty} n|\alpha_n|^2 < \infty \tag{2.1}$$

On $\partial\mathbb{D}$, D^{-1} is defined a.e. $d\theta$ as boundary values of D, the Szegő function (see [3, 8]), and so a.e. $d\mu_{\rm ac}$. We extend it to $L^2(\partial\mathbb{D}, d\mu) = L^2(\partial\mathbb{D}, d\mu_{\rm ac}) \oplus L^2(\partial\mathbb{D}, d\mu_{\rm s})$ by setting to 0 on the singular subspace. Then Golinskii-Ibragimov [4] prove that

$$\|\varphi_n^* - D^{-1}\|_{L^2(\partial\mathbb{D},d\mu)} \le Cn^{-1/2} \tag{2.2}$$

and that

$$\|(\varphi_n^*)^{-1}\|_\infty \le C_1 \exp(C_2\sqrt{\log n}\,) \tag{2.3}$$

where $\|\cdot\|_\infty$ means sup over $\partial\mathbb{D}$ or $\bar{\mathbb{D}}$ (equal since $(\varphi_n^*)^{-1}$ is analytic in a neighborhood of $\bar{\mathbb{D}}$). In (2.2), the Szegő function exists since (2.1) implies $\sum_{n=0}^\infty |\alpha_n|^2 < \infty$. We will actually prove results like this below.

They then write

$$\begin{aligned} \||\varphi_n^*|^{-2} - |D|^2\|_{L^1(\partial\mathbb{D},d\theta/2\pi)} & \\ &\le \|(\varphi_n^*)^{-1}\|_\infty^2 \|(|\varphi_n|^2 - |D|^{-2})D^2\|_{L^1(\partial\mathbb{D},d\theta/2\pi)} \\ &\le \|(\varphi_n^*)^{-1}\|_\infty^2 \|(|\varphi_n|^2 - |D|^{-2})\|_{L^1(\partial\mathbb{D};d\mu)} \qquad (2.4) \\ &\le \|(\varphi_n^*)^{-1}\|_\infty^2 \|\varphi_n - D^{-1}\|_{L^2(\partial\mathbb{D};d\mu)} \||\varphi_n| + |D|^{-1}\|_{L^2(\partial\mathbb{D};d\mu)} \qquad (2.5) \end{aligned}$$

In (2.4), we use $D^2 \frac{d\theta}{2\pi} = d\mu_{\rm ac} \le d\mu$ and in (2.5) that $||\varphi_n| - |D|^{-1}| \le |\varphi_n - D^{-1}|$. Thus, by (2.2) and (2.3), $|\varphi_n|^{-2}\frac{d\theta}{2\pi} \to |D|^2 \frac{d\theta}{2\pi}$ in norm on measures. But it is known that $|\varphi_n|^{-2}\frac{d\theta}{2\pi} \to d\mu$ weakly (see, e.g., [2]). Thus $d\mu = |D|^2\frac{d\theta}{2\pi}$ so $d\mu_{\rm s} = 0$.

What is especially interesting about this approach is that it uses the divergent estimate (2.3). Clearly, we can have much more rapid growth of $\|(\varphi_n^*)^{-1}\|_\infty$ than in (2.3) and still have convergence. Basically, it suffices to have $\|(\varphi_n^*)^{-1}\|_\infty \le Cn^\beta$, with $2\beta < \frac{1}{2}$. If one looks at the proof of (2.3) in [4], that translates to a bound like (1.4) with $\sqrt{A} < \frac{1}{4}$, that is, $A < \frac{1}{16}$. Our first observation is that instead of estimating L^1 norms as GI do, it pays to estimate L^2 norms. $\|(\varphi_n^*)^{-1}\|_\infty$ will then occur as a first power, not second. Here are the key facts:

Theorem 2.1. *Let μ obey the Szegő condition. Then*
(a) $d\mu_{\rm s} = 0$ *if and only if* $\|(\varphi_n^*)^{-1} - D\|_{L^2(\partial\mathbb{D},d\theta/2\pi)} \to 0$.

