ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE: NEW RESULTS

BARRY SIMON

ABSTRACT. We announce numerous new results in the theory of orthogonal polynomials on the unit circle.

1. Introduction

I am completing a comprehensive look at the theory of orthogonal polynomials on the unit circle (OPUC; we'll use OPRL for the real-line case). These two 500+-page volumes [124, 125] to appear in the same AMS series that includes Szegő's celebrated 1939 book [137] contain numerous new results. Our purpose here is to discuss the most significant of these new results. Besides what we say here, some joint new results appear instead in papers with I. Nenciu [93], Totik [128], and Zlatoš [130]. We also note that some of the results I discuss in this article are unpublished joint work with L. Golinskii (Section 3.2) and with Denisov (Section 4.2). Some other new results appear in [126].

Throughout, $d\mu$ will denote a nontrivial (i.e., not a finite combination of delta functions) probability measure on $\partial\mathbb{D}$, the boundary of $\mathbb{D} = \{z \mid |z| < 1\}$. We'll write

$$d\mu(\theta) = w(\theta)\,\frac{d\theta}{2\pi} + d\mu_{\rm s}(\theta) \tag{1.1}$$

where $d\mu_{\rm s}$ is singular and $w \in L^1(\partial\mathbb{D}, \frac{d\theta}{2\pi})$.

Given $d\mu$, one forms the monic orthogonal polynomials, $\Phi_n(z; d\mu)$, and orthonormal polynomials

$$\varphi_n(z; d\mu) = \frac{\Phi_n(z; d\mu)}{\|\Phi_n\|_{L^2}} \tag{1.2}$$

If one defines

$$\alpha_n = -\overline{\Phi_{n+1}(0)} \tag{1.3}$$

Date: May 5, 2004.

* Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125. E-mail: bsimon@caltech.edu. Supported in part by NSF grant DMS-0140592.

then the Φ's obey a recursion relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z) \tag{1.4}$$

where * is defined on degree n polynomials by

$$P_n^*(z) = z^n \, \overline{P_n(1/\bar{z})} \tag{1.5}$$

(1.4) is due to Szegő [137]. The cleanest proofs are in Atkinson [3] and Landau [77]. The α_n are called Verblunsky coefficients after [146]. Since Φ_n^* is orthogonal to Φ_{n+1}, (1.4) implies

$$\|\Phi_{n+1}\|^2 = (1-|\alpha_n|^2)\|\Phi_n\|^2 \tag{1.6}$$

$$= \prod_{j=0}^{n} (1-|\alpha_j|^2) \tag{1.7}$$

It is a fundamental result of Verblunsky [146] that $\mu \mapsto \{\alpha_n\}_{n=0}^\infty$ sets up a one-one correspondence between nontrivial probability measures and $\times_{n=0}^\infty \mathbb{D}$.

A major focus in the book [124, 125] and in our new results is the view of $\{\alpha_n\}_{n=0}^\infty \leftrightarrow \mu$ as a spectral theory problem analogous to the association of V to the spectral measure $-\frac{d}{dx^2} + V(x)$ or of Jacobi parameters to a measure in the theory of OPRL.

We divide the new results in major sections: Section 2 involving relations to Szegő's theorem, Section 3 to the CMV matrix, Section 4 on miscellaneous results, Section 5 on the case of periodic Verblunsky coefficients, and Section 6 to some interesting spectral theory results in special classes of Verblunsky coefficients.

I'd like to thank P. Deift, S. Denisov, L. Golinskii, S. Khruschchev, R. Killip, I. Nenciu, P. Nevai, F. Peherstorfer, V. Totik, and A. Zlatoš for useful discussions.

2. Szegő's Theorem

In the form first given by Verblunsky [147], this says, with μ given by (1.1), that

$$\prod_{j=0}^{\infty} (1-|\alpha_j|^2) = \exp\left(\int_0^{2\pi} \log(w(\theta)) \, \frac{d\theta}{2\pi} \right) \tag{2.1}$$

2.1. **Szegő's Theorem via Entropy.** The sum rules of Killip-Simon [70] can be viewed as an OPRL analog of (2.1) so, not surprisingly, (2.1) has a "new" proof that mimics that in [70]. Interestingly enough, while the proof in [70] has an easy half that depends on semicontinuity of the entropy and a hard half (that even after simplifications in [129, 123] is not so short), the analog of the hard half for (2.1) follows in a few lines from Jensen's inequality and goes back to Szegő in 1920 [134, 135]. Here's how this analogous proof goes (see [124, Section 2.3] for details):

(a) (well-known, goes back to Szegő [134, 135]). By (1.7),

$$\prod_{j=0}^{n}(1-|\alpha_j|^2) \geq \int \exp[\log(w(\theta)) + \log|\Phi_n^*(e^{i\theta})|^2]\,\frac{d\theta}{2\pi} \tag{2.2}$$

$$\geq \exp\biggl(\int \log(w(\theta)) + 2\log|\Phi_n^*(e^{i\theta})|\biggr)\frac{d\theta}{2\pi} \tag{2.3}$$

$$= \exp\biggl(\int_0^{2\pi} \log(w(\theta))\,\frac{d\theta}{2\pi}\biggr) \tag{2.4}$$

where (2.2) uses $d\mu \geq w(\theta)\frac{d\theta}{2\pi}$, (2.3) is Jensen's inequality, and (2.4) uses the fact that since Φ_n^* is nonvanishing in $\bar{\mathbb{D}}$, $\log|\Phi_n^*(z)|$ is harmonic there and $\Phi_n^*(0)=1$.

(b) The map $d\mu \mapsto \int_0^{2\pi}\log(w(\theta))\frac{d\theta}{2\pi}$ is a relative entropy and so weakly upper semicontinuous in μ by a Gibbs variational principle:

$$\int_0^{2\pi} \log(w(\theta))\,\frac{d\theta}{2\pi} = \inf_{\substack{f\in C(\partial\mathbb{D})\\ f>0}} \biggl[\int f(\theta)\,d\mu(\theta) - 1 - \int \log(f(\theta))\,\frac{d\theta}{2\pi}\biggr] \tag{2.5}$$

(c) By a theorem of Geronimus [37], if

$$d\mu_n(\theta) = \frac{d\theta}{2\pi|\varphi_n(e^{i\theta})|^2} \tag{2.6}$$

(the Bernstein-Szegő approximations), then $d\mu_n \to d\mu$ weakly and the Verblunsky coefficients of $d\mu_n$ obey

$$\alpha_j(d\mu_n) = \begin{cases} \alpha_j(d\mu) & j=0,\dots,n-1 \\ 0 & j\geq n \end{cases} \tag{2.7}$$

Therefore, by the weak upper semicontinuity of (b),

$$\int_0^{2\pi} \log(w(\theta))\,d\mu \geq \limsup_{n\to\infty}\int_0^{2\pi} -\log(|\varphi_n(e^{i\theta})|^2)\,\frac{d\theta}{2\pi} \tag{2.8}$$

(d) Since $|\varphi_n(e^{i\theta})| = |\varphi_n^*(e^{i\theta})| = \prod_{j=0}^{n-1}(1-|\alpha_j|^2)^{-1/2}|\Phi_n^*(e^{i\theta})|$, the same calculation that went from (2.3) to (2.4) shows

$$\exp\biggl[\int_0^{2\pi} -\log(|\varphi_n(e^{i\theta})|^2)\,\frac{d\theta}{2\pi}\biggr] = \prod_{j=0}^{n-1}(1-|\alpha_j|^2) \tag{2.9}$$

(2.4), (2.8), and (2.9) imply (2.1) and complete the sketch of this proof.

We put "new" in front of this proof because it is closely related to the almost-forgotten proof of Verblunsky [147] who, without realizing he was dealing with an entropy or a Gibbs principle, used a formula close to (2.5) in his initial proof of (2.1)

The interesting aspect of this entropy proof is how $d\mu_{\rm s}$ is handled en passant — its irrelevance is hidden in (2.5).

2.2. **A Higher-Order Szegő Theorem.** (2.1) implies

$$\sum_{j=0}^\infty |\alpha_j|^2 < \infty \Leftrightarrow \int_0^{2\pi} \log(w(\theta))\,\frac{d\theta}{2\pi} > -\infty \tag{2.10}$$

The following result of the same genre is proven as Theorem 2.8.1 in [124]:

Theorem 2.1. *For any Verblunsky coefficients* $\{\alpha_j\}_{j=0}^\infty$,

$$\sum_{j=0}^\infty |\alpha_{j+1}-\alpha_j|^2 + \sum_{j=0}^\infty |\alpha_j|^4 < \infty \Leftrightarrow \int_0^{2\pi} (1-\cos(\theta))\log(w(\theta))\,\frac{d\theta}{2\pi} > -\infty \tag{2.11}$$

The proof follows the proof of (2.10) using the sum rule

$$\exp\biggl(-\tfrac{1}{2}\,|\alpha_0|^2 - \text{Re}(\alpha_0) + \tfrac{1}{2}\sum_{j=0}^\infty |\alpha_{j+1}-\alpha_j|^2\biggr)\prod_{j=0}^\infty (1-|\alpha_j|^2)e^{|\alpha_j|^2} = \exp\biggl(\int_0^{2\pi}(1-\cos(\theta))\log(w(\theta))\,\frac{d\theta}{2\pi}\biggr) \tag{2.12}$$

in place of (2.1) The proof of (2.12) is similar to the proof of (2.1) sketched in Section 2.1. For details, see [124, Section 2.8].

Earlier than this work, Denisov [25] proved that when the left side of (2.11) is finite, then $w(\theta) > 0$ for a.e. θ. In looking for results like (2.10), we were motivated in part by attempts of Kupin [75, 76] and Latpev et al. [78] to extend the OPUC results of Killip-Simon (see also

[92]). After Theorem 2.1 appeared in a draft of [124], Denisov-Kupin [28] and Simon-Zlatos [130] discussed higher-order analogs.

2.3. **Relative Szegő Function.** In the approach to sum rules for OPRL called step-by-step, a critical role is played by the fact that if m is the m-function for a Jacobi matrix, J, and m_1 is the m-function for J_1, the matrix obtained from J by removing one row and column, then

$$\frac{\operatorname{Im} m_1(E+i0)}{\operatorname{Im} m(E+i0)} = |a_1 m(E+i0)|^2 \tag{2.13}$$

The most obvious analog of the m-function for OPUC is the Carathéodory function

$$F(z) = \int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta) \tag{2.14}$$

If $\{\alpha_j\}_{j=0}^\infty$ are the Verblunsky coefficients of $d\mu$, the analog of m_1 is obtained by letting $\beta_j = \alpha_{j+1}$ and $d\mu_1$ the measure with $\alpha_j(d\mu_1) = \beta_j$ and $d\mu_1 = w_1(\theta)\frac{d\theta}{2\pi} + d\mu_{1,\mathrm{s}}$.

For $\frac{d\theta}{2\pi}$-a.e. $\theta \in \partial\mathbb{D}$, $F(e^{i\theta}) \equiv \lim_{r\uparrow 1} F(re^{i\theta})$ has a limit and

$$w(\theta) = \operatorname{Re} F(e^{i\theta}) \tag{2.15}$$

Thus, as in (2.13), we are interested in $\operatorname{Re} F(e^{i\theta})/\operatorname{Re} F_1(e^{i\theta})$ which, unlike (2.13), is not simply related to $F(e^{i\theta})$. Rather, there is a new object $(\delta_0 D)(z)$ which we have found whose boundary values have a magnitude equal to the square root of $\operatorname{Re} F(e^{i\theta})/\operatorname{Re} F_1(e^{i\theta})$.

To define $\delta_0 D$, we recall the Schur function, f, of $d\mu$ is defined by

$$F(z) = \frac{1+zf(z)}{1-zf(z)} \tag{2.16}$$

f maps $\mathbb{D}$ to $\mathbb{D}$ and (2.14)/(2.16) set up a one-one correspondence between such f's and probability measures on $\partial\mathbb{D}$.

$\delta_0 D$, the relative Szegő function, is defined by

$$(\delta_0 D)(z) = \frac{1-\bar\alpha_0 f(z)}{\rho_0}\,\frac{1-zf_1}{1-zf} \tag{2.17}$$

where f_1 is the Schur function of $d\mu_1$. One has the following:

Theorem 2.2. *Let $d\mu$ be a nontrivial probability measure on $\partial\mathbb{D}$ and $\delta_0 D$ defined by* (2.17)*. Then*

(i) *$\delta_0 D$ is analytic and nonvanishing on $\mathbb{D}$.*
(ii) $\log(\delta_0 D) \in \cap_{p=1}^\infty H^p(\mathbb{D})$

(iii) *For* $\frac{d\theta}{2\pi}$*-a.e.* $e^{i\theta} \in \partial\mathbb{D}$ *with* $w(\theta) \neq 0$,

$$\frac{w(\theta)}{w_1(\theta)} = |\delta_0 D(e^{i\theta})|^2 \tag{2.18}$$

and, in particular,

$$\int_{w_1(\theta)\neq 0} \left| \log\left(\frac{w(\theta)}{w_1(\theta)} \right) \right|^p \frac{d\theta}{2\pi} < \infty$$

for all $p \in [1, \infty)$.

(iv) *If* $\sum_{j=0}^\infty |\alpha_j|^2 < \infty$, *then*

$$(\delta_0 D)(z) = \frac{D(z; d\mu)}{D(z; d\mu_1)}$$

where D *is the Szegő function.*

(v) *If* $\varphi_j(z; d\mu_1)$ *are the OPUC for* $d\mu_1$, *then for* $z \in \mathbb{D}$,

$$\lim_{n\to\infty} \frac{\varphi_{n-1}^*(z; d\mu_1)}{\varphi_n^*(z; d\mu)} = (\delta_0 D)(z)$$

For a proof, see [124, Section 2.9]. The key fact is the calculation in $\mathbb{D}$ that

$$\frac{\operatorname{Re} F(z)}{\operatorname{Re} F_1(z)} = \frac{|1 - \bar\alpha_0 f|^2}{1 - |\alpha_0|^2} \, \frac{|1 - z f_1|^2}{|1 - z f|^2} \, \frac{1 - |z|^2 |f|^2}{1 - |f|^2}$$

which follows from

$$\operatorname{Re} F(z) = \frac{1 - |z|^2 |f(z)|^2}{1 - |f(z)|^2}$$

and the Schur algorithm relating f and f_1,

$$z f_1 = \frac{f - \alpha_0}{1 - \bar\alpha_0 f} \tag{2.19}$$

One consequence of using $\delta_0 D$ is

Corollary 2.3. *Let* $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_\mathrm{s}$ *and* $d\nu = x(\theta)\frac{d\theta}{2\pi} + d\nu_\mathrm{s}$ *and suppose that for some* N *and* k,

$$\alpha_{n+k}(d\mu) = \alpha_n(d\nu)$$

for all $n > N$ *and that* $w(\theta) \neq 0$ *for a.e.* θ. *Then,* $\log(x(\theta)/w(\theta)) \in L^1$ *and*

$$\lim_{n\to\infty} \frac{\|\Phi_n(d\nu)\|^2}{\|\Phi_{n+k}(d\mu)\|^2} = \exp\left(\int \log\left(\frac{x(\theta)}{w(\theta)} \right) \frac{d\theta}{2\pi} \right)$$

$\delta_0 D$ is also central in the forthcoming paper of Simon-Zlatoš [130].

