

**FINE STRUCTURE OF THE ZEROS OF  
ORTHOGONAL POLYNOMIALS,  
II. OPUC WITH COMPETING EXPONENTIAL DECAY**

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ABSTRACT. We present a complete theory of the asymptotics of the zeros of OPUC with Verblunsky coefficients  $\alpha_n = \sum_{\ell=1}^L C_\ell b_\ell^n + O((b\Delta)^n)$  where  $\Delta < 1$  and  $|b_\ell| = b < 1$ .

1. INTRODUCTION

This paper is the second in a series [7, 8] that discusses detailed asymptotics of the zeros of orthogonal polynomials with special emphasis on distances between nearby zeros. We will focus here on OPUC, orthogonal polynomials on the unit circle; see [9, 2, 5, 6] for background. The polynomials are described by the recursion coefficients,  $\{\alpha_n\}_{n=0}^\infty$ , called Verblunsky coefficients, that give the monic OPUC,  $\Phi_n(z)$ , by

$$\Phi_n(z) = z\Phi_n(z) - \bar{\alpha}_n\Phi_n^*(z) \tag{1.1}$$

where

$$\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})} \tag{1.2}$$

One of the examples discussed in the first paper [7] is where  $0 < b < 1$  and

$$\alpha_n = Cb^n + O((b\Delta)^n) \tag{1.3}$$

for  $0 < \Delta < 1$ . A typical example is shown in Figure 1 where

$$\alpha_n = \left(\frac{1}{2}\right)^{n+1} \tag{1.4}$$

and zeros of  $\Phi_{22}$  are shown. Figures and numeric zeros, which appear in [5, 7] and here, are computed using Mathematica and code written by M. Stoiciu.

Critical aspects of the zeros in this case are:

- (a) Finitely many zeros outside  $|z| = b + O(\log n/n)$  at the Nevai-Totik points, that is, solutions of  $D(1/\bar{z})^{-1} = 0$  where  $D$  is the Szegő function; this is due to Nevai-Totik [4].

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- (b) No zeros in  $|z| < b - O(\log n/n)$ . This is due to Barrios-López-Saff [1].
- (c) Zeros near  $|z| = b$  are asymptotically a distance  $2\pi b/n + o(1/n)$  from each other except for a single gap at  $z = b$ , where the nearby zeros are  $be^{\pm 2\pi i/n} + O(1/n^2)$ , and the distance between these neighboring zeros is  $2(2\pi b/n) + O(1/n^2)$ . This is proven in [7].

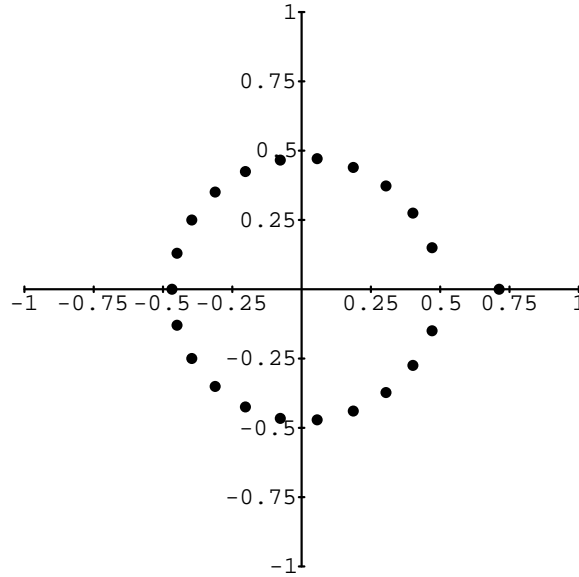


FIGURE 1

In this paper, I will analyze the case of Verblunsky coefficients of the form

$$\alpha_n = \sum_{\ell=1}^L C_\ell b_\ell^n + O((b\Delta)^n) \quad (1.5)$$

where the  $b_\ell$ 's are distinct,  $C_\ell \neq 0$  for all  $\ell$ , and

$$|b_\ell| = b \quad \ell = 1, \dots, L \quad (1.6)$$

If the  $b_\ell$ 's obey  $b_\ell^p = b^p$  for some  $p$ , then  $\alpha_{n+1}/\alpha_n$  is periodic of period  $p$ , and this overlaps examples of BLS [1] discussed later.