(b) *If I is an open interval and $\|\chi_I[(\varphi_n^*)^{-1} - D]\|_{L^2(\partial\mathbb{D},d\theta/2\pi)} \to 0$, then $\mu_{\rm s}(I) = 0$.*

Proof. (a) As is well known (see [2]), $\|(\varphi_n^*)^{-1}\|_{L^2(\partial\mathbb{D},d\theta/2\pi)} = 1$. Moreover, $(\varphi_n^*)^{-1}(z) \to D(z)$ uniformly on compact subsets of $\mathbb{D}$, so $(\varphi_n^*)^{-1} \to D$ weakly in L^2. Thus $(\varphi_n^*)^{-1} \to D$ in norm if and only if $1 = \|(\varphi_n)^{-1}\|^2_{L^2(\partial\mathbb{D},d\theta/2\pi)} = \|D\|^2_{L^2(\partial\mathbb{D},d\theta/2\pi)}$. But $\|D\|^2 = \int |D(\theta)|^2 \frac{d\theta}{2\pi} = \mu_{\rm ac}(\partial\mathbb{D})$, so norm convergence is equivalent to $\mu_{\rm s}(\partial\mathbb{D}) = 0$.

(b) Let f be continuous and have support in I. Then

$$\begin{aligned} f|\varphi_n^*|^{-2} &= (f(\varphi_n^*)^{-1}\chi_1)(\overline{(\varphi_n^*)^{-1}}\,\chi_I) \\ &\to f|D|^2 \end{aligned}$$

in L^1 and thus, by the weak convergence of $|\varphi_n^*|^{-2}\frac{d\theta}{2\pi}$ to $d\mu$, we have $\int f d\mu = \int f|D|^2 \frac{d\theta}{2\pi}$, that is, $d\mu_{\rm s}(I) = 0$. □

Theorem 2.2. *If*

$$\|\chi_I(\varphi_n^*)^{-1}\|_\infty \|(\varphi_n^* - D^{-1})\|_{L^2(\partial\mathbb{D},d\mu)} \to 0 \tag{2.6}$$

for some open interval I (including $I = \partial\mathbb{D}$), then $\mu_{\rm s}(I) = 0$.

Proof.

$$\begin{aligned} \|\chi_I[(\varphi_n^*)^{-1} - D]\|_{L^2(\partial\mathbb{D},d\theta/2\pi)} &= \|\chi_I(\varphi_n^*)^{-1}(\varphi_n^* - D^{-1})D\|_{L^2(\partial\mathbb{D},d\theta/2\pi)} \\ &\le \|\chi_I(\varphi_n^*)^{-1}\|_\infty \|(\varphi_n^* - D^{-1})\|_{L^2(\partial\mathbb{D},d\mu_{\rm ac})} \\ &\le \text{LHS of (2.6)} \end{aligned}$$

Thus the result follows from Theorem 2.1. □

Define $\rho_n = (1 - |\alpha_n|^2)^{1/2}$. A straightforward calculation shows that for $n < m$,

$$\langle \varphi_n^*, \varphi_m^* \rangle = \prod_{\ell=n}^{m-1} \rho_\ell$$

and, of course, $\langle \varphi_n^*, \varphi_n^* \rangle = 1$. Since [3] $\varphi_n^* \to D^{-1}$ in $L^2(\partial\mathbb{D}, d\mu)$, we have that

$$\|\varphi_n^* - D^{-1}\|^2_{L^2(\partial\mathbb{D},d\mu)} = 2\biggl(1 - \prod_{\ell=n}^\infty \rho_\ell\biggr) \tag{2.7}$$

Proposition 2.3. (a) $\|\varphi_n^* - D^{-1}\|^2_{L^2(\partial\mathbb{D},d\mu)} \le 2\sum_{\ell=n}^\infty |\alpha_\ell|^2$

(b) *If (1.4) holds, then $\sum_{n=0}^\infty |\alpha_n|^2 < \infty$.*

(c) *If (1.4) holds for any A, then*

$$\|\varphi_n^* - D^{-1}\|_{L^2(\partial\mathbb{D},d\mu)} \le \frac{C\log n}{n^{1/2}} \tag{2.8}$$

for all large n.

Proof. (a) $1-\prod_{\ell=n}^{\infty}\rho_\ell \leq 1-\prod_{\ell=n}^{\infty}\rho_\ell^2$ since $\rho_\ell^2 \leq \rho_\ell$. Moreover, $1-\prod_{\mathrm{J}=1}^{\ell} x_j \leq \sum_{j=1}^{\ell}(1-x_j)$ by a simple induction. Thus (2.3) implies (a).