2.4. **Totik's Workshop.** In [143], Totik proved the following:

Theorem 2.4 (Totik [143])**.** *Let $d\mu$ be any measure on $\partial\mathbb{D}$ with* $\operatorname{supp}(d\mu) = \partial\mathbb{D}$*. Then there exists a measure $d\nu$ equivalent to $d\mu$ so that*

$$\lim_{n\to\infty} \alpha_n(d\nu) = 0 \tag{2.20}$$

This is in a section on Szegő's theorem because Totik's proof uses Szegő's theorem. Essentially, the fact that $\sum_{j=0}^\infty |\alpha_j|^2$ doesn't depend on $d\mu_{\text{s}}$ lets one control the a.c. part of the measure and changes of $\sum_{j=0}^\infty |\alpha_j|^2$. By redoing Totik's estimates carefully, one can prove the stronger (see [124, Section 2.10]):

Theorem 2.5. *Let $d\mu$ be any measure on $\partial\mathbb{D}$ with* $\operatorname{supp}(d\mu) = \partial\mathbb{D}$*. Then there exists a measure $d\nu$ equivalent to $d\mu$ so that for all $p > 2$,*

$$\sum_{n=0}^\infty |\alpha_n(d\nu)|^p < \infty \tag{2.21}$$

It is easy to extend this to OPRL and there is also a variant for Schrödinger operators; see Killip-Simon [71].

3. The CMV Matrix

One of the most interesting developments in the theory of OPUC in recent years is the discovery by Cantero, Moral, and Velázquez [14] of a matrix realization for multiplication by z on $L^2(\partial\mathbb{D}, d\mu)$ which is of finite width (i.e., $|\langle \chi_n, z\chi_m\rangle| = 0$ if $|m-n| > k$ for some k; in this case, $k = 2$ to be compared with $k = 1$ for OPRL). The obvious choice for basis, $\{\varphi_n\}_{n=0}^\infty$, yields a matrix (which [124] calls GGT after Geronimus [37], Gragg [52], and Teplyaev [139]) with two defects: If the Szegő condition, $\sum_{j=0}^\infty |\alpha_j|^2 < \infty$, holds, $\{\varphi_n\}_{n=0}^\infty$ is not a basis and $\mathcal{G}_{k\ell} = \langle \varphi_k, z\varphi_\ell\rangle$ is not unitary. In addition, the rows of $\mathcal{G}$ are infinite, although the columns are finite, so $\mathcal{G}$ is not finite width.

What CMV discovered is that if χ_n is obtained by orthonormalizing the sequence $1, z, z^{-1}, z^2, z^{-2}, \dots$, we always get a basis $\{\chi_n\}_{n=0}^\infty$, in which

$$\mathcal{C}_{nm} = \langle \chi_n, z\chi_m\rangle \tag{3.1}$$

is five-diagonal. The χ's can be written in terms of the φ's and φ^* (indeed, $\chi_{2n} = z^{-n}\varphi_{2n}^*$ and $\chi_{2n-1} = z^{-n+1}\varphi_{2n-1}$) and $\mathcal{C}$ in terms of the α's. The most elegant way of doing this was also found by CMV [14]; one can write

$$\mathcal{C} = \mathcal{L}\mathcal{M} \tag{3.2}$$

with

$$\mathcal{M} = \begin{pmatrix} 1 & & & \\ & \Theta_1 & & \\ & & \Theta_3 & \\ & & & \ddots \end{pmatrix} \qquad \mathcal{L} = \begin{pmatrix} \Theta_0 & & & \\ & \Theta_2 & & \\ & & \Theta_4 & \\ & & & \ddots \end{pmatrix} \tag{3.3}$$

where the 1 in $\mathcal{M}$ is a 1×1 block and all Θ's are the 2×2 block

$$\Theta_j = \begin{pmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix} \tag{3.4}$$

We let $\mathcal{C}_0$ denote the CMV matrix for $\alpha_j \equiv 0$.

The CMV matrix is an analog of the Jacobi matrix for OPRL and it has many uses; since [14, 15] only presented the formalization and a very few applications, the section provides numerous new OPUC results based on the CMV matrix.

3.1. **The CMV Matrix and the Szegő Function.** If the Szegő condition holds, one can define the Szegő function

$$D(z) = \exp\biggl(\int \frac{e^{i\theta}+z}{e^{i\theta}-z} \, \log(w(\theta)) \, \frac{d\theta}{4\pi}\biggr) \tag{3.5}$$

One can express D in terms of $\mathcal{C}$. We use the fact, a special case of Lemma 3.2 below, that

$$\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty \Rightarrow \mathcal{C} - \mathcal{C}_0 \text{ is Hilbert-Schmidt} \tag{3.6}$$

$$\sum_{j=0}^{\infty} |\alpha_j| < \infty \Rightarrow \mathcal{C} - \mathcal{C}_0 \text{ is trace class} \tag{3.7}$$

We also use the fact that if A is trace class, one can define [43, 120] $\det(1+A)$, and if A is Hilbert-Schmidt, $\det_2$ by

$$\det_2(1+A) \equiv \det((1+A)e^{-A}) \tag{3.8}$$

We also define w_n by

$$\log(D(z)) = \tfrac{1}{2}\, w_0 + \sum_{n=1}^{\infty} z^n w_n \tag{3.9}$$

so

$$w_n = \int e^{-in\theta} \log(w(\theta)) \, \frac{d\theta}{2\pi} \tag{3.10}$$

Here's the result:

Theorem 3.1. *Suppose* $\{\alpha_n(d\mu)\}_{n=1}^\infty$ *obeys the Szegő condition*

$$\sum_{n=0}^\infty |\alpha_n|^2 < \infty \tag{3.11}$$

Then the Szegő function, D, obeys for $z \in \mathbb{D}$,

$$D(0)D(z)^{-1} = \det{}_2\biggl(\frac{(1 - z\bar{\mathcal{C}})}{(1 - z\bar{\mathcal{C}}_0)}\biggr) e^{+zw_1} \tag{3.12}$$

where

$$w_1 = \alpha_0 - \sum_{n=1}^\infty \alpha_n \bar\alpha_{n-1} \tag{3.13}$$

If

$$\sum_{n=0}^\infty |\alpha_n| < \infty \tag{3.14}$$

then

$$D(0)D(z)^{-1} = \det\biggl(\frac{(1 - z\bar{\mathcal{C}})}{(1 - z\bar{\mathcal{C}}_0)}\biggr) \tag{3.15}$$

The coefficients w_n of (3.9) *are given by*

$$w_n = \frac{\overline{\mathrm{Tr}(\mathcal{C}^n - \mathcal{C}_0^n)}}{n} \tag{3.16}$$

for all $n \geq 1$ if (3.14) *holds and for $n \geq 2$ if* (3.11) *holds. In all cases, one has*

$$w_n = \sum_{j=0}^\infty \frac{\overline{(\mathcal{C}^n)}_{jj}}{n} \tag{3.17}$$

Remark. $\bar{\mathcal{C}}$ is the matrix $(\bar{\mathcal{C}})_{k\ell} = \overline{(\mathcal{C}_{k\ell})}$.

The proof (given in [124, Section 4.2]) is simple: by (4.12) below, Φ_n can be written as a determinant of a cutoff CMV matrix, which gives a formula for φ_n^*. Since $\varphi_n^* \to D^{-1}$, the cutoff matrices converge in Hilbert-Schmidt and trace norm and since $\det/\det_2$ are continuous, one can take limits of the finite formulae.

3.2. **CMV Matrices and Spectral Analysis.** The results in this subsection are joint with Leonid Golinskii. The CMV matrix provides a powerful tool for the comparison of properties of two measures $d\mu$, $d\nu$ on $\partial\mathbb{D}$ if we know something about $\alpha_n(d\nu)$ as a perturbation of $\alpha_n(d\mu)$. Of course, this idea is standard in OPRL and Schrödinger operators. For example, Krein [2] proved a theorem of Stieltjes [132] that $\mathrm{supp}(d\mu)$ has a single non-isolated point λ if and only if the Jacobi parameters $a_n \to 0$ and $b_n \to \lambda$ by noting both statements are equivalent to $J - \lambda 1$ being constant. Prior to results in this section, many results were proven using the GGT representation, but typically, they required $\liminf_{n\to\infty}|\alpha_n| > 0$ to handle the infinite rows.

Throughout this section, we let $d\mu$ (resp. $d\nu$) have Verblunsky coefficient α_n (resp. β_n) and we define $\rho_n = (1-|\alpha_n|^2)^{1/2}$, $\sigma_n = (1-|\beta_n|)^{1/2}$. An easy estimate using the $\mathcal{LM}$ factorization shows with $\|\cdot\|_p$ the $\mathcal{I}_p$ trace ideal norm [43, 120]:

Lemma 3.2. *There exists a universal constant C so that for all $1 \leq p \leq \infty$,*

$$\|\mathcal{C}(d\mu) - \mathcal{C}(d\nu)\|_p \leq C\biggl(\sum_{n=0}^{\infty} |\alpha_n - \beta_n|^p + |\rho_n - \sigma_n|^p\biggr)^{1/p} \tag{3.18}$$

Remark. One can take $C = 6$. For $p = \infty$, the right side of (3.18) is interpreted as $\sup_n(\max(|\alpha_n - \beta_n|, |\rho_n - \sigma_n|))$.

This result allows one to translate the ideas of Simon-Spencer [127] to a new proof of the following result of Rakhmanov [114] (sometimes called Rakhmanov's lemma):

Theorem 3.3. *If* $\limsup|\alpha_n| = 1$, *$d\mu$ is purely singular.*

Sketch. Pick a subsequence n_j so

$$\sum_{j=0}^{\infty} (1 - |\alpha_{n_j}|)^{1/2} < \infty \tag{3.19}$$

Let $\beta_k = \alpha_k$ if $k \neq n_j$ and $\beta_k = \alpha_k/|\alpha_k|$ if $k = n_j$. There is a limiting unitary $\tilde{\mathcal{C}}$ with those values of β. It is a direct sum of finite rank matrices since $|\beta_{n_j}| = 1$ forces $\mathcal{L}$ or $\mathcal{M}$ to have some zero matrix elements. Thus $\tilde{\mathcal{C}}$ has no a.c. spectrum.

By (3.19) and (3.18), $\mathcal{C} - \tilde{\mathcal{C}}$ is trace class, so by the the Kato-Birman theorem for unitaries [11], $\mathcal{C}$ has simply a.c. spectrum. □

Golinskii-Nevai [49] already remarked that Rakhmanov's lemma is an analog of [127]. For the next pair of results, the special case $\lambda_n \equiv 1$ are analogs of extended results of Weyl and Kato-Birman but for OPUC are new even in this case with the generality we have.

Theorem 3.4. *Suppose* $\{\lambda_n\}_{n=0}^{\infty} \in \partial\mathbb{D}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \in \mathbb{D}^{\infty}$ *and*

$$\text{(i)} \qquad \beta_n\lambda_n - \alpha_n \to 0$$

$$\text{(ii)} \qquad \lambda_{n-1}\bar{\lambda}_n \to 1$$

Then the derived sets of $\operatorname{supp}(d\mu)$ *and* $\operatorname{supp}(d\nu)$ *are equal, that is, up to a discrete set,* $\operatorname{supp}(d\mu)$ *and* $\operatorname{supp}(d\nu)$ *are equal.*

Theorem 3.5. *Suppose* $\{\lambda_n\}_{n=0}^\infty \in \partial\mathbb{D}^\infty$ *and* α_n, β_n *are the Verblunsky coefficients of* $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_{\rm s}$ *and* $d\nu = f(\theta)\frac{d\theta}{2\pi} + d\nu_{\rm s}$*. Suppose that*

$$\sum_{j=0}^\infty |\lambda_j \alpha_j - \beta_j| + |\lambda_{j+1}\bar\lambda_j - 1| < \infty$$

Then $\{\theta \mid w(\theta) \neq 0\} = \{\theta \mid f(\theta) \neq 0\}$ *(up to sets of* $d\theta/2\pi$ *measure* 0*).*

The proofs (see [124, Section 4.3]) combine the estimates of Lemma 3.2 and the fact that conjugation of CMV matrices with diagonal matrices can be realized as phase changes. That $\mathrm{supp}(d\mu) = \partial\mathbb{D}$ if $|\alpha_j| \to 0$ (special case of Theorem 3.4) is due to Geronimus [39]. Other special cases can be found in [7, 47].

[124, Section 4.3] also has results that use trial functions and CMV matrices. Trial functions are easier to use for unitary operators than for selfadjoint ones since linear variational principles for selfadjoint operators only work at the ends of the spectrum. But because $\partial\mathbb{D}$ is curved, linear variational principles work at any point in $\partial\mathbb{D}$. For example, $(\theta_0 - \varepsilon, \theta_0 + \varepsilon) \cap \mathrm{supp}(d\mu) = \emptyset$ if and only if

$$\mathrm{Re}(e^{-i\theta_0}\langle \psi, (e^{i\theta_0} - \mathcal{C})\psi\rangle) \geq 2\sin^2\left(\frac{\varepsilon}{2}\right)\|\psi\|^2$$

for all ψ. Typical of the results one can prove using trial functions is:

Theorem 3.6. *Suppose there exists* $N_j \to \infty$ *and* k_j *so*

$$\frac{1}{N_j}\sum_{\ell=1}^{N_j} |\alpha_{k_j+\ell}|^2 \to 0$$

Then $\mathrm{supp}(d\mu) = \partial\mathbb{D}$*.*

3.3. **CMV Matrices and the Density of Zeros.** A fundamental object of previous study is the density of zeros, $d\nu_n(z; d\mu)$, defined to give weight k/n to a zero of $\Phi_n(z; d\mu)$ of multiplicity k. One is interested in its limit or limit points as $n \to \infty$. A basic difference from OPRL is that for OPRL, any limit point is supported on $\mathrm{supp}(d\mu)$, while limits of $d\nu_n$ need not be supported on $\partial\mathbb{D}$. Indeed, for $d\mu = d\theta/2\pi$, $d\nu_n$ is a delta mass at $z = 0$ and [128] have found $d\mu$'s for which the limit points of $d\nu_n$ are all measures on $\bar{\mathbb{D}}$!