A typical example is shown in Figure 2 where

$$\alpha_n = \left(\frac{1}{2}\right)^{n+1} (1 + 2 \cos(\frac{\pi}{2}(n+1))) \quad (1.7)$$

and again, zeros of  $\Phi_{22}$  are shown. This has  $b_1 = 1/2$ ,  $b_2 = i/2$ ,  $b_3 = -i/2$ .

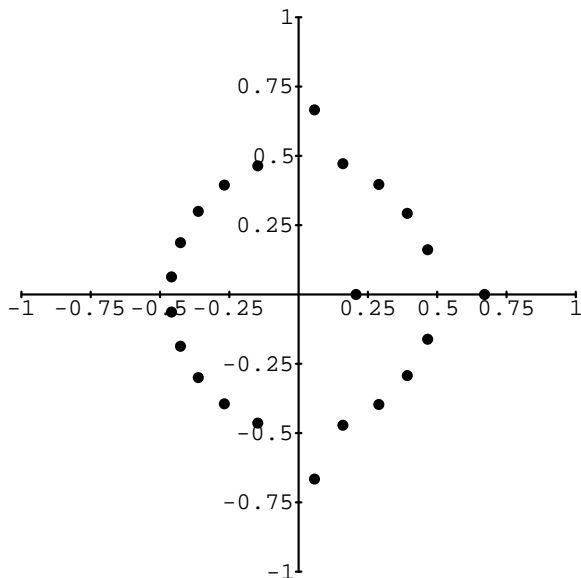


FIGURE 2

Here is what happens to (a)–(c) above:

- (a') The Nevai-Totik theory still applies. There are three NT zeros in the example in Figure 2.
- (b') There are at most  $L - 1$  zeros in  $|z| < b - O(\log n/n)$  and they can be described explicitly. In the case of periodic  $\alpha_{n+1}/\alpha_n$ , the interior zeros for  $\Phi_n$  with  $n = mp + q$  and  $m \rightarrow \infty$  have explicitly computable limits (limits, not merely accumulation points). For the example in Figure 2,  $\Phi_n$ ,  $n \equiv 2 \pmod{4}$ , have a zero approaching  $\frac{1}{2}(\sqrt{2} - 1) = 0.20710678118\dots$ . The actual zero in Figure 2 is at  $0.20710678374\dots$ .
- (c') There are gaps at each  $\bar{b}_\ell$ .

The method used in [7] to prove (a)–(c) exploits a second-order difference equation that  $\Phi_n$  obeys, relating  $\Phi_{n+1}$  to  $\Phi_n$  and  $\Phi_{n-1}$ . That method may extend to the periodic case,  $b_\ell^p = b$ , but will have small divisor problems if  $\alpha_n/b^n$  is only almost periodic. Instead, this paper will use a different and potentially more powerful and illuminating method that views (1.1) with  $\Phi_n^*(z)$  fixed as an inhomogeneous first-order difference equation for  $\Phi_n(z)$ .

In Section 2, we discuss asymptotics of  $\Phi_n$  for  $|z| < b$ . In Section 3, we discuss asymptotics of  $\Phi_n$  in the critical region  $b\Delta_1 < |z| < b\Delta_1^{-1}$ . In Section 4, we study zeros in  $|z| < b$ , and in Section 5, the zeros near  $|z| = b$ . Given Section 2, Section 4 is straightforward. Section 5 will

use the ideas in [7]. Finally, Section 6 makes various remarks about the connection to [1].

Figures 1 and 2 suggest that there might be a connection between the gaps in the clock and the Nevai-Totik zeros since in these two cases the number of NT zeros equals the number of gaps, and the zeros are near the gaps. There is certainly something to this notion. If a coefficient  $C_\ell$  in (1.5) is changed from zero to nonzero, for large  $n$ , the zero that was in the gap at zero value of  $C_\ell$  must stop being on the critical circle, and so presumably turns into an NT zero (this is an expectation, not a proof, since a proof involves controlling an interchange of limits). But the connection is not always there. Figure 8.3 in [5] shows an example where there is a gap but no associated NT zero. Moreover, the discussion in Section 13 of [7] makes it clear that the number of NT zeros can be arbitrary and is not, in general, the same as the number of gaps in the clock.

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## 2. ASYMPTOTICS IN $|z| < b - \varepsilon$

Our main goal here is to give asymptotics of  $\varphi_n(z)$  in the region  $|z| < b$  in case (1.5) holds. Our methods will also allow us to say something when weaker asymptotics (ratio asymptotics) holds and, in particular, to improve a result of BLS [1].