(b),(c)

$$\begin{aligned}\sum_{n=2^\ell}^{2^{\ell+1}-1}|\alpha_n|^2 &\leq 2^{-\ell}\sum_{n=0}^{2^{\ell+1}} n|\alpha_n|^2 \\ &\leq 2^{-\ell}[A\log(2^{\ell+1})+C]\end{aligned}$$

Since

$$\sum_{\ell=k}^{\infty}\ell 2^{-\ell} \leq Ck2^{-k}$$

we see that (1.4) implies $\sum_{n=0}^{\infty}|\alpha_n|^2<\infty$ and

$$\sum_{\ell=n}^{\infty}|\alpha_\ell|^2 \leq Cn^{-1}\log n$$

Given (a), we get (c). □

Theorem 2.4. *If the α's obey* (1.4) *and for some open interval* $I\subset\partial\mathbb{D}$*, and* $B<\frac{1}{2}$*, we have*

$$\sup_{z\in I}[|\varphi_n(z)|^{-1}] \leq C(n+1)^B \tag{2.9}$$

for a constant C*, then* $\mu_{\mathrm{s}}(I)=0$*.*

Proof. For $z\in\partial\mathbb{D}$, $|\varphi_n^*(z)|=|\varphi_n(z)|$ so (2.9) says $\|\chi_I(\varphi_n^*)^{-1}\|_\infty \leq C(n+1)^B$. This and (2.8) implies that (2.6) holds, so $\mu_{\mathrm{s}}(I)=0$. □

The second kind polynomials, ψ_n,Ψ_n, are defined by reversing the signs of all the Verblunsky coefficients. It is known (see, e.g., [5]) that for $z\in\partial\mathbb{D}$, $\mathrm{Re}(\varphi_n(z)\bar\psi_n(z))=1$, so $|\varphi_n(z)|^{-1}\leq|\psi_n(z)|$.

Since $\psi_n=(\prod_{j=0}^{n-1}\rho_j^{-1})\Psi_n$ and $\sum\alpha_j^2<\infty$ implies $\prod_{j=0}^{\infty}\rho_j^{-1}<\infty$, we see $|\psi_n(z)|\leq C|\Psi_n|$. Thus, Theorem 2.4 can be rewritten:

Corollary 2.5. *If the α's obey* (1.4) *and for some open interval* $I\subset\partial\mathbb{D}$*, we have that*

$$\sup_{z\in I}|\Psi_n(z)| \leq C(n+1)^B$$

for a constant C*, and* $B<\frac{1}{2}$*, then* $\mu_{\mathrm{s}}(I)=0$*.*

Proof of Theorem 2. (1.2) implies $|\Psi_n^*|=|\Psi_n|$ on $\partial\mathbb{D}$ and so (1.1) implies $|\Psi_{n+1}|\leq(1+|\alpha_n|)|\Psi_n|$. Thus

$$\sup_{z\in\partial\mathbb{D}}|\Psi_n(z)| \leq \prod_{j=0}^{n-1}1+|\alpha_j| \leq \exp\biggl(\sum_{j=0}^{n-1}|\alpha_j|\biggr) \tag{2.10}$$

But, by the Schwartz inequality, (C is a "variable constant" and ε arbitrarily small)

$$\begin{aligned}\sum_{j=0}^{n}|\alpha_j| &\leq \biggl(\sum_{j=0}^{n}(j+1)|\alpha_j|^2\biggr)^{1/2}\biggl(\sum_{j=0}^{n}(j+1)^{-1}\biggr)^{1/2} \\ &\leq [A\log(n)+C]^{1/2}\,[\log(n)+C]^{1/2} \\ &\leq \sqrt{A+\varepsilon}\,\log(n)+C\end{aligned} \tag{2.11}$$

Thus, by (2.10),

$$\sup_{z\in\mathbb{D}}|\Psi_n(z)| \leq Cn^{\sqrt{A+\varepsilon}} \tag{2.12}$$

If $\sqrt{A+\varepsilon} < \frac{1}{2}$, that is, $A < \frac{1}{4}$, Corollary 2.5 is applicable. □

We emphasize that, in essence, the calculation in Proposition 2.3(a), (2.10), and the basic strategy are all from [4]; the only real advance in this section is the use of L^2-norms allowing $A < \frac{1}{4}$, where the method of [4] gets $A < \frac{1}{16}$.