As suggested by consideration of the "density of states" for Schrödinger operators and OPRL (see [101, 5]), moments of the density of zeros are related to traces of powers of a truncated CMV matrix. Define $\mathcal{C}^{(n)}$ to be the matrix obtained from the topmost n rows and leftmost n columns of $\mathcal{C}$. Moreover, let $d\gamma_n$ be the Cesàro mean of $|\varphi_j|^2\,d\mu$, that is,

$$d\gamma_n(\theta) = \frac{1}{n}\sum_{j=0}^{n-1} |\varphi_j(e^{i\theta}, d\mu)|^2\, d\mu(\theta) \tag{3.20}$$

Then:

Theorem 3.7. *For any $k \geq 0$,*

$$\int z^k\, d\nu_n(z) = \frac{1}{n}\operatorname{Tr}((\mathcal{C}^{(n)})^k) \tag{3.21}$$

Moreover,

$$\lim_{n\to\infty}\biggl[\biggl(\int z^k\, d\nu_n(z)\biggr) - \biggl(\int z^k\, d\gamma_n(z)\biggr)\biggr] = 0 \tag{3.22}$$

Sketch. (For details, see [124, Section 8.2].) We'll see in Theorem 4.5 that the eigenvalues of $\mathcal{C}^{(n)}$ (counting geometric multiplicity) are the zeros of $\Phi_n(z;d\mu)$ from which (3.21) is immediate.

Under the CMV representation, δ_j corresponds to $z^\ell\varphi_j$ or $z^\ell\varphi_j^*$ for suitable ℓ (see the discussion after (3.1)) so

$$(\mathcal{C}^k)_{jj} = \int e^{ik\theta}|\varphi_j(e^{i\theta})|^2\, d\mu(\theta)$$

and thus

$$\int z^k\, d\gamma_n(z) = \frac{1}{n}\sum_{j=0}^{n-1}(\mathcal{C}^k)_{jj} \tag{3.23}$$

If $\ell < n-2k$,

$$([\mathcal{C}^{(n)}]^k)_{\ell\ell} = (\mathcal{C}^k)_{\ell\ell}$$

so that (3.22) follows from (3.21) and (3.23). □

From (3.21) and (3.18), we immediately get

Corollary 3.8. *If $\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}|\alpha_j - \beta_j| \to 0$, then for any k,*

$$\lim_{N\to\infty}\int z^k[d\nu_N(z;\{\alpha_j\}_{j=0}^\infty) - d\nu_N(z;\{\beta_j\}_{j=0}^\infty)] = 0 \tag{3.24}$$

One application of this is to a partially alternative proof of a theorem of Mhaskar-Saff [86]. They start with an easy argument that uses a theorem of Nevai-Totik [96] and the fact that $(-1)^{n+1}\bar\alpha_{n-1}$ is the product of zeros of $\Phi_n(z)$ to prove

Lemma 3.9. *Let*

$$A = \limsup |\alpha_n|^{1/n} \tag{3.25}$$

and pick n_j so

$$|\alpha_{n_j-1}|^{1/n_j-1} \to A \tag{3.26}$$

Then any limit points of $d\nu_{n_j}$ is supported on $\{z \mid |z| = A\}$.

They then use potential theory to prove the following, which can be proven instead using the CMV matrix:

Theorem 3.10 (Mhaskar-Saff [86])**.** *Suppose* (3.25) *and* (3.26) *hold and that either $A < 1$ or*

$$\lim_{n\to\infty} \frac{1}{N} \sum_{j=0}^{n-1} |\alpha_j| = 0 \tag{3.27}$$

Then $d\nu_{n_j}$ converges weakly to the uniform measure on the set $\{z \mid |z| = A\}$.

Sketch of New Proof. Since $d\theta/2\pi$ is the unique measure with $\int z^k \frac{d\theta}{2\pi} = \delta_{k0}$ for $k \geq 0$, it suffices to show that for $k \geq 1$,

$$\int z^k \, d\nu_{n_j} \to 0$$

This is immediate from Corollary 3.8 and the fact that $\int z^k d\tilde{\nu}_n = 0$ if $\tilde{\nu}_n$ is the zero's measure for $d\theta/2\pi$. □

3.4. **CMV and Wave operators.** In [125, Section 10.7], we prove the following:

Theorem 3.11. *Suppose $\sum_{n=0}^\infty |\alpha_n|^2 < \infty$. Let $\mathcal{C}$ be the CMV matrix for $\{\alpha_n\}_{n=0}^\infty$ and $\mathcal{C}_0$ the CMV matrix for $\alpha_j \equiv 0$. Then*

$$\operatorname*{s-lim}_{n\to\pm\infty} \mathcal{C}^n \mathcal{C}_0^{-n} = \Omega^\pm$$

exists and its range is $\chi_S(\mathcal{C})$ where S is a set with $d\mu_{\rm s}(S) = 0$, $|\partial\mathbb{D}\backslash S| = 0$.

The proof depends on finding an explicit formula for $\Omega^\pm$ (in terms of $D(z)$, the Szegő function); equivalently, from the fact that in a suitable sense, $\mathcal{C}$ has no dispersion. The surprise is that one only needs $\sum_{n=0}^\infty |\alpha_n|^2 < \infty$, not $\sum_{n=0}^\infty |\alpha_n| < \infty$. Some insight can be obtained from the formulae Geronimus [38] found mapping to a Jacobi matrix when the α's are real. The corresponding a's and b's have the form $c_{n+1} - c_n + d_n$ where $d_n \in \ell^1$ and $c_n \in \ell^2$, so there are expected to be modified wave operators with finite modifications since $c_{n+1} - c_n$ is conditionally summable.

Simultaneous with our discovery of Theorem 3.11, Denisov [27] found a similar result for Dirac operators.

3.5. **The Resolvent of the CMV Matrix.** I have found an explicit formula for the resolvent of the CMV matrix $(\mathcal{C}-z)^{-1}_{k\ell}$ when $z\in\mathbb{D}$ (and for some suitable limits as $z\to\partial\mathbb{D}$), not unrelated to a formula for the resolvent of the GGT matrix found by Geronimo-Teplyaev [36] (see also [34, 35]).

Just as the CMV basis, χ_n, is the result of applying Gram-Schmidt to orthonormalize $\{1,z,z^{-1},z^2,z^{-2},\dots\}$, the alternate CMV basis, x_n, is what we get by orthonormalizing $\{1,z^{-1},z,z^{-2},z^2,\dots\}$. (One can show $\tilde{\mathcal{C}}=\langle x.,zx.\rangle=\mathcal{M}\mathcal{L}$.) Similarly, let y_n,Υ_n be the CMV and alternate CMV bases associated to $(\psi_n,-\psi_n^*)$. Define

$$p_n=y_n+F(z)x_n \tag{3.28}$$

$$\pi_n=\Upsilon_n+F(z)\chi_n \tag{3.29}$$

Then

Theorem 3.12. *We have that for $z\in\mathbb{D}$,*

$$[(\mathcal{C}-z)^{-1}]_{k\ell}=\begin{cases}(2z)^{-1}\chi_\ell(z)p_k(z) & k>\ell \text{ or } k=\ell=2n-1\\ (2z)^{-1}\pi_\ell(z)x_k(z) & \ell>k \text{ or } k=\ell=2n\end{cases} \tag{3.30}$$

This is proven in [124, Section 4.4]. It can be used to prove Khrushchev's formula [68] that the Schur function for $|\varphi_n|^2\,d\mu$ is $\varphi_n(\varphi_n^*)^{-1}f(z;\{\alpha_{n+j}\}_{j=0}^\infty)$; see [125, Section 9.2].

3.6. **Rank Two Perturbations and CMV Matrices.** We have uncovered some remarkably simple formulae for finite rank perturbations of unitaries. If U and V are unitary so $U\varphi=V\varphi$ for $\varphi\in\operatorname{Ran}(1-P)$ where P is a finite-dimensional orthogonal projection, then there is a unitary $\Lambda=P\mathcal{H}\to P\mathcal{H}$ so that

$$V=U(1-P)+U\Lambda P \tag{3.31}$$

For $z\in\mathbb{D}$, define $G_0(z)$, $G(z)$, $g_0(z)$, $g(z)$ mapping $P\mathcal{H}$ to $P\mathcal{H}$ by

$$G(z)=P\left[\frac{V+z}{V-z}\right]P \tag{3.32}$$

$$G_0(z)=P\left[\frac{U+z}{U-z}\right]P \tag{3.33}$$

$$G(z)=\frac{1+zg(z)}{1-zg(z)}\qquad G_0(z)=\frac{1+zg_0(z)}{1-zg_0(z)} \tag{3.34}$$

As operators on $P\mathcal{H}$, $\|g(z)\| < 1$, $\|g_0(z)\| < 1$ on $\mathbb{D}$. A direct calculation (see [124, Section 4.5]) proves that

$$g(z) = \Lambda^{-1} g_0(z) \tag{3.35}$$

This can be used to provide, via a rank two decoupling of a CMV matrix (change a $\Theta(\alpha)$ to $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$), new proofs of Geronimus' theorem and of Khrushchev's formula; see [124, Section 4.5].

3.7. **Extended and Periodized CMV Matrices.** The CMV matrix is defined on $\ell^2(\{0,1,\dots\})$. It is natural to define an extended CMV matrix associated to $\{\alpha_j\}_{j=-\infty}^{\infty}$ on $\ell^2(\mathbb{Z})$ by extending $\mathcal{L}$ and $\mathcal{M}$ to $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{M}}$ on $\ell^2(\mathbb{Z})$ as direct sums of Θ's and letting $\mathcal{E} = \tilde{\mathcal{L}}\tilde{\mathcal{M}}$.

This is an analog of whole-line discrete Schrödinger operators. It is useful in the study of OPUC with ergodic Verblunsky coefficients as well as a natural object in its own right. [124, 125] have numerous results about this subject introduced here for the first time.

If $\{\alpha_j\}_{j=-\infty}^{\infty}$ is periodic of period p, $\mathcal{E}$ commutes with translations and so is a direct integral of $p \times p$ periodized CMV matrices depending on $\beta \in \partial\mathbb{D}$: essentially to restrictions of $\mathcal{E}$ to sequences in ℓ^∞ with $u_{n+kp} = \beta^k u_n$. In [125, Section 12.1], these are linked to Floquet theory and to the discriminant, as discussed below in Section 5.1.

4. Miscellaneous Results

In this section, we discuss a number of results that don't fit into the themes of the prior sections and don't involve explicit classes of Verblunsky coefficients, the subject of the final two sections.

4.1. **Jitomirskaya-Last Inequalities.** In a fundamental paper intended to understand the subordinacy results of Gilbert-Pearson [42] and extend the theory to understand Hausdorff dimensionality, Jitomirskaya-Last [64, 65] proved some basic inequalities about singularities of the m-function as energy approaches the spectrum.

In [125, Section 10.8], we prove an analog of their result for OPUC. First, we need some notation. $\psi.$ denotes the second polynomial, that is, the OPUC with sign flipped α_j's. For $x \in [0,\infty)$, let $[x]$ be the integral part of x and define for a sequence a:

$$\|a\|_x^2 = \sum_{j=0}^{[x]} |a_j|^2 + (x - [x])|a_{j+1}|^2 \tag{4.1}$$

We prove

Theorem 4.1. *For $z \in \partial\mathbb{D}$ and $r \in [0,1)$, define $x(r)$ to be the unique solution of*

$$(1-r)\|\varphi.(z)\|_{x(r)}\|\psi.(z)\|_{x(r)} = \sqrt{2} \tag{4.2}$$

Then

$$A^{-1}\left[\frac{\|\psi.(z)\|_{x(r)}}{\|\varphi.(z)\|_{x(r)}}\right] \le |F(rz)| \le A\left[\frac{\|\psi.(z)\|_{x(r)}}{\|\varphi.(z)\|_{x(r)}}\right] \tag{4.3}$$

where A is a universal constant in $(1,\infty)$.

Remark. One can take $A = 6.65$; no attempt was made to optimize A.

This result allows one to extend the Gilbert-Pearson subordinacy theory [42] to OPUC. Such an extension was accomplished by Golinskii-Nevai [49] under an extra assumption that

$$\limsup |\alpha_n| < 1 \tag{4.4}$$

We do not need this assumption, but the reason is subtle as we now explain.

Solutions of (1.4) and its * viewed as an equation for $\binom{\varphi}{\varphi^*}$ are given by a transfer matrix

$$T_n(z) = A(\alpha_{n-1},z)A(\alpha_{n-2},z)\dots A(\alpha_0,z) \tag{4.5}$$

where $\rho = (1-|\alpha|^2)^{1/2}$ and

$$A(\alpha,z) = \rho^{-1}\begin{pmatrix} z & -\bar\alpha \\ -\alpha z & 1 \end{pmatrix} \tag{4.6}$$

In the discrete Schrödinger case, the transfer matrix is a product of $A(v,e) = \left(\begin{smallmatrix} e-v & -1 \\ 1 & 0 \end{smallmatrix}\right)$. A key role in the proof in [65] is that $A(v,e') - A(v,e)$ depends only on e and e' and not on v. For OPUC, the A has the form (4.6). [49] requires (4.4) because $A(\alpha,z) - A(\alpha,z')$ has a ρ^{-1} divergence, and (4.4) controlled that. The key to avoiding (4.4) is to note that

$$A(\alpha,z) - A(\alpha,w) = (1-z^{-1}w)A(\alpha,z)P$$

where $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

4.2. **Isolated Pure Points.** Part of this section is joint work with S. Denisov. These results extend beyond the unit circle. We'll be interested in general measures on $\mathbb{C}$ with nontrivial probability measures

$$\int |z|^j \, d\mu(z) < \infty \tag{4.7}$$

for all $j = 0,1,2,\dots$. In that case, one can define monic orthogonal polynomials $\Phi_n(z)$, $n = 0,1,2,\dots$. Recall the following theorem of Fejér [30]:

Theorem 4.2 (Fejér [30])**.** *All the zeros of* Φ_n *lie in the convex hull of* $\operatorname{supp}(d\mu)$.

We remark that this theorem has an operator theoretic interpretation. If M_z is the operator of multiplication by z on $L^2(\mathbb{C}, d\mu)$, and if P_n is the projection onto the span of $\{z^j\}_{j=0}^{n-1}$, then we'll see (4.10) that the eigenvalues of $P_n M_z P_n$ are precisely the zeros of Φ_n. If $\eta(\,\cdot\,)$ denotes numerical range, $\eta(M_z)$ is the convex hull of $\operatorname{supp}(d\mu)$, so Fejér's theorem follows from $\eta(P_n M_z P_n) \subseteq \eta(M_z)$ and the fact that eigenvalues lie in the numerical range.