Finally, we note $A = \frac{1}{4}$ is a critical value for the appearance of bound states, for

Theorem 2.6. *If all α_n are real, then $\varphi_n(1) > 0$ and*

$$\varphi_n(1) = \prod_{j=0}^{n-1}(1-\alpha_j) \tag{2.13}$$

In addition, $(-1)^n\varphi_n(-1) > 0$ and

$$(-1)^n\varphi_n(-1) = \prod_{j=0}^{n-1}(1-(-1)^{j+1}\alpha_j) \tag{2.14}$$

Remark. In particular, if $\alpha_j = B(j+1)^{-1}$, then $\varphi_n(1) \sim n^{-B}$ and $\sum|\varphi_n(1)|^2 < \infty$ if $B > \frac{1}{2}$. For that α, (1.4) holds for $A = B^2$, that is, there are examples with $d\mu_{\rm s} \neq 0$ and (1.4) holding for any $A > \frac{1}{4}$. At $A = \frac{1}{4}$, our proof of Theorem 2 shows $|\alpha_n(z)| \geq Cn^{-1/2}$, so at least there are no eigenvalues.

Proof. $\varphi_n(1)$ is real by induction and then (1.1) says

$$\varphi_{n+1}(1) = (1-\alpha_n)\varphi_n(1)$$

proving (2.13). Similarly, $\varphi_n(-1)$ is real. (1.1) for $z=-1$ says

$$(-1)^{n+1}\varphi_{n+1}(-1) = (-1)^n\varphi_n(-1) - \alpha_n(-1)^{n+1}(-1)^n\varphi_n(-1)$$

since $\varphi_n^*(-1) = (-1)^n\varphi_n(-1)$, which yields (2.14). □

3. Prüfer Variables and the Proof of Theorem 3

Write $\Psi_n(z) = R_n(z)e^{i[\theta_n(z)+n\eta(z)]}$ where $e^{i\eta(z)} = z$ and θ_n is defined initially only modulo 2π. If α_n is real, (1.1) with $\alpha_n \to -\alpha_n$ becomes $R_{n+1}^2 = R_n^2|e^{i\eta}e^{i[\theta_n+n\eta]} + \alpha_n e^{-i\theta_n}|^2$.

Thus,

$$\frac{R_{n+1}^2}{R_n^2} = 1 + \alpha_n^2 + 2\alpha_n \cos((n+1)\eta + 2\theta_n) \tag{3.1}$$

Secondly,

$$e^{i(\theta_{n+1}(z)-\theta_n(z))} = \frac{1+\alpha_n e^{-i(2\theta_n+(n+1)\theta_n)}}{[1+\alpha_n^2+2\alpha_n\cos((n+1)\eta+2\theta_n)]^{1/2}} \tag{3.2}$$

These are the Prüfer variable equations for Ψ. (3.2) implies $\cos(\theta_{n+1} - \theta_n) > 0$, so we can always pick θ_{n+1} so $|\theta_{n+1} - \theta_n| < \frac{\pi}{2}$, settling the 2π ambiguity.

Proposition 3.1. *Let α_n be real and $\sum \alpha_n^2 < \infty$. Define $Q = \sup_n |\alpha_n| < 1$. Then*

(a) *For a z independent constant $C \in (1,\infty)$,*

$$C^{-1} \leq R_n \exp\biggl(-\sum_{j=0}^{n-1} \alpha_j \cos((j+1)\eta + 2\theta_j(z))\biggr) \leq C \tag{3.3}$$

(b) *For all n,*

$$|\theta_{n+1} - \theta_n| \leq \tfrac{\pi}{2}\, \tfrac{1}{1-Q}\, |\alpha_n| \tag{3.4}$$

Remark. C depends only on Q and $\sum_{j=0}^\infty \alpha_j^2$.