[124, Section 1.7] contains the following result I proved with Denisov:

Theorem 4.3. *Let* μ *obey* (4.7) *and suppose* z_0 *is an isolated point of* $\operatorname{supp}(d\mu)$. *Define* $\Gamma = \operatorname{supp}(d\mu)\backslash\{z_0\}$ *and* $\operatorname{ch}(\Gamma)$, *the convex hull of* Γ. *Suppose* $\delta \equiv \operatorname{dist}(z_0, \operatorname{ch}(\Gamma)) > 0$. *Then* Φ_n *has at most one zero in* $\{z \mid |z - z_0| < \delta/3\}$.

Remarks. 1. In case $\operatorname{supp}(d\mu) \subset \partial\mathbb{D}$, any isolated point has $\delta > 0$. Indeed, if $d = \operatorname{dist}(z_0, \Gamma)$, $\delta \geq d^2/2$ and so, Theorem 4.3 says that there is at most one zero in the circle of radius $d^2/6$.

2. If $d\mu$ is a measure on $[-1, -\frac{1}{2}] \cup \{0\} \cup [\frac{1}{2}, 1]$ and symmetric under x, and $\mu(\{0\}) > 0$, it can be easily shown that $P_{2n}(x)$ has two zeros near 0 for n large. Thus, for a result like Theorem 4.3, it is not enough that z_0 be an isolated point of $\operatorname{supp}(d\mu)$; note in this example that 0 is in the convex hull of $\operatorname{supp}(d\mu)\backslash\{0\}$.

The other side of this picture is the following result proven in [124, Section 8.1] using potential theoretic ideas of the sort exposed in [116, 131]:

Theorem 4.4. *Let* μ *be a nontrivial probability measure on* $\partial\mathbb{D}$ *and let* z_0 *be an isolated point of* $\operatorname{supp}(d\mu)$. *Then there exist* $C > 0$, $a > 0$, *and a zero* z_n *of* $\Phi_n(z; d\mu)$ *so that*

$$|z_n - z_0| \leq Ce^{-a|n|} \tag{4.8}$$

There is an explicit formula for a in terms of the equilibrium potential for $\operatorname{supp}(d\mu)$ at z_0. The pair of theorems in this section shows that any isolated mass point, z_0, of $d\mu$ on $\partial\mathbb{D}$ has exactly one zero near z_0 for n large.

4.3. **Determinant Theorem.** It is a well-known fact that if $J^{(n)}$ is the $n \times n$ truncated Jacobi matrix and P_n the monic polynomial associated to J, then

$$P_n(x) = \det(x - J^{(n)}) \tag{4.9}$$

The usual proofs of (4.9) use the selfadjointness of $J^{(n)}$ but there is a generalization to OPs for measures on $\mathbb{C}$:

Theorem 4.5. *Let $d\mu$ be a measure on $\mathbb{C}$ obeying* (4.7)*. Let P_n be the projection onto the span of $\{z^j\}_{j=0}^{n-1}$, M_z be multiplication by z, and $M^{(n)} = P_n M_z P_n$. Then*

$$\Phi_n(z) = \det(z - M^{(n)}) \tag{4.10}$$

Sketch. Suppose z_0 is an eigenvalue of $M^{(n)}$. Then there exists Q, a polynomial of degree at most $n-1$, so $P_n(z-z_0)Q(z) = 0$. Since Φ_n is up to a constant, the only polynomial, S, of degree n with $P_n(S) = 0$, we see

$$(z - z_0)Q(z) = c\Phi_n(z) \tag{4.11}$$

It follows that $\Phi_n(z_0) = 0$, and conversely, if $\Phi_n(z_0) = 0$, $\Phi_n(z)/(z - z_0) \equiv Q$ provides an eigenfunction. Thus, the eigenvalues of $M^{(n)}$ are exactly the zeros of Φ_n. This proves (4.10) if Φ_n has simple zeros. In general, by perturbing $d\mu$, we can get Φ_n as a limit of other Φ_n's with simple zeros. □

In the case of $\partial\mathbb{D}$, z^ℓ is unitary on $L^2(\partial\mathbb{D}, d\mu)$, so P_n in defining $M^{(n)}$ can be replaced by the projection onto the span of $\{z^{j+\ell}\}_{j=0}^{n-1}$ for any ℓ, in particular, the span onto the first n of $1, z, z^{-1}, z^2, \dots$, so

Corollary 4.6. *If $\mathcal{C}^{(n)}$ is the truncated $n \times n$ CMV matrix, then*

$$\Phi_n(z) = \det(z - \mathcal{C}^{(n)}) \tag{4.12}$$

4.4. **Geronimus' Theorem and Taylor Series.** Given a Schur function, that is, f mapping $\mathbb{D}$ to $\mathbb{D}$ analytically, one defines γ_0 and f_1 by

$$f(z) = \frac{\gamma_0 + z f_1(z)}{1 + \bar{\gamma}_0 z f_1(z)} \tag{4.13}$$

so $\gamma_0 = f(0)$ and f_1 is either a new Schur function or a constant in $\partial\mathbb{D}$. The later combines the fact that $\omega \to (\gamma_0 + \omega)/(1 + \bar{\gamma}_0\omega)$ is a bijection of $\mathbb{D}$ to $\mathbb{D}$ and the Schur lemma that if g is a Schur function with $g(0) = 0$, then $g(z)z^{-1}$ is also a Schur function. If one iterates, one gets either a finite sequence $\gamma_0, \dots, \gamma_{n-1} \in \mathbb{D}^n$ and $\gamma_n \in \partial\mathbb{D}$ or an infinite sequence $\{\gamma_j\}_{j=0}^\infty \in \mathbb{D}^\infty$. It is a theorem of Schur that this sets up a one-one correspondence between the Schur functions and such γ-sequences. The finite sequences correspond to finite Blaschke products.

In 1944, Geronimus proved

Theorem 4.7 (Geronimus' Theorem [37])**.** *Let $d\mu$ be a nontrivial probability measure on $\partial\mathbb{D}$ with Verblunsky coefficients $\{\alpha_j\}_{j=0}^\infty$. Let f be* *the Schur function associated to $d\mu$ by* (2.14)/(2.16) *and let* $\{\gamma_n\}_{n=0}^\infty$ *be its Schur parameters. Then*

$$\gamma_n = \alpha_n \tag{4.14}$$

[124] has several new proofs of this theorem (see [44, 111, 67] for other proofs, some of them also discussed in [124]). We want to describe here one proof that is really elementary and should have been found in 1935! Indeed, it is obvious to anyone who knows Schur's paper [117] and Verblunsky [146] — but apparently Verblunsky didn't absorb that part of Schur's work, and Verblunsky's paper seems to have been widely unknown and unappreciated!

This new proof depends on writing the Taylor coefficients of $F(z)$ in terms of the α's and the γ's. Since

$$\frac{e^{i\theta}+z}{e^{i\theta}-z} = 1 + 2\sum_{n=1}^\infty e^{-in\theta} z^n$$

we have

$$F(z) = 1 + 2\sum_{n=1}^\infty c_n z^n \tag{4.15}$$

with c_n given by

$$c_n = \int e^{-in\theta}\, d\mu(\theta) \tag{4.16}$$

Define $s_n(f)$ by $f(z) = \sum_{n=0}^\infty s_n(f) z^n$. Then Schur [117] noted that $(1+\bar\gamma_0 z f_1) f = \gamma_0 + z f_1$ implies

$$s_n(f) = (1-|\gamma_0|^2) s_{n-1}(f_1) - \bar\gamma_0 \sum_{j=1}^n s_j(f) s_{n-1-j}(f_1)$$

so that, by induction,

$$s_n(f) = \prod_{j=0}^{n-1} (1-|\gamma_j|^2)\gamma_n + r_n(\gamma_0, \bar\gamma_0, \dots, \gamma_{n-1}, \bar\gamma_{n-1})$$

with r_n a polynomial. This formula is in Schur [117]. Since $F(z) = 1 + 2\sum_{n=1}^\infty (zf)^n$, we find that $c_n = s_{n-1}(f) +$ polynomial in $(s_0(f), \dots, s_{n-1}(f))$, and thus

$$c_n(f) = \prod_{j=0}^{n-2} (1-|\gamma_j|^2)\gamma_{n-1} + \tilde r_{n-1}(\gamma_0, \bar\gamma_0, \dots, \gamma_{n-2}, \bar\gamma_{n-2}) \tag{4.17}$$

for a suitable polynomial $\tilde r_{n-1}$.

On the other hand, Verblunsky [146] had the formula relating his parameters and $c_n(f)$:

$$c_n(f) = \prod_{j=0}^{n-2}(1-|\alpha_n|^2)\alpha_{n-1} + \tilde{q}_{n-1}(\alpha_0, \bar{\alpha}_0, \dots, \alpha_{n-2}, \bar{\alpha}_{n-2}) \qquad (4.18)$$

For Verblunsky, (4.18) was actually the definition of α_{n-1}, that is, he showed (as did Akhiezer-Krein [1]) that, given $c_0, \dots, c_{n-1}$, the set of allowed c_n's for a positive Toeplitz determinant is a circle of radius inductively given by $\prod_{j=0}^{n-1}(1-|\alpha_j|^2)$, which led him to define parameters α_{n-1}.

On the other hand, it is a few lines to go from the Szegő recursion (1.4) to (4.18). For we note that

$$\int \Phi_{n+1}(z)\, d\mu(z) = \langle 1, \Phi_{n+1}\rangle = 0$$

while

$$\langle 1, \Phi_n^* \rangle = \langle \Phi_n, z^n \rangle = \langle \Phi_n, \Phi_n \rangle = \prod_{j=1}^{n-1}(1-|\alpha_j|^2)$$

by (1.7). Thus

$$\langle 1, z\Phi_n \rangle = \bar{\alpha}_n \prod_{j=1}^{n-1}(1-|\alpha_j|^2) \qquad (4.19)$$

But since $z\Phi_n = z^{n+1} +$ lower order,

$$\langle 1, z\Phi_n \rangle = \bar{c}_{n+1} + \text{ polynomial in } (c_0, c_1, \dots, c_n, \bar{c}_1, \dots, \bar{c}_n) \qquad (4.20)$$

This plus induction implies (4.18).

Given (4.17) and (4.18) plus the theorem of Schur that any $\{\gamma_j\}_{j=0}^{n-1}$ in $\mathbb{D}^n$ is allowed, and the theorem of Verblunsky that any $\{\alpha_j\}_{j=0}^{n-1}$ in $\mathbb{D}^n$ is allowed, we get (4.14) inductively.

4.5. **Improved Exponential Decay Estimates.** In [96], Nevai-Totik proved that

$$\limsup_{n\to\infty} |\alpha_n|^{1/n} = A < 1 \Leftrightarrow d\mu_{\mathrm{s}} = 0 \text{ and } D^{-1}(z) \text{ is analytic in } \{z \mid |z| < A^{-1}\} \qquad (4.21)$$

providing a formula for the exact rate of exponential decay in terms of properties of D^{-1}. By analyzing their proof carefully, [124, Section 7.2] refines this to prove

Theorem 4.8. *Suppose*

$$\lim_{n\to\infty} |\alpha_n|^{1/n} = A < 1 \qquad (4.22)$$

and define

$$S(z)=\sum_{n=0}^\infty \alpha_n z^n \tag{4.23}$$

Then $S(z)+\overline{D(1/\bar z)}\,D(z)^{-1}$ *has an analytic continuation to* $\{z \mid A < |z| < A^{-2}\}$.

The point of this theorem is that both $S(z)$ and $\overline{D(1/\bar z)}\,D(z)^{-1}$ have singularities on the circle of radius A^{-1} (S by (4.22) and D^{-1} by (4.21)), so the fact that the combination has the continuation is a strong statement.

Theorem 4.8 comes from the same formula that Nevai-Totik [96] use, namely, if $d\mu_{\rm s}=0$ and $\kappa_\infty=\prod_{n=0}^\infty(1-|\alpha_n|^2)^{1/2}$, then

$$\alpha_n=-\kappa_\infty\int \overline{\Phi_{n+1}(e^{i\theta})}\,D(e^{i\theta})^{-1}\,d\mu(\theta) \tag{4.24}$$

We combine this with an estimate of Geronimus [39] that

$$\|\varphi_{n+1}^*-D^{-1}\|_{L^2(\partial\mathbb{D},d\mu)}\leq\sqrt{2}\biggl(\sum_{j=n+1}^\infty|\alpha_j|^2\biggr)^{1/2} \tag{4.25}$$

and $D^{-1}\,d\mu=\bar D\frac{d\theta}{2\pi}$ to get

$$\alpha_n+\sum_{j=n}^\infty d_{j,-1}\bar d_{j-n,1}=O((A^{-1}-\varepsilon)^{-2n}) \tag{4.26}$$

where $D(z)=\sum_{j=0}^\infty d_{j,1}z^j$, $D(z)^{-1}=\sum_{j=0}^\infty d_{j,-1}z^j$. (4.26) is equivalent to analyticity of $S(z)+\overline{D(1/\bar z)}\,D(z)^{-1}$ in the stated region.

One consequence of Theorem 4.8 is

Corollary 4.9. *Let* $b\in\mathbb{D}$. *Then*

$$\frac{\alpha_{n+1}}{\alpha_n}=b+O(\delta^n) \tag{4.27}$$

for some $\delta<1$ *if and only if* $D^{-1}(z)$ *is meromorphic in* $\{z \mid |z|<|b|^{-1}+\delta'\}$ *for some* δ' *and* $D(z)^{-1}$ *has only a single pole at* $z=1/b$ *in this disk.*

This result is not new; it is proven by other means in Barrios-López-Saff [8]. Our approach leads to a refined form of (4.27), namely,

$$\alpha_n=-Cb^n+O((b\delta)^n) \tag{4.28}$$

with

$$C=\bigl[\lim_{z\to b^{-1}}(1-zb)D(z)^{-1}\bigr]\,\overline{D(\bar b)} \tag{4.29}$$

One can get more. If $D(z)^{-1}$ is meromorphic in $\{z \mid |z| < A^{-2}\}$, one gets an asymptotic expansion of α_n of the form

$$\alpha_n = \sum_{j=1}^{\ell} P_{m_j}(n) z_j^n + O((A^{-2} - \varepsilon)^{-n})$$

where the z_j are the poles of D^{-1} in $\{z \mid |z| < A^{-2}\}$ and P_{m_j} are polynomials of degree $m_j =$ the order of the pole at m_j. There are also results relating asymptotics of α_n of the form $\alpha_n = Cb^n n^k(1 + o(1))$ to asymptotics of $d_{n,-1}$ or the form $d_{n,-1} = C_1 b^n n^k (1 + o(1))$.