Proof. (a) Define $b_n(z)$ so the right side of (3.1) is $1 + b_n$. Then $(1-Q)^2 \leq (1-|\alpha_n|)^2 \leq 1 + b_n \leq (1+\alpha_n)^2 \leq (1+Q)^2$. It follows that for a Q dependent constant K, we have $e^{-Kb_n^2} \leq (1+b_n)e^{-b_n} \leq e^{Kb_n^2}$. Since $b_n \leq 3|\alpha_n|$ and b_n has an α_n^2 in it, we have

$$\exp(-(9K+1)\alpha_n^2) \leq (1+b_n)\exp(-2\alpha_n\cos((n+1)\eta+2\theta_n)) \leq \exp((9K+1)\alpha_n^2)$$

Thus, by (3.1) and $R_0 = 1$, we have (3.3) with

$$C = \exp\biggl((9K+1)\sum_{j=0}^\infty \alpha_j^2\biggr)$$

(b) Taking imaginary parts of both sides of (3.2) and using the lower bound $1-Q$ on the denominator, we get

$$|\sin(\theta_{n+1} - \theta_n)| \leq |\alpha_n|(1-Q)^{-1} \tag{3.5}$$

(3.2) also implies $\cos(\theta_{n+1} - \theta_n) > \frac{(1-Q)}{(1+Q)} > 0$, so $|\theta_{n+1} - \theta_n| < \frac{\pi}{2}$. Since $|x_n| < \frac{\pi}{2}$ implies $|x| \leq \frac{\pi}{2}|\sin x|$, (3.4) follows from (3.5). □

The point of (3.3) is to control $|R_n|$, we need to control $\sum_{j=0}^{n-1} \alpha_j \cos((j+1)\eta + 2\theta_j)$. In using the Schwartz inequality, we will decouple the α_j's and the cosines so the key will be the following (essentially in [1]):

Lemma 3.2. (i) *If k is not a multiple of 2π,*

$$\left|\sum_{j=1}^{n} \frac{\cos(kj+\theta_j)}{j}\right| \leq \frac{1}{|\sin(\frac{k}{2})|}\left[1+\sum_{j=1}^{n-1}\frac{|\theta_{j+1}-\theta_j|}{j+1}\right] \tag{3.6}$$

(ii) *If k is not a multiple of π,*

$$\sum_{j=1}^{N} \frac{\cos^2(kj+\theta_j)}{j} \leq \tfrac{1}{2}\left(\log N + 1 + C\right) \tag{3.7}$$

where

$$C = \frac{1}{|\sin(k)|}\left[1+2\sum_{j=1}^{\infty}\frac{|\theta_{j+1}-\theta_j|}{j+1}\right] \tag{3.8}$$

Proof. Since $\cos(kj+\theta_j) = \mathrm{Re}(\exp(ikj)\exp(i\theta_j))$, it suffices to prove (3.6) with cosines replaced by complex exponentials . Define $b_n = \sum_{j=1}^{n} e^{ikj}$ so, by summing the geometric series,

$$|b_n| \leq \frac{1}{|\sin(\frac{k}{2})|} \tag{3.9}$$

If

$$a_j = \frac{e^{i\theta_j}}{j}$$

then the sum we want to control is $\sum_{j=1}^{n}(b_j - b_{j-1})a_j$ with $b_0 = 0$. But

$$\sum_{j=1}^{n}(b_j - b_{j-1})a_j = -\sum_{j=1}^{n} b_j(a_{j+1}-a_j)$$

where $a_{n+1} = 0$. Thus

$$\left|\sum_{j=1}^{n} \frac{\cos(kj+\theta_j)}{j}\right| \leq \frac{1}{\sin(\frac{k}{2})}\left[\sum_{j=1}^{n-1}|a_{j+1}-a_j| + |a_n|\right] \tag{3.10}$$

Clearly,

$$\begin{aligned} |a_{j+1}-a_j| &\leq |(e^{i\theta_{j+1}} - e^{i\theta_j})(j+1)^{-1}| + |e^{i\theta_j}(j^{-1}-(j+1)^{-1})| \\ &\leq |\theta_{j+1}-\theta_j|(j+1)^{-1} + j^{-1} - (j+1)^{-1} \end{aligned} \tag{3.11}$$

Since $\sum_{j=1}^{n-1} j^{-1} - (j+1)^{-1} + n^{-1} = 1$, (3.10) and (3.11) yield (3.6).

(b) Since $\cos^2(x) = \frac{1}{2}(1+\cos(2x))$ and

$$\sum_{j=1}^{n} \tfrac{1}{j} \leq 1 + \int_1^n \tfrac{dx}{x} = 1 + \log(n)$$

(3.6) implies (3.7). □

Proof of Theorem 3. Write

$$\begin{aligned} &\Big|\sum_{j=0}^{n-1} \alpha_j \cos((j+1)\eta + 2\theta_j)\Big| \\ &\qquad \leq \bigg(\sum_{j=0}^{n-1} (j+1)|\alpha_j|^2\bigg)^{1/2} \bigg(\sum_{j=0}^{n-1} \frac{\cos^2((j+1)\eta + 2\theta_j)}{(j+1)}\bigg)^{1/2} \end{aligned} \tag{3.12}$$

By hypothesis, the first sum on the right side of (3.12) is bounded by $A\log N + C$. By the lemma, if η is not a multiple of π (i.e., $z \neq \pm 1$), the second sum is bounded by

$$\tfrac{1}{2}\,(\log n + \widetilde{C}|\sin(\eta)|^{-1})$$

where $\widetilde{C}$ can be chosen independently of z, since

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{|\theta_{j+1}-\theta_j|}{j+1} &\leq C \sum_{j=0}^{\infty} \frac{|\alpha_j|}{j+2} \qquad \text{(by (3.5))} \\ &\leq C\bigg(\sum_{j=0}^{\infty} |\alpha_j|^2\bigg)^{1/2} \bigg(\sum_{j=0}^{\infty} \frac{1}{(j+2)^2}\bigg)^{1/2} \end{aligned}$$

is finite. Thus, for $\eta \in [\theta_0, \pi - \theta_0]$,

$$\begin{aligned} |R_n| &\leq C \exp|[A(\log n) + C]|^{1/2} [\tfrac{1}{2}\log n + \widetilde{C}|\sin\theta_0|^{-1}]^{1/2} \\ &\leq C_\varepsilon n^{\sqrt{\frac{1}{2}A + \varepsilon}} \end{aligned}$$

where C_ε depends on ε and θ_0. So long as $\sqrt{\frac{1}{2}A} < \frac{1}{2}$, we can apply Corollary 2.5, that is, $A < \frac{1}{2}$. We conclude $d\mu_{\rm s}(I) = 0$ for $I = \pm(\theta_0, \pi - \theta_0)$, that is, $d\mu_{\rm s}$ is supported on $\{\pm 1\}$. As already noted, at ± 1, $|\varphi_n(\pm 1)| \geq n^{-\sqrt{A}}$, so if $A = \frac{1}{4}$, $\varphi_n(\pm 1)$ are not in L^2 and $d\mu_{\rm s} = 0$. □

4. Sequences of Bounded Variation and the Proof of Theorem 4

To obtain Theorem 4, we need one more summation-by-parts argument that will supplement Lemma 3.2:

Lemma 4.1. *If k is not a multiple of 2π,*

$$\begin{aligned}\Big|\sum_{j=1}^{n} c_j \cos(kj+\theta_j)\Big| \\ \leq \frac{1}{\sin(\frac{k}{2})}\Big[\sum_{j=1}^{\infty}|c_{j+1}-c_j| + \sup_n |c_n| + \sum_{j=1}^{\infty}|c_{j+1}|\,|\theta_{j+1}-\theta_j|\Big]\end{aligned} \tag{4.1}$$

Proof. As in the proof of Lemma 3.2, let $b_n = \sum_{j=1}^{n} e^{ikj}$, so (3.9) holds and $a_k = c_j e^{i\theta_j}$. Then summing by parts as in the earlier lemma,

$$\Big|\sum_{j=1}^{n} c_j \cos(kj+\theta_j)\Big| \leq \frac{1}{\sin(\frac{k}{2})}\Big[|c_n| + \sum_{j=1}^{n-1}|a_{j+1}-a_j|\Big]$$

But

$$|a_{j+1}-a_j| \leq |a_{j+1}|\,|e^{i\theta_{j+1}} - e^{i\theta_j}| + |c_{j+1}-c_j|$$

so (4.1) follows. □

Proof of Theorem 4. By (3.3) and the hypothesis that $\sum_{j=0}^{\infty}|\alpha_j|^2$,

$$|\Psi_n(z)| \leq C \exp\Big(\sum_{j=0}^{n-1}\alpha_j \cos((j+1)\eta + 2\theta_j)\Big)$$

Write

$$\sum_{j=0}^{n-1}\alpha_j\cos((j+1)\eta+2\theta_j) = O_n + E_n \tag{4.2}$$

where O_n is the sum over odd values of j and E_n over even values.