4.6. **Rakhmanov's Theorem on an Arc with Eigenvalues in the Gap.** Rakhmanov [113] proved a theorem that if (1.1) holds with $w(\theta) \neq 0$ for a.e. θ, then $\lim_{n\to\infty} |\alpha_n| = 0$ (see also [114, 84, 95, 68]). In [125, Section 13.4], we prove the following new result related to this. Define for $a \in (0,1)$ and $\lambda \in \partial\mathbb{D}$

$$\Gamma_{a,\lambda} = \{z \in \partial\mathbb{D} \mid \arg(\lambda z) > 2\arcsin(a)\} \tag{4.30}$$

and $\operatorname{ess\,supp}(d\mu)$ of a measure as points z_0 with $\{z \mid |z - z_0| < \varepsilon\} \cap \operatorname{supp}(d\mu)$ an infinite set for all $\varepsilon > 0$. Then

Theorem 4.10. *Let $d\mu$ be given by* (1.1) *so that* $\operatorname{ess\,supp}(d\mu) = \Gamma_{a,\lambda}$ *and $w(\theta) > 0$ for a.e. $e^{i\theta} \in \Gamma_{a,\lambda}$. Then*

$$\lim_{n\to\infty} |\alpha_n(d\mu)| = a \qquad \lim_{n\to\infty} \overline{\alpha_{n+1}(d\mu)}\, \alpha_n(d\mu) = a^2\lambda \tag{4.31}$$

We note that, by rotation invariance, one need only look at $\lambda = 1$. $\Gamma_a \cup \{1\}$ is known (Geronimus [40, 41]; see also [45, 46, 50, 51, 107, 108, 109, 110]) to be exactly the spectrum for $\alpha_n \equiv a$ and the spectrum on Γ_a is purely a.c. with $w(\theta) > 0$ on Γ_a^{int}.

Theorem 4.10 can be viewed as a combination of two previous extensions of Rakhmanov's theorem. First, Bello-López [7] proved (4.31) if $\operatorname{ess\,supp}(d\mu) = \Gamma_{a,\lambda}$ is replaced by $\operatorname{supp}(d\mu) = \Gamma_{a,\lambda}$. Second, Denisov [26] proved an analog of Rakhmanov's theorem for OPRL. By the mapping of measures on $\partial\mathbb{D}$ to measures on $[-2,2]$ due to Szegő [136] and the mapping of Jacobi coefficients to Verblunsky coefficients due to Geronimus [38], Rakhmanov's theorem immediately implies that if a Jacobi matrix has $\operatorname{supp}(d\gamma) = [-2,2]$ and $d\gamma = f(E) + d\gamma_{\rm s}$ with $f(E) > 0$ on $[-2,2]$, then $a_n \to 1$ and $b_n \to 0$. What Denisov [26] did is extend this result to only require $\operatorname{ess\,supp}(d\gamma) = [-2,2]$.

Thus, Theorem 4.10 is essentially a synthesis of the Bello-López [7] and Denisov [26] results. One difficulty in such a synthesis is that Denisov relies on Sturm oscillation theorems and such a theorem does not seem to be applicable for OPUC. Fortunately, Nevai-Totik [97] have provided an alternate approach to Denisov's result using variational principles, and their approach — albeit with some extra complications — allows one to prove Theorem 4.10. The details are in [125, Section 13.4].

4.7. **A Birman-Schwinger Principle for OPUC.** Almost all quantitative results on the number of discrete eigenvalues for Schrödinger operators and OPRL depend on a counting principle of Birman [10] and Schwinger [118]. In [125, Section 10.15], we have found an analog for OPUC by using a Cayley transform and applying the Birman-Schwinger idea to it. Because of the need to use a point in $\partial\mathbb{D}$ about which to base the Cayley transform, the constants that arise are not universal. Still, the method allows the proof of perturbation results like the following from [125, Section 12.2]:

Theorem 4.11. *Suppose $d\mu$ has Verblunsky coefficients α_j and there exists β_j with $\beta_{j+p} = \beta_j$ for some p and*

$$\sum_{j=0}^{\infty} j|\alpha_j - \beta_j| < \infty \tag{4.32}$$

Then $d\mu$ has an essential support whose complement has at most p gaps, and each gap has only finitely many mass points.

Theorem 4.12. *Suppose α and β are as in Theorem 4.11, but* (4.32) *is replaced by*

$$\sum_{j=0}^{\infty} |\alpha_j - \beta_j|^p < \infty \tag{4.33}$$

for some $p \geq 1$. Then

$$\sum_{z_j = \text{ mass points in gaps}} \text{dist}(z_j, \text{ess sup}(d\mu))^q < \infty \tag{4.34}$$

where $q > \frac{1}{2}$ if $p = 1$ and $q \geq p - \frac{1}{2}$ if $p > 1$.

Theorem 4.11 is a bound of Bargmann type [6], while Theorem 4.12 is of Lieb-Thirring type [81]. We have not succeeded in proving $q = \frac{1}{2}$ for $p = 1$ whose analog is known for Schrödinger operators [148, 62] and OPRL [63].

4.8. **Rotation Number for OPUC.** Rotation numbers and their connection to the density of states have been an important tool in the theory of Schrödinger operators and OPRL (see Johnson-Moser [66]). Their analog for OPUC has a twist, as seen from the following theorem from [124, Section 8.3]:

Theorem 4.13. $\arg(\Phi_n(e^{i\theta}))$ *is monotone increasing in* θ *on* $\partial\mathbb{D}$ *and defines a measure* $d\arg(\Phi_n(e^{i\theta}))/d\theta$ *of total mass* $2\pi n$. *If the density of zeros* $d\nu_n$ *has a limit* $d\nu$ *supported on* $\partial\mathbb{D}$, *then*

$$\frac{1}{2\pi n}\,\frac{d\arg(\Phi_n(e^{i\theta}))}{d\theta} \to \frac{1}{2}\,d\nu + \frac{1}{2}\,\frac{d\theta}{2\pi} \tag{4.35}$$

weakly.

Given the OPRL result, the $\frac{1}{2}\frac{d\theta}{2\pi}$ is surprising. In a sense, it comes from the fact that the transfer matrix obeys $\det(T_n) = z^n$ rather than determinant 1. The proof of Theorem 4.13 comes from an exact result that in turn comes from looking at $\arg(e^{i\theta} - z_0)$ for $z_0 \in \mathbb{D}$:

$$\frac{1}{2\pi n}\,\frac{d\arg(\Phi_n(e^{i\theta}))}{d\theta} = \frac{1}{2}\,\mathcal{P}(d\nu_n) + \frac{1}{2}\,\frac{d\theta}{2\pi} \tag{4.36}$$

where $\mathcal{P}$ is the dual of Poisson kernel viewed as a map of $C(\partial\mathbb{D})$ to $C(\mathbb{D})$, that is,

$$\mathcal{P}(d\gamma) = \frac{1}{2\pi}\int \frac{1-|r|^2}{1+r^2-2r\cos\theta}\,d\gamma(re^{i\theta}) \tag{4.37}$$

5. Periodic Verblunsky Coefficients

In this section, we describe some new results/approaches for Verblunsky coefficients $\{\alpha_n\}_{n=0}^\infty$ that obey

$$\alpha_{n+p} = \alpha_n \tag{5.1}$$

for some p. We'll normally suppose p is even. If it is not, one can use the fact that $(\alpha_0, 0, \alpha_1, 0, \alpha_2, 0, \dots)$ is the Verblunsky coefficients of the measure $\frac{1}{2}d\mu(e^{2i\theta})$ and it has (5.1) with p even, so one can read off results for p odd from p even.

The literature is vast for Schrödinger operators with periodic potential called Hill's equation after Hill [53]. The theory up to the 1950's is summarized in Magnus-Winkler [83] whose key tool is the discriminant; see also Reed-Simon [115]. There was an explosion of ideas following the KdV revolution, including spectrally invariant flows and abelian functions on hyperelliptic Riemann surfaces. Key papers include McKean-van Moerbeke [85], Dubrovin et al. [29], and Trubowitz [144]. Their ideas have been discussed for OPRL; see especially Toda [142], van Moerbeke [145], and Flaschka-McLaughlin [32].

For OPUC, the study of measures associated with (5.1) goes back to Geronimus [37] with a fundamental series of papers by Peherstorfer-Steinbauer [103, 104, 107, 108, 109, 110, 105, 106] and considerable literature on the case $p = 1$ (i.e., constant α); see, for example, [40, 41, 45, 46, 51, 50, 68, 69]. The aforementioned literature on OPUC used little from the the work on Hill's equation; work that does make a partial link is Geronimo-Johnson [35], which discussed almost periodic Verblunsky coefficients using abelian functions. Simultaneous with our work reported here, Geronimo-Gesztesy-Holden [33] have discussed this further, including work on isospectral flows. Besides the work reported here, Nenciu-Simon [93] have found a symplectic structure on $\mathbb{D}^p$ for which the coefficients of the discriminant Poisson commute (this is discussed in [125, Section 11.11].

5.1. **Discriminant and Floquet Theory.** For Schrödinger operators, it is known that the discriminant is just the trace of the transfer matrix. Since the transfer matrix has determinant one in this case, the eigenvalues obey $x^2 - \mathrm{Tr}(T)x + 1 = 0$, which is the starting point for Floquet theory. For OPUC, the transfer matrix, $T_p(z)$, of (4.5) has $\det(T_p(z)) = z^p$, so it is natural to define the discriminant by

$$\Delta(z) = z^{-p/2}\mathrm{Tr}(T_p(z)) \tag{5.2}$$

which explains why we take p even. Because for $z = e^{i\theta}$, $A(\alpha, z) \in \mathbb{U}(1,1)$ (see [125, Section 10.4] for a discussion of $\mathbb{U}(1,1)$), $\Delta(z)$ is real on $\partial\mathbb{D}$, so

$$\Delta(1/\bar{z}) = \overline{\Delta(z)} \tag{5.3}$$

Here are the basic properties of Δ:

Theorem 5.1. (a) *All solutions of $\Delta(z) - w = 0$ with $w \in (-2,2)$ are simple zeros and lie in $\partial\mathbb{D}$ (so are p in number).*

(b) *$\{z \mid \Delta(z) \in (-2,2)\}$ is p disjoint intervals on $\partial\mathbb{D}$ whose closures $B_1, \dots, B_p$ can overlap at most in single points. The complements where $|\Delta(z)| > 2$ and $z \in \partial\mathbb{D}$ are "gaps," at most p in number.*

(c) *On $\cup B_j$, $d\mu$ is purely a.c. (i.e., in terms of (1.1), $\mu_{\mathrm{s}}(\cup_{j=1}^p B_j) = 0$ and $w(\theta) > 0$ for a.e. $\theta \in \cup_{j=1}^p B_j$).*

(d) *$\mu \restriction (\partial\mathbb{D} \setminus \cup_{j=1}^p B_j)$ consists of pure points only with at most one pure point per gap.*

(e) *For all $z \in \mathbb{C}\setminus\{0\}$, the Lyapunov exponent $\lim_{n\to\infty} \|T_n(z)\|^{1/n}$ exists and obeys*

$$\gamma(z) = \frac{1}{2}\log(z) + \frac{1}{p}\log\left|\frac{\Delta(z)}{2} + \sqrt{\frac{\Delta^2}{4} - 1}\,\right| \tag{5.4}$$

where the branch of square root is taken that maximizes the log.

(f) *If* $B = \cup_{j=1}^{p} B_j$*, then the logarithmic capacity of* B *is given by*

$$C_B = \prod_{j=0}^{p-1} (1 - |\alpha_j|^2)^{1/p} \tag{5.5}$$

and $-[\gamma(z) + \log C_B]$ *is the equilibrium potential for* B*.*

(g) *The density of zeros is the equilibrium measure for* B *and given in terms of* Δ *by*

$$d\nu(\theta) = V(\theta)\, \frac{d\theta}{2\pi} \tag{5.6}$$

where $V(\theta) = 0$ *on* $\partial\mathbb{D} \setminus \cup_{j=1}^{p} B_j$*, and on* B_j *is given by*

$$V(\theta) = \frac{1}{p}\, \frac{|\Delta'(e^{i\theta})|}{\sqrt{4 - \Delta^2(e^{i\theta})}} \tag{5.7}$$

where $\Delta'(e^{i\theta}) = \frac{\partial}{\partial\theta}\, \Delta(e^{i\theta})$*.*

(h) $\nu(B_j) = 1/p$

For proofs, see [125, Section 11.1]. The proofs are similar to those for Schrödinger operators. That the density of zeros is an equilibrium measure has been emphasized by Saff, Stahl, and Totik [116, 131]. While not expressed as the trace of a transfer matrix, Δ is related to the (monic) Tchebychev polynomial, T, of Peherstorfer-Steinbauer [108] by

$$\Delta(z) = z^{-p/2} C_B^{-1/2} T(z)$$

and some of the results in Theorem 5.1 are in their papers.

One can also relate Δ to periodized CMV matrices, an OPUC version of Floquet theory. As discussed in Section 3.7, $\mathcal{E}_p(\beta)$ is defined by restricting $\mathcal{E}$ to sequences obeying $u_{n+p} = \beta u_n$ for all n. $\mathcal{E}_p$ can be written as a $p \times p$ matrix with an $\mathcal{LM}$ factorization. With Θ given by

(3.4), $\mathcal{E}_p(\beta) = \mathcal{L}_p\mathcal{M}_p(\beta)$

$$\mathcal{M}_p(\beta) = \begin{pmatrix} -\alpha_{p-1} & & & & \rho_p\beta_{-1} \\ & \Theta_1 & & & \\ & & \ddots & & \\ & & & \Theta_{p-3} & \\ \rho_{p-1}\beta & & & & \bar{\alpha}_{p-1} \end{pmatrix}$$

$$\mathcal{L}_p = \begin{pmatrix} \Theta_0 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \Theta_{p-2} \end{pmatrix} \tag{5.8}$$

Then:

Theorem 5.2. (a) *The following holds:*

$$\det(z - \mathcal{E}_p(\beta)) = \prod_{j=0}^{p-1}(1 - |\alpha_j|^2)^{1/2p} z^{p/2}[\Delta(z) - \beta - \beta^{-1}] \tag{5.9}$$

(b) $\mathcal{E}$ *is a direct integral of* $\mathcal{E}_p(\beta)$.

5.2. **Generic Potentials.** The following seems to be new; it is an analog of a result [119] for Schrödinger operators.

Notice that for any $\{\alpha_j\}_{j=0}^{p} \in \mathbb{D}^p$, one can define a discriminant $\Delta(z, \{\alpha_j\}_{j=0}^{p-1})$ for the period p Verblunsky coefficients that agree with $\{\alpha_j\}_{j=0}^{p-1}$ for $j = 0, \dots, p-1$.

Theorem 5.3. *The set of* $\{\alpha_j\}_{j=0}^{p} \in \mathbb{D}^p$ *for which* $\Delta(z)$ *has all gaps open is a dense open set.*

[125] has two proofs of this theorem: one in Section 11.10 uses Sard's theorem and one is perturbation theoretic calculation that if $|\mathrm{Tr}(z, \{\alpha_j^{(0)}\}_{j=0}^{p-1})| = 2$, then $|\mathrm{Tr}(z, \{\alpha_j^{(0)} + (e^{i\eta} - 1)\delta_{jk}\alpha_k^{(0)}\}_{j=0}^{p-1})| = 2 + 2\eta^2(\rho_k^{(0)})^2|\alpha_k^{(0)}| + O(\eta^3)$.

5.3. **Borg's Theorems.** In [12], Borg proved several theorems about the implication of closed gaps. Further developments of Borg's results for Schrödinger equations or for OPRL are in Hochstadt [54, 55, 56, 57, 58, 59, 60, 61], Clark et al. [16], Trubowitz [144], and Flaschka [31]. In [125, Section 11.14], we prove the following analogs of these results:

Theorem 5.4. *If* $\{\alpha_j\}_{j=0}^\infty$ *is a periodic sequence of Verblunsky coefficients so* $\text{supp}(d\mu) = \partial\mathbb{D}$ *(i.e., all gaps are closed), then* $\alpha_j \equiv 0$.

[125] has three proofs of this: one uses an analog of a theorem of Deift-Simon [22] that $d\mu/d\theta \geq 1/2\pi$ on the essential support of the a.c. spectrum of any ergodic system, one tracks zeros of the Wall polynomials, and one uses the analog of Tchebychev's theorem for the circle that any monic Laurent polynomial real on $\partial\mathbb{D}$ has $\max_{z\in\partial\mathbb{D}}|L(z)| \geq 2$.

Theorem 5.5. *If* p *is even and* $\{\alpha_j\}_{j=0}^\infty$ *has period* $2p$*, then if all gaps with* $\Delta(z) = -2$ *are closed, we have* $\alpha_{j+p} = \alpha_j$*, and if all gaps with* $\Delta(z) = 2$ *are closed, then* $\alpha_{j+p} = -\alpha_j$.

Theorem 5.6. *Let* p *be even and suppose for some* k *that* $\alpha_{kp+j} = \alpha_j$ *for all* j*. Suppose for some labelling of* $\{w_j\}_{j=0}^{k_p-1}$ *of the zeros of the derivative* $\partial\Delta/\partial\theta$ *labelled counterclockwise, we have* $|\Delta(w_j)| = 2$ *if* $j \not\equiv 0 \mod k$*. Then* $\alpha_{p+j} = \omega\alpha_j$ *where* ω *is a* k*-th root of unity.*

The proof of these last two theorems depends on the study of the Carathéodory function for periodic Verblunsky coefficients as meromorphic functions on a suitable hyperelliptic Riemann surface.

5.4. **Green's Function Bounds.** In [125, Section 10.14], we develop the analog of the Combes-Thomas [17] method for OPUC and prove, for points in $\partial\mathbb{D}\backslash\text{supp}(d\mu)$, the Green's function (resolvent matrix elements of $(\mathcal{C}-z)^{-1}$ with $\mathcal{C}$ the CMV matrix) decays exponentially in $|n-m|$. The rate of decay in these estimates goes to zero at a rate faster than expected in nice cases. For periodic Verblunsky coefficients, one expects behavior similar to the free case for OPRL or Schrödinger operators — and that is what we discuss here. An energy $z_0 \in \partial\mathbb{D}$ at the edge of a band is called a resonance if $\sup_n|\varphi_n(z_0)| < \infty$. For the family of measures, $d\mu_\lambda$, with Verblunsky coefficients $\alpha_n = \lambda\alpha_n^{(0)}$ and a given z_0, there is exactly one λ for which z_0 is a resonance (for the other values, $\varphi_n(z_0, d\mu_\lambda)$ grows linearly in n). Here is the bound we prove in [125, Section 11.12]:

Theorem 5.7. *Let* $\{\alpha_n\}_{n=0}^\infty$ *be a periodic family of Verblunsky coefficients. Suppose* $G = \{z = e^{i\theta} \mid \theta_0 < \theta < \theta_1\}$ *is an open gap and* $e^{i\theta_0}$ *is not a resonance. Let*

$$G_{nm}(z) = \langle\delta_n, (\mathcal{C}(\alpha) - z)^{-1}\delta_m\rangle$$

Then for $z = e^{i\theta}$ *with* $z \in G$ *and* $|\theta-\theta_0| < |\theta-\theta_1|$*, we have*

$$\sup_{n,m}|G_{nm}(z)| \leq C_1|z-e^{i\theta_0}|^{-1/2}$$

$$\sup_{\text{such } z} |G_{nm}(z)| \leq C_2(n+1)^{1/2}(m+1)^{1/2}$$

and similarly for z approaching $e^{i\theta_1}$.

The proof depends on bounds on polynomials in the bands of some independent interest.

Theorem 5.8. *Let $\{\alpha_n\}_{n=0}^\infty$ be a sequence of periodic Verblunsky coefficients, and let B^{int} be the union of the interior of the bands. Let $\mathcal{E}_1$ be the set of band edges by open gaps and $\mathcal{E}_2$ the set of band edges by closed gaps. Define*

$$d(z) = \min(\text{dist}(z, \mathcal{E}_1), \text{dist}(z, \mathcal{E}_2)^2)$$

Then

$$\text{(1)} \qquad \sup_n |\varphi_n(z)| \leq C_1\, d(z)^{-1/2}$$

$$\text{(2)} \qquad \sup_{z \in B^{\text{int}}} |\varphi_n(z)| \leq C_2 n$$

where C_1 and C_2 are $\{\alpha_n\}_{n=0}^\infty$ dependent constants.

Remark. One can, with an extra argument, show $d(z)$ can be replaced by $\text{dist}(z, \mathcal{E}_1)$ which differs from $d(z)$ only when there is a closed gap. That is, there is no singularity in $\sup_n |\varphi_n(z)|$ at band edges next to closed gaps.

5.5. **Isospectral Results.** In [125, Chapter 11], we prove the following theorem:

Theorem 5.9. *Let $\{\alpha_j\}_{j=0}^{p-1}$ be a sequence in $\mathbb{D}^p$ so $\Delta(z, \{\alpha_j\}_{j=0}^{p-1})$ has k open gaps. Then $\{\{\beta_j\}_{j=0}^{p-1} \in \mathbb{D}^p \mid \Delta(z, \{\beta_j\}_{j=0}^{p-1}) = \Delta(z, \{\alpha_j\}_{j=0}^{p-1})\}$ is a k-dimensional torus.*

This result for OPUC seems to be new, although its analog for finite-gap Jacobi matrices and Schrödinger operators (see, e.g., [85, 29, 145]) is well known and it is related to results on almost periodic OPUC by Geronimo-Johnson [35].

There is one important difference between OPUC and the Jacobi/Schrödinger case. In the later, the infinite gap doesn't count in the calculation of dimension of torus, so the torus has a dimension equal to the genus of the Riemann surface for the m-function. In the OPUC case, all gaps count and the torus has dimension one more than the genus.

The torus can be defined explicitly in terms of natural additional data associated to $\{\alpha_j\}_{j=0}^{p-1}$. One way to define the data is to analytically continue the Carathéodory function, F, for the periodic sequence.

One cuts $\mathbb{C}$ on the "combined bands," that is, connected components of $\{e^{i\theta} \mid |\Delta(e^{i\theta})| \leq 2\}$, and forms the two-sheeted Riemann surface associated to $\sqrt{\Delta^2 - 4}$. On this surface, F is meromorphic with exactly one pole on each "extended gap." By extended gap, we mean the closure of the two images of a gap on each of two sheets of the Riemann surface. The ends of the gap are branch points and join the two images into a circle. The p points, one on each gap, are thus p-dimensional torus, and the refined version of Theorem 5.9 is that there is exactly one Carathéodory function associated to a period p set of Verblunsky coefficients with specified poles.

Alternately, the points in the gaps are solutions of $\Phi_p(z) - \Phi_p^*(z) = 0$ with sheets determined by whether the points are pure points of the associated measure or not.

[125] has two proofs of Theorem 5.9: one using the Abel map on the above referenced Riemann surface and one using Sard's theorem.

5.6. **Perturbation Conjectures.** [124, 125] have numerous conjectures and open problems. We want to end this section with a discussion of conjectures that describe perturbations of periodic Verblunsky coefficients. We discuss the Weyl-type conjecture in detail. As a model, consider Theorem 3.4 when $\alpha_n \equiv a \neq 0$. For $\operatorname{ess\,supp}(d\nu)$ to be $\Gamma_{a,1}$, the essential support for $\alpha_n \equiv a$, it suffices that $|\alpha_n| \to a$ and $\alpha_{n+1}/\alpha_n \to 1$. This suggests

Conjecture 5.10. Fix a period p set of Verblunsky coefficients with discriminant Δ. Let M be the set of period p (semi-infinite) sequences with discriminant Δ and let $S \subset \partial\mathbb{D}$ be their common essential support. Suppose

$$\lim_{j\to\infty} \inf_{\alpha\in M} \left[\sum_{n=1}^{\infty} e^{-n} |\beta_{j+n} - \alpha_n| \right] = 0$$

Then if ν is the measure with Verblunsky coefficients β, then $\operatorname{ess\,supp}(d\nu) = S$.

Thus, limit results only hold in the sense of approach to the isospectral manifold. There are also conjectures in [125] for extensions of Szegő's and Rakmanov's theorems in this context.

6. Spectral Theory Examples

[125, Chapter 12] is devoted to analysis of specific classes of Verblunsky coefficients, mainly finding analogs of known results for Schrödinger or discrete Schrödinger equations. Most of these are reasonably straightforward, but there are often some extra tricks needed and the results are of interest.

6.1. **Sparse and Decaying Random Verblunsky Coefficients.** In [72], Kiselev, Last, and Simon presented a thorough analysis of continuum and discrete Schrödinger operators with sparse or decaying random potentials, subjects with earlier work by Pearson [102], Simon [121], Delyon [23, 24], and Kotani-Ushiroya [74]. In [125, Sections 12.3 and 12.7], I have found analogs of these results for OPUC:

Theorem 6.1. *Let $d\mu$ have the form* (1.1). *Let $\{n_\ell\}_{\ell=1}^\infty$ be a monotone sequence of positive integers with $\liminf_{\ell\to\infty} \frac{n_{\ell+1}}{n_\ell} > 1$ and*

$$\alpha_j(d\mu) = 0 \qquad \text{if } j \notin \{n_\ell\} \tag{6.1}$$

and

$$\sum_{j=0}^\infty |\alpha_j(d\mu)|^2 < \infty \tag{6.2}$$

Then $\mu_{\rm s} = 0$, $\operatorname{supp}(d\mu) = \partial\mathbb{D}$, and $w, w^{-1} \in \cap_{p=1}^\infty L^p(\partial\mathbb{D}, \frac{d\theta}{2\pi})$.

This result was recently independently obtained by Golinskii [48]:

Theorem 6.2. *Let $\{n_\ell\}_{\ell=1}^\infty$ be a monotone sequence of positive integers with $\lim \frac{n_{\ell+1}}{n_\ell} = \infty$ so that* (6.1) *holds. Suppose $\lim_{j\to\infty}|\alpha_j(d\mu)| = 0$ and* (6.2) *fails. Then $d\mu$ is purely singular continuous.*

Theorem 6.3. *Let $\{\alpha_j(\omega)\}_{j=0}^\infty$ be a family of independent random variables with values in $\mathbb{D}$ with*

$$\mathbb{E}(\alpha_j(\omega)) = 0 \tag{6.3}$$

and

$$\sum_{j=0}^\infty \mathbb{E}(|\alpha_j(\omega)|^2) < \infty \tag{6.4}$$

Let $d\mu_\omega$ be the measure with $\alpha_j(d\mu_\omega) = \alpha_j(\omega)$. Then for a.e. ω, $d\mu_\omega$ has the form (1.1) *with $d\mu_{\omega,\rm s}$ and $w(\theta) > 0$ for a.e. θ.*

This result is not new; it is a result of Teplyaev, with earlier results of Nikishin [99] (see Teplyaev [138, 139, 140, 141]). We state it for comparison with the next two theorems.

The theorems assume (6.3) and also

$$\sup_{\omega,j} |\alpha_j(\omega)| < 1 \quad \sup_{\omega} |\alpha_j(\omega)| \to 0 \qquad \text{as } j \to \infty \tag{6.5}$$

$$\mathbb{E}(\alpha_j(\omega)^2) = 0 \tag{6.6}$$

$$\mathbb{E}(|\alpha_j(\omega)|^2)^{1/2} = \Gamma j^{-\gamma} \qquad \text{if } j > J_0 \tag{6.7}$$

Theorem 6.4. *If $\{\alpha_j(\omega)\}_{j=0}^\infty$ is a family of independent random variables so* (6.3), (6.5), (6.6), *and* (6.7) *hold and* $\Gamma > 0$, $\gamma < \frac{1}{2}$, *then for a.e. pairs* ω *and* $\lambda \in \partial\mathbb{D}$, $d\mu_{\lambda,\omega}$, *the measure with* $\alpha_j(d\mu_{\lambda,\omega}) = \lambda\alpha_j(\omega)$, *is pure point with support equal to* $\partial\mathbb{D}$ (*i.e., dense mass points*).

Theorem 6.5. *If $\{\alpha_j(\omega)\}_{j=0}^\infty$ is a family of independent random variables so* (6.3), (6.5), (6.6), *and* (6.7) *hold for* $\Gamma > 0$, $\gamma = \frac{1}{2}$, *and*

$$\sup_{n,\omega} n^{1/2}|\alpha_n(\omega)| < \infty \tag{6.8}$$

Then
 (i) *If* $\Gamma^2 > 1$, *then for a.e. pairs* $\lambda \in \partial\mathbb{D}$, $\omega \in \Omega$, $d\mu_{\lambda,\omega}$ *has dense pure point spectrum.*
 (ii) *If* $\Gamma^2 \leq 1$, *then for a.e. pairs* $\lambda \in \partial\mathbb{D}$, $\omega \in \Omega$, $d\mu_{\lambda,\omega}$ *has purely singular continuous spectrum of exact Hausdorff dimension* $1 - \Gamma^2$ *in that* $d\mu_{\lambda,\omega}$ *is supported on a set of dimension* $1 - \Gamma^2$ *and gives zero weight to any set* S *with* $\dim(S) < 1 - \Gamma^2$.

For the last two theorems, a model to think of is to let $\{\beta_n\}_{n=0}^\infty$ be identically distributed random variables on $\{z \mid |z| \leq r\}$ for some $r < 1$ with a rotationally invariant distribution and to let $\alpha_n = \Gamma^{1/2}\mathbb{E}(|\beta_1|^2)^{-1/2}\max(n,1)^{-\gamma}\beta_n$.

The proofs of these results exploit Prüfer variables, which for OPUC go back to Nikishin [98] and Nevai [94].