By Lemma 4.1,

$$\begin{aligned}|O_n| \leq \frac{1}{|\sin(\frac{\eta}{2})|}\Big[\sup_n|\alpha_{2n-1}| &+ \sum_{n=1}^{\infty}|\alpha_{2n+1}-\alpha_{2n-1}| \\ &+ \sum_{n=1}^{\infty}|\alpha_{2n+1}|\,|\theta_{2n+1}-\theta_{2n-1}|\Big]\end{aligned} \tag{4.3}$$

Since α_{2n-1} is monotone, $\sum_{n=1}^{\infty}|\alpha_{2n+1}-\alpha_{2n-1}| = |\alpha_1|$. By (3.4), $|\theta_{2n+1}-\theta_{2n-1}| \leq C(|\alpha_{2n-1}|+|\alpha_{2n}|)$ so, since $\sum_{n=0}^{\infty}|\alpha_n|^2<\infty$, (4.3) implies

$$|O_n| \leq C|\sin(\tfrac{\eta}{2})|^{-1} \tag{4.4}$$

For E_n, we use the same argument as in the proof of Theorem 3 taking into account that the change in frequency from $2n$ to $2n+2$ is $2\eta + (\theta_{2n+2}-\theta_{2n})$. Thus $|\sin(\eta)|^{-1}$ becomes $|\sin(2\eta)|^{-1}$, and we find that

$$|E_n| \leq (A\log N + C)^{1/2}(\tfrac{1}{2}\log N + C|\sin(2\eta)|^{-1})^{1/2} \tag{4.5}$$

(4.4) and (4.5) imply that for any $\theta_0 > 0$ for all $z \in (\theta_0, \frac{\pi}{2} - \theta_0) \cup (\frac{\pi}{2} + \theta_0, \pi - \theta_0)$,

$$|\Psi_n| \leq C_{\theta_0,\varepsilon} n^{\varepsilon + \sqrt{A/2}}$$

which, by Corollary 2.5, implies $\mu_{\rm s}$ is restricted to $\pm 1, \pm i$.

To obtain the result on eigenvalues when $A = \frac{1}{4}$, note first that since $|E_n| \leq \sum_{m=1}^{[n/2]} |\alpha_{2m}|$, uniformly in z,

$$e^{|E_n|} \leq C n^{1/2}$$

if $A = \frac{1}{4}$. At $z = \pm i$, (4.4) implies $e^{|O_n|}$ is bounded, so $|\varphi_n(\pm i)| \geq C n^{-1/2}$ is not in L^2.

At ± 1, we use Theorem 2.6. Since $\alpha_{2j-1} \leq 0$, (2.13) implies

$$|\varphi_n(\pm 1)| \geq \prod_{\substack{j=0 \\ j \text{ even}}}^{n} (1 - |\alpha_j|) \geq C n^{-1/2}$$

since $1 - \alpha_{2j-1} \geq 1$. So ± 1 are not eigenvalues if $\alpha_{2j-1} \leq 0$. □

References

[1] D. Damanik and R. Killip, *Half-line Schrödinger operators with no bound states*, preprint.

[2] T. Erdélyi, P. Nevai, J. Zhang, and J. Geronimo, *A simple proof of "Favard's theorem" on the unit circle*, Atti Sem. Mat. Fis. Univ. Modena **39** (1991), 551–556.

[3] G. Freud, *Orthogonal Polynomials*, Pergamon Press, Oxford-New York, 1971.

[4] B.L. Golinskii and I.A. Ibragimov, *On Szegő's limit theorem*, Math. USSR Izv. **5** (1971), 421–444.

[5] L. Golinskii and P. Nevai, *Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle*, Comm. Math. Phys. **223** (2001), 223–259.

[6] S. Jitomirskaya and Y. Last, *Power-law subordinacy and singular spectra, I. Half-line operators*, Acta Math. **183** (1999), 171–189.

[7] C. Remling, *The absolutely continuous spectrum of one-dimensional Schrödinger operators with decaying potentials*, Comm. Math. Phys. **193** (1998), 151–170.

[8] B. Simon, *Orthogonal Polynomials on the Unit Circle*, AMS Book Series, expected 2004.

[9] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, American Mathematical Society, Providence, R.I., 1939; 3rd edition, 1967.