6.2. **Fibonacci Subshifts.** For discrete Schrödinger operators, there is an extensive literature [4, 73, 100, 133, 9, 13, 19, 20, 18, 21, 79, 80] on subshifts (see [125, 112, 82] for a definition of subshifts). In [125, Section 12.8], we have analyzed the OPUC analog of the most heavily studied of these subshifts, defined as follows: Pick $\alpha, \beta \in \mathbb{D}$. Let $F_1 = \alpha$, $F_2 = \alpha\beta$, and $F_{n+1} = F_nF_{n-1}$ for $n = 2, 3, \dots$. F_{n+1} is a sequence which starts with F_n and so there is a limit $F = \alpha, \beta, \alpha, \alpha, \beta, \alpha, \beta, \alpha, \alpha, \beta, \alpha, \alpha, \dots$. We write $F(\alpha, \beta)$ when we want to vary α and β.

Theorem 6.6. *The essential support of the measure* μ *with* $\alpha.(d\mu) = F(\alpha,\beta)$ *is a closed perfect set of Lebesgue measure zero for any* $\alpha \neq \beta$. *For fixed* α_0, β_0 *and a.e.* $\lambda \in \partial\mathbb{D}$, *the measure with* $\alpha.(d\mu) = F(\lambda\alpha_0, \lambda\beta_0)$ *is a pure point measure, with each pure point isolated and the limit points of the pure points a perfect set of* $\frac{d\theta}{2\pi}$*-measure zero.*

The proof follows that for Schrödinger operators with a few additional tricks needed.

6.3. **Dense Embedded Point Spectrum.** Naboko [87, 88, 89, 90, 91] and Simon [122] constructed Schrödinger operators $-\frac{d^2}{dx^2} + V(x)$ with $V(x)$ decaying only slightly slower than $|x|^{-1}$ so there is dense embedded point spectrum. Naboko's method extends to OPUC.

Theorem 6.7. *Let $g(n)$ be an arbitrary function with $0 < g(n) \leq g(n+1)$ and $g(n) \to \infty$ as $n \to \infty$. Let $\{\omega_j\}_{j=0}^{\infty}$ be an arbitrary subset of $\partial\mathbb{D}$ which are multiplicatively rationally independent, that is, for no $n_1, n_2, \dots, n_k \in \mathbb{Z}$ other than $(0, 0, \dots, 0)$, is it true that $\prod_{j=1}^{k}(\omega_j\omega_0^{-1})^{n_j} = 1$. Then there exists a sequence $\{\alpha_j\}_{j=0}^{\infty}$ of Verblunsky coefficients with*

$$|\alpha_n| \leq \frac{g(n)}{n}$$

for all n so that the measure $d\mu$ with $\alpha_j(d\mu) = \alpha_j$ has pure points at each ω_j.

Remark. If $g(n) \leq n^{1/2-\varepsilon}$, then $|\alpha_n| \in \ell^2$ so, by Szegő's theorem, $d\mu$ has the form (1.1) with $w(\theta) > 0$ for a.e. θ, that is, the point masses are embedded in a.c. spectrum.

6.4. **High Barriers.** Jitomirskaya-Last [65] analyzed sparse high barriers to get discrete Schrödinger operators with fractional-dimensional spectrum. Their methods can be applied to OPUC. Let $0 < a < 1$ and

$$L = 2^{n^n} \tag{6.9}$$

$$\alpha_j = (1 - \rho_j^2)^{-1/2} \tag{6.10}$$

$$\rho_j = \begin{cases} L_n^{-(1-a)/2a} & j = L_n \\ 0 & \text{otherwise} \end{cases} \tag{6.11}$$

Theorem 6.8. *Let α_j be given by (6.10)/(6.11) and let $d\mu_\lambda$ be the Aleksandrov measures with $\alpha_j(d\mu_\lambda) = \lambda\alpha_j$. Then for Lebesgue a.e. λ, $d\mu_\lambda$ has exact dimension a in the sense that $d\mu_\lambda$ is supported on a set of Hausdorff dimension a and gives zero weight to any set B of Hausdorff dimension strictly less than a.*

References

[1] N.I. Akhiezer and M. Krein, *Das Momentenproblem bei der zusätzlichen Bedingung von A. Markoff*, Zap. Har'kov. Math. Obšč. **12** (1936), 13–36.

[2] N.I. Akhiezer and M. Krein, *Some Questions in the Theory of Moments*, Transl. Math. Monographs, Vol. 2, American Mathematical Society, Providence, R.I., 1962; Russian original, 1938.

[3] F.V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.

[4] S. Aubry, *Metal insulator transition in one-dimensional deformable lattices*, in "Bifurcation Phenomena in Mathematical Physics and Related Topics," (C. Bardos and D. Bessis, eds.), pp. 163–184, NATO Advanced Study Institute Series, Ser. C: Mathematical and Physical Sciences, **54**, D. Reidel Publishing, Dordrecht-Boston, 1980.

[5] J. Avron and B. Simon, *Almost periodic Schrödinger operators, II. The integrated density of states*, Duke Math. J. **50** (1983), 369–391.

[6] V. Bargmann, *On the number of bound states in a central field of force*, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 961–966.

[7] D. Barrios Rolanía and G. López Lagomasino, *Ratio asymptotics for polynomials orthogonal on arcs of the unit circle*, Constr. Approx. **15** (1999), 1–31.

[8] D. Barrios Rolanía, G. López Lagomasino, and E.B. Saff, *Asymptotics of orthogonal polynomials inside the unit circle and Szegő-Padé approximants*, J. Comput. Appl. Math. **133** (2001), 171–181.

[9] J. Bellissard, B. Iochum, E. Scoppola, and D. Testard, *Spectral properties of one-dimensional quasi-crystals*, Comm. Math. Phys. **125** (1989), 527–543.

[10] M.S. Birman, *On the spectrum of singular boundary-value problems*, Mat. Sb. (N.S.) **55 (97)** (1961), 125–174; translated in Amer. Math. Soc. Transl. **53** (1966), 23–80.

[11] M.S. Birman and M.G. Krein, *On the theory of wave operators and scattering operators*, Dokl. Akad. Nauk SSSR **144** (1962), 475–478. [Russian]

[12] G. Borg, *Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe*, Acta Math. **78** (1946), no.1, 1–96.

[13] A. Bovier and J.-M. Ghez, *Spectral properties of one-dimensional Schrödinger operators with potentials generated by substitutions*, Comm. Math. Phys. **158** (1993), 45–66.

[14] M.J. Cantero, L. Moral, and L. Velázquez, *Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle*, Linear Algebra Appl. **362** (2003), 29–56.

[15] M.J. Cantero, L. Moral, and L. Velázquez, *Unitary five-diagonal matrices, para-orthogonal polynomials and measures on the unit circle*, preprint.

[16] S. Clark, F. Gesztesy, H. Holden, and B.M. Levitan, *Borg-type theorems for matrix-valued Schrödinger operators*, J. Differential Equations **167** (2000), 181–210.

[17] J.M. Combes and L. Thomas, *Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators*, Comm. Math. Phys. **34** (1973), 251–270.

[18] D. Damanik, R. Killip, and D. Lenz, *Uniform spectral properties of one-dimensional quasicrystals. III. α-continuity*, Comm. Math. Phys. **212** (2000), 191–204.

[19] D. Damanik and D. Lenz, *Uniform spectral properties of one-dimensional quasicrystals. I. Absence of eigenvalues*, Comm. Math. Phys. **207** (1999), 687–696.

[20] D. Damanik and D. Lenz, *Uniform spectral properties of one-dimensional quasicrystals. II. The Lyapunov exponent*, Lett. Math. Phys. **50** (1999), 245–257.

[21] D. Damanik and D. Lenz, *Uniform spectral properties of one-dimensional quasicrystals. IV. Quasi-Sturmian potentials*, J. Anal. Math. **90** (2003), 115–139.

[22] P.A. Deift and B. Simon, *Almost periodic Schrödinger operators, III. The absolutely continuous spectrum in one dimension*, Commun. Math. Phys. **90** (1983), 389–411.

[23] F. Delyon, *Appearance of a purely singular continuous spectrum in a class of random Schrödinger operators*, J. Statist. Phys. **40** (1985), 621–630.

[24] F. Delyon, B. Simon, and B. Souillard, *From power pure point to continuous spectrum in disordered systems*, Ann. Inst. H. Poincaré **42** (1985), 283–309.

[25] S.A. Denisov, *Probability measures with reflection coefficients* $\{a_n\} \in \ell^4$ *and* $\{a_{n+1} - a_n\} \in \ell^2$ *are Erdős measures*, J. Approx. Theory **117** (2002), 42–54.

[26] S.A. Denisov, *On Rakhmanov's theorem for Jacobi matrices*, Proc. Amer. Math. Soc. **132** (2004), 847–852.

[27] S.A. Denisov, *On the existence of wave operators for some Dirac operators with square summable potential*, to appear in Geom. Funct. Anal.

[28] S.A. Denisov and S. Kupin, *Asymptotics of the orthogonal polynomials from the Szegő class with a polynomial weight*, preprint.

[29] B.A. Dubrovin, V.B. Matveev, and S.P. Novikov, *Nonlinear equations of Korteweg-de Vries type, finite-band linear operators and Abelian varieties*, Uspehi Mat. Nauk **31** (1976), no. 1(187), 55–136. [Russian]

[30] L. Fejér, *Über die Lage der Nullstellen von Polynomen, die aus Minimumforderungen gewisser Art entspringen*, Math. Ann. **85** (1922), 41–48.

[31] H. Flaschka, *Discrete and periodic illustrations of some aspects of the inverse method*, in "Dynamical Systems, Theory and Applications" (J. Moser, ed.), Lecture Notes In Physics, Vol. 38, p. 441–466, Springer Verlag, Berlin, 1975.

[32] H. Flaschka and D.W. McLaughlin, *Canonically conjugate variables for the Korteweg-de Vries equation and the Toda lattice with periodic boundary conditions*, Progr. Theoret. Phys. **55** (1976), 438–456.

[33] J.S. Geronimo, F. Gesztesy, and H. Holden, *A new integrable hierarchy of differential-difference equations and its algebro-geometric solutions*, preprint.

[34] J.S. Geronimo and R. Johnson, *Rotation number associated with difference equations satisfied by polynomials orthogonal on the unit circle*, J. Differential Equations **132** (1996), 140–178.

[35] J.S. Geronimo and R. Johnson, *An inverse problem associated with polynomials orthogonal on the unit circle*, Comm. Math. Phys. **193** (1998), 125–150.

[36] J.S. Geronimo and A. Teplyaev, *A difference equation arising from the trigonometric moment problem having random reflection coefficients—an operator-theoretic approach*, J. Funct. Anal. **123** (1994), 12–45.

[37] Ya. L. Geronimus, *On polynomials orthogonal on the circle, on trigonometric moment problem, and on allied Carathéodory and Schur functions*, Mat. Sb. **15** (1944), 99–130. [Russian]

[38] J. Geronimus, *On the trigonometric moment problem*, Ann. of Math. (2) **47** (1946), 742–761.

[39] Ya. L. Geronimus, *Polynomials Orthogonal on a Circle and Their Applications*, Amer. Math. Soc. Translation **1954** (1954), no. 104, 79 pp.

[40] Ya. L. Geronimus, *Certain limiting properties of orthogonal polynomials*, Vest. Kharkov. Gos. Univ. **32** (1966), 40–50. [Russian]

[41] Ya. L. Geronimus, *Orthogonal polynomials*, Engl. translation of the appendix to the Russian translation of Szegő's book [137], in "Two Papers on Special Functions," Amer. Math. Soc. Transl., Ser. 2, Vol 108, pp. 37–130, American Mathematical Society, Providence, R.I., 1977.

[42] D.J. Gilbert and D.B. Pearson, *On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators*, J. Math. Anal. Appl. **128** (1987), 30–56.

[43] I.C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Selfadjoint Operators*, Transl. Math. Monographs, Vol. 18, American Mathematical Society, Providence, R.I., 1969.

[44] L. Golinskii, *Schur functions, Schur parameters and orthogonal polynomials on the unit circle*, Z. Anal. Anwendungen **12** (1993), 457–469.

[45] L. Golinskii, *Geronimus polynomials and weak convergence on a circular arc*, Methods Appl. Anal. **6** (1999), 421–436.

[46] L. Golinskii, *The Christoffel function for orthogonal polynomials on a circular arc*, J. Approx. Theory **101** (1999), 165–174.

[47] L. Golinskii, *Operator theoretic approach to orthogonal polynomials on an arc of the unit circle*, Mat. Fiz. Anal. Geom. **7** (2000), 3–34.

[48] L. Golinskii, *Absolutely continuous measures on the unit circle with sparse Verblunsky coefficients*, to appear in Mat. Fiz. Anal. Geom.

[49] L. Golinskii and P. Nevai, *Szegő difference equations, transfer matrices and orthogonal polynomials on the unit circle*, Comm. Math. Phys. **223** (2001), 223–259.

[50] L. Golinskii, P. Nevai, F. Pintér, and W. Van Assche, *Perturbation of orthogonal polynomials on an arc of the unit circle, II*, J. Approx. Theory **96** (1999), 1–32.

[51] L. Golinskii, P. Nevai, and W. Van Assche, *Perturbation of orthogonal polynomials on an arc of the unit circle*, J. Approx. Theory **83** (1995), 392–422.

[52] W.B. Gragg, *Positive definite Toeplitz matrices, the Arnoldi process for isometric operators, and Gaussian quadrature on the unit circle*, J. Comput. Appl. Math. **46** (1993), 183–198; Russian original in "Numerical methods of linear algebra," pp. 16–32, Moskov. Gos. Univ., Moscow, 1982.

[53] G.W. Hill, *On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon*, Acta Math. **8** (1886), 1–36.

[54] H. Hochstadt, *Functiontheoretic properties of the discriminant of Hill's equation*, Math. Z. **82** (1963), 237–242.

[55] H. Hochstadt, *Results, old and new, in the theory of Hill's equation*, Trans. New York Acad. Sci. (2) **26** (1963/1964), 887–901.

[56] H. Hochstadt, *On the determination of a Hill's equation from its spectrum*, Arch. Rational Mech. Anal. **19** (1965), 353–362.

[57] H. Hochstadt, *On the determination of a Hill's equation from its spectrum. II*, Arch. Rational Mech. Anal. **23** (1966), 237–238.

[58] H. Hochstadt, *On a Hill's equation with double eigenvalues*, Proc. Amer. Math. Soc. **65** (1977), 373–374.

[59] H. Hochstadt, *A generalization of Borg's inverse theorem for Hill's equations*, J. Math. Anal. Appl. **102** (1984), 599–605.

[60] H. Hochstadt, *A direct and inverse problem for a Hill's equation with double eigenvalues*, J. Math. Anal. Appl. **66** (1978), 507–513.

[61] H. Hochstadt, *On the theory of Hill's matrices and related inverse spectral problems*, Linear Algebra and Appl. **11** (1975), 41–52.
[62] D. Hundertmark, E.H. Lieb, and L.E. Thomas, *A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator* Adv. Theor. Math. Phys. **2** (1998), 719–731.
[63] D. Hundertmark and B. Simon, *Lieb-Thirring inequalities for Jacobi matrices* J. Approx. Theory **118** (2002), 106–130.
[64] S. Jitomirskaya and Y. Last, *Dimensional Hausdorff properties of singular continuous spectra*, Phys. Rev. Lett. **76** (1996), 1765–1769.
[65] S. Jitomirskaya and Y. Last, *Power-law subordinacy and singular spectra, I. Half-line operators*, Acta Math. **183** (1999), 171–189.
[66] R. Johnson and J. Moser, *The rotation number for almost periodic potentials*, Comm. Math. Phys. **84** (1982), 403–438.
[67] S. Khrushchev, *Parameters of orthogonal polynomials*, In "Methods of Approximation Theory in Complex Analysis and Mathematical Physics" (Leningrad, 1991), pp. 185–191, Lecture Notes in Math. 1550, Springer, Berlin, 1993.
[68] S. Khrushchev, *Schur's algorithm, orthogonal polynomials, and convergence of Wall's continued fractions in* $L^2(\mathbb{T})$, J. Approx. Theory **108** (2001), 161–248.
[69] S. Khrushchev, *Classification theorems for general orthogonal polynomials on the unit circle*, J. Approx. Theory **116** (2002), 268–342.
[70] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Ann. of Math. **158** (2003), 253–321.
[71] R. Killip and B. Simon, in preparation.
[72] A. Kiselev, Y. Last, and B. Simon, *Modified Prüfer and EFGP transforms and the spectral analysis of one-dimensional Schrödinger operators*, Comm. Math. Phys. **194** (1998), 1–45.
[73] M. Kohmoto, L.P. Kadanoff, and C. Tang, *Localization problem in one dimension: Mapping and escape*, Phys. Rev. Lett. **50** (1983), 1870–1872.
[74] S. Kotani and N. Ushiroya, *One-dimensional Schrödinger operators with random decaying potentials*, Comm. Math. Phys. **115** (1988), 247–266.
[75] S. Kupin, *On a spectral property of Jacobi matrices*, preprint.
[76] S. Kupin, *Spectral properties of Jacobi matrices and sum rules of special form*, preprint.
[77] H.J. Landau, *Maximum entropy and the moment problem*, Bull. Amer. Math. Soc. **16** (1987), 47–77.
[78] A. Laptev, S. Naboko, and O. Safronov, *On new relations between spectral properties of Jacobi matrices and their coefficients*, to appear in Comm. Math. Phys.
[79] D. Lenz, *Singular spectrum of Lebesgue measure zero for one-dimensional quasicrystals*, Comm. Math. Phys. **227** (2002), 119–130.
[80] D. Lenz, *Uniform ergodic theorems on subshifts over a finite alphabet*, Ergodic Theory Dynam. Systems **22** (2002), 245–255.
[81] E.H. Lieb, *Bounds on the eigenvalues of the Laplace and Schrödinger operators*, Bull. Amer. Math. Soc. **82** (1976), 751–753.
[82] M. Lothaire, *Algebraic Combinatorics on Words*, Encyclopedia of Mathematics and Its Applications, 90, Cambridge University Press, Cambridge, 2002.

[83] W. Magnus and S. Winkler, *Hill's Equation*, Interscience Tracts in Pure and Applied Mathematics, No. 20, Interscience Publishers, New York, 1966.

[84] A. Máté, P. Nevai, and V. Totik, *Asymptotics for the ratio of leading coefficients of orthonormal polynomials on the unit circle*, Constr. Approx. **1** (1985), 63–69.

[85] H.P. McKean and P. van Moerbeke, *The spectrum of Hill's equation*, Invent. Math. 30 (1975), 217–274.

[86] H.N. Mhaskar and E.B. Saff, *On the distribution of zeros of polynomials orthogonal on the unit circle*, J. Approx. Theory **63** (1990), 30–38.

[87] S.N. Naboko, *Schrödinger operators with decreasing potential and with dense point spectrum*, Soviet Math. Dokl. **29** (1984), 688–691; Rusiian original in Dokl. Akad. Nauk SSSR **276** (1984), 1312–1315.

[88] S.N. Naboko, *On the dense point spectrum of Schrödinger and Dirac operators* Theoret. and Math. Phys. **68** (1986), 646–653; Russian original in Teoret. Mat. Fiz. **68** (1986), 18–28.

[89] S.N. Naboko, *On the singular spectrum of discrete Schrödinger operator*, Séminaire sur les Équations aux Dérivées Partielles, pp. 1993–1994, Exp. No. XII, École Polytech., Palaiseau, 1994.

[90] S.N. Naboko and S.I. Yakovlev, *The point spectrum of a discrete Schrödinger operator*, Funct. Anal. Appl. **26** (1992), 145-147; Russian original in Funktsional. Anal. i Prilozhen. **26** (1992), 85–88.

[91] S.N. Naboko and S.I. Yakovlev, *The discrete Schrödinger operator. A point spectrum lying in the continuous spectrum*, St. Petersburg Math. J. **4** (1993), 559-568; Russian original in Algebra i Analiz **4** (1992), 183–195.

[92] F. Nazarov, F. Peherstorfer, A. Volberg, and P. Yuditskii, *On generalized sum rules for Jacobi matrices*, preprint.

[93] I. Nenciu and B. Simon, in preparation.

[94] P. Nevai, *Orthogonal polynomials, measures and recurrences on the unit circle*, Trans. Amer. Math. Soc. **300** (1987), 175–189.

[95] P. Nevai, *Weakly convergent sequences of functions and orthogonal polynomials*, J. Approx. Theory **65** (1991), 322–340.

[96] P. Nevai and V. Totik, *Orthogonal polynomials and their zeros*, Acta Sci. Math. (Szeged) **53** (1989), 99–104.

[97] P. Nevai and V. Totik, *Denisov's theorem on recurrent coefficients*, preprint

[98] E.M. Nikishin, *An estimate for orthogonal polynomials*, Acta Sci. Math. (Szeged) **48** (1985), 395–399. [Russian]

[99] E.M. Nikishin, *Random orthogonal polynomials on the circle*, Moscow Univ. Math. Bull. **42** (1987), 42–45; Russian original in Vestnik Moskov. Univ. Ser. I Mat. Mekh. **42** (1987), 52–55.

[100] S. Ostlund, R. Pandit, D. Rand, H.J. Schellnhuber, and E.D. Siggia, *One-dimensional Schrödinger equation with an almost periodic potential*, Phys. Rev. Lett. **50** (1983), 1873–1876.

[101] L.A. Pastur, *Spectra of random selfadjoint operators*, Uspehi Mat. Nauk **28** (1973), 3–64.

[102] D.B. Pearson, *Singular continuous measures in scattering theory*, Comm. Math. Phys. **60** (1978), 13–36.

[103] F. Peherstorfer, *Deformation of minimal polynomials and approximation of several intervals by an inverse polynomial mapping*, J. Approx. Theory **111** (2001), 180–195.

[104] F. Peherstorfer, *Inverse images of polynomial mappings and polynomials orthogonal on them*, in "Proc. Sixth International Symposium on Orthogonal Polynomials, Special Functions and their Applications" (Rome, 2001), J. Comput. Appl. Math. **153** (2003), 371–385.

[105] F. Peherstorfer and R. Steinbauer, *Perturbation of orthogonal polynomials on the unit circle—a survey*, In "Orthogonal Polynomials on the Unit Circle: Theory and Applications" (Madrid, 1994), pp. 97–119, Univ. Carlos III Madrid, Leganés, 1994.

[106] F. Peherstorfer and R. Steinbauer, *Characterization of general orthogonal polynomials with respect to a functional*, J. Comp. Appl. Math. **65** (1995), 339–355.

[107] F. Peherstorfer and R. Steinbauer, *Orthogonal polynomials on arcs of the unit circle, I*, J. Approx. Theory **85** (1996), 140–184.

[108] F. Peherstorfer and R. Steinbauer, *Orthogonal polynomials on arcs of the unit circle, II. Orthogonal polynomials with periodic reflection coefficients*, J. Approx. Theory **87** (1996), 60–102.

[109] F. Peherstorfer and R. Steinbauer, *Asymptotic behaviour of orthogonal polynomials on the unit circle with asymptotically periodic reflection coefficients*, J. Approx. Theory **88** (1997), 316–353.

[110] F. Peherstorfer and R. Steinbauer, *Asymptotic behaviour of orthogonal polynomials on the unit circle with asymptotically periodic reflection coefficients, II. Weak asymptotics*, J. Approx. Theory **105** (2000), 102–128.

[111] F. Pintér and P. Nevai, *Schur functions and orthogonal polynomials on the unit circle*, in "Approximation Theory and Function Series," Bolyai Soc. Math. Stud. 5, pp. 293–306, János Bolyai Math. Soc., Budapest, 1996.

[112] M. Queffélec, *Substitution Dynamical Systems—Spectral Analysis*, Lecture Notes in Mathematics, 1294, Springer-Verlag, Berlin, 1987.

[113] E.A. Rakhmanov, *On the asymptotics of the ratio of orthogonal polynomials*, Math. USSR Sb. **32** (1977), 199–213.

[114] E.A. Rakhmanov, *On the asymptotics of the ratio of orthogonal polynomials, II*, Math. USSR Sb. **46** (1983), 105–117.

[115] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. 4: Analysis of Operators*, Academic Press, New York, 1978.

[116] E.B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Grundlehren der Mathematischen Wissenschaften, Band 316, Springer, Berlin-Heiderlberg, 1997.

[117] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, I*, J. Reine Angew. Math. **147** (1917), 205–232. English translation in "Schur methods in operator theory and signal processing" (edited by I. Gohberg), Operator Theory: Advances and Applications **18** Birkhäuser Verlag, Basel, 1986.

[118] J. Schwinger, *On the bound states of a given potential*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 122–129.

[119] B. Simon, *On the genericity of nonvanishing instability intervals in Hill's equation*, Ann. Inst. H. Poincaré Sect. A (N.S.) **24** (1976), 91–93.

[120] B. Simon, *Trace Ideals and Their Applications*, London Mathematical Society Lecture Note Series, 35, Cambridge University Press, Cambridge-New York, 1979.

[121] B. Simon, *Some Jacobi matrices with decaying potential and dense point spectrum*, Comm. Math. Phys. **87** (1982), 253–258.

[122] B. Simon, *Some Schrödinger operators with dense point spectrum*, Proc. Amer. Math. Soc. **125** (1997), 203–208.

[123] B. Simon, *A canonical factorization for meromorphic Herglotz functions on the unit disk and sum rules for Jacobi matrices*, to appear in J. Funct. Anal.

[124] B. Simon, *Orthogonal Polynomials on the Unit Circle, Vol. 1*, AMS Colloquium Series, American Mathematical Society, Providence, RI, expected 2004.

[125] B. Simon, *Orthogonal Polynomials on the Unit Circle, Vol. 2*, AMS Colloquium Series, American Mathematical Society, Providence, RI, expected 2004.

[126] B. Simon, *The sharp form of the strong Szegő theorem*, to appear in Proc. Conf. on Geometry and Spectral Theory (Haifa, 2004).

[127] B. Simon and T. Spencer, *Trace class perturbations and the absence of absolutely continuous spectrum*, Comm. Math. Phys. **125** (1989), 113–126.

[128] B. Simon and V. Totik, *Limits of zeros of orthogonal polynomials on the circle*, preprint.

[129] B. Simon and A. Zlatoš, *Sum rules and the Szegő condition for orthogonal polynomials on the real line*, to appear in Comm. Math. Phys.

[130] B. Simon and A. Zlatoš, in preparation.

[131] H. Stahl and V. Totik, *General Orthogonal Polynomials*, Cambridge Univ. Press, Cambridge, 1992.

[132] T. Stieltjes, *Recherches sur les fractions continues*, Anns. Fac. Sci. Univ. Toulouse **8** (1894–1895), J1–J122; **9**, A5–A47.

[133] A. Sütő, *The spectrum of a quasiperiodic Schrödinger operator*, Comm. Math. Phys. **111** (1987), 409–415.

[134] G. Szegő, *Beiträge zur Theorie der Teoplitzschen Formen, I*, Math. Z. **6** (1920), 167–202.

[135] G. Szegő, *Beiträge zur Theorie der Teoplitzschen Formen, II*, Math. Z. **9** (1921), 167–190.

[136] G. Szegő, *Über den asymptotischen Ausdruck von Polynomen, die durch eine Orthogonalitätseigenschaft definiert sind*, Math. Ann. **85** (1922), 114–139.

[137] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, American Mathematical Society, Providence, R.I., 1939; 3rd edition, 1967.

[138] A.V. Teplyaev, *Properties of polynomials that are orthogonal on the circle with random parameters*, J. Soviet Math. **61** (1992), 1931–1935; Russian original in Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **177** (1989), Problemy Teorii Veroyatnost. Raspred. XI, 157–162, 191–192.

[139] A.V. Teplyaev, *The pure point spectrum of random orthogonal polynomials on the circle*, Soviet Math. Dokl. **44** (1992), 407–411; Russian original in Dokl. Akad. Nauk SSSR **320** (1991), 49–53.

[140] A.V. Teplyaev, *Absolute continuity of the spectrum of random polynomials that are orthogonal on the circle and their continual analogues*, J. Math. Sci. **75** (1995), 1982–1984; Russian original in Zap. Nauchn. Sem. S.-Peterburg.

Otdel. Mat. Inst. Steklov. (POMI) **194** (1992), Problemy Teorii Veroyatnost. Raspred. **12**, 170–173, 180–181.

[141] A.V. Teplyaev, *Continuous analogues of random polynomials that are orthogonal on the circle*, Theory Probab. Appl. **39** (1994), 476–489; Russian original in Teor. Veroyatnost. i Primenen. **39** (1994), 588–604.

[142] M. Toda, *Theory of Nonlinear Lattices*, second edition, Springer Series in Solid-State Sciences, **20**, Springer-Verlag, Berlin, 1989.

[143] V. Totik, *Orthogonal polynomials with ratio asymptotics*, Proc. Amer. Math. Soc. **114** (1992), 491–495.

[144] E. Trubowitz, *The inverse problem for periodic potentials*, Comm. Pure Appl Math. **30** (1977), 321–337.

[145] P. van Moerbeke, *The spectrum of Jacobi matrices*, Invent. Math. **37** (1976), 45–81.

[146] S. Verblunsky, *On positive harmonic functions: A contribution to the algebra of Fourier series*, Proc. London Math. Soc. **38** (1935), 125–157.

[147] S. Verblunsky, *On positive harmonic functions* (*second part*), Proc. London Math. Soc. **40** (1936), 290–320.

[148] T. Weidl, *On the Lieb-Thirring constants* $L_{\gamma,1}$ *for* $\gamma \geq \frac{1}{2}$, Comm. Math. Phys. **178** (1996), 135–146.