We begin with an analysis of some bounds and rate of convergence of  $\Phi_n^*$ . Estimates similar to these appear in [4, 3, 1, 7]. We use  $q$  rather than  $b$  since sometimes  $q = b + \varepsilon$ .

**Proposition 2.1.** *Suppose that*

$$|\alpha_n| \leq Cq^n \tag{2.1}$$

for some  $q \in (0, 1)$ . Then

(i) *With  $C_1 = \prod_{j=0}^{\infty} (1 + Cq^j) < \infty$ , we have*

$$|z| \leq 1 \Rightarrow |\Phi_n^*(z)| \leq C_1 \tag{2.2}$$

$$|z| \geq 1 \Rightarrow |\Phi_n(z)| \leq C_1 |z|^n \tag{2.3}$$

(ii) *For  $1 \leq |z| < q^{-1}$ ,*

$$|\Phi_n^*(z)| \leq 1 + C_1 C |z| (1 - q|z|)^{-1} \tag{2.4}$$

(iii) *For any  $q' > q$ , there is  $C_{q'}$  with*

$$|\Phi_n(z)| \leq C_{q'} (\max(|z|, q')^n) \tag{2.5}$$

for  $|z| < 1$ .

(iv) For  $|z| \leq q'$ ,

$$|\Phi_n^*(z) - D(z)^{-1}D(0)| \leq \tilde{C}_{q'}(qq')^n \quad (2.6)$$

(v)  $D(z)^{-1}$  has an analytic continuation to  $\{z \mid |z| < q^{-1}\}$ , and in that region,

$$\Phi_n^*(z) \rightarrow D(0)D(z)^{-1} \quad (2.7)$$

(vi) For  $|z| > q$ ,

$$\lim_{n \rightarrow \infty} z^{-n}\Phi_n(z) = D(0)\overline{D(1/\bar{z})}^{-1} \quad (2.8)$$

*Proof.* (i) For  $|z| = 1$ ,  $|\Phi_n(z)| = |\Phi_n^*(z)|$ , so (2.2) holds by induction from (1.1). (2.2) for  $|z| < 1$  follows from the maximum principle. (2.3) then follows from (1.2).

(ii) By (2.3) and the \* of (1.1),

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n(z) \quad (2.9)$$

$$\begin{aligned} |\Phi_{n+1}^*(z)| &\leq 1 + C_1 \sum_{j=0}^n |\alpha_j| |z|^{n+1} \\ &\leq 1 + CC_1 |z| \sum_{j=0}^{\infty} q^j |z|^j \end{aligned}$$

proving (2.4).

(iii) By (1.2) and (2.4), we have (2.5) for  $q' \leq |z| \leq 1$ . The result for  $|z| < q'$  then follows from the maximum principle.

(iv) By (2.9) and (2.5),

$$\begin{aligned} \sum_{m=n}^{\infty} |\Phi_{m+1}^*(z) - \Phi_m^*(z)| &\leq \sum_{m=n}^{\infty} C_q C q^m |z| (q')^m \\ &\leq \tilde{C}_{q'} (qq')^n \end{aligned}$$

Since  $\Phi_m^*(z) \rightarrow D(0)D(z)^{-1}$ , (2.6) holds.

(v) Following [4], we note that (2.9) implies that if  $|z| < q^{-1}$ , then  $\sum_n |\Phi_{n+1}^*(z) - \Phi_n^*(z)| < \infty$ , so  $\Phi_n^*(z)$  has a limit. Since Szegő's theorem holds if  $|z| < 1$ , we conclude that  $D(z)^{-1}$  has a continuation and (2.7) holds.

(vi) is immediate from (v) and (1.2).  $\square$

By iterating (1.1), we obtain

$$\Phi_n(z) = z^n - \sum_{j=1}^n \bar{\alpha}_{n-j} z^{j-1} \Phi_{n-j}^*(z) \quad (2.10)$$

We conclude that

**Theorem 2.2.** *Suppose (1.5) holds. Then, uniformly in each disk  $|z| < b - \varepsilon$ , we have that*

$$\begin{aligned} \varphi_n(z) &= \left[ \sum_{\ell=1}^L \bar{C}_\ell \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1} \right] D(z)^{-1} \\ &= O(n(b\Delta)^n) + O(nb^{2n}) + O\left( nb^n \left(1 - \frac{\varepsilon}{b}\right)^n \right) \end{aligned} \quad (2.11)$$

In particular, uniformly on the disk,

$$\lim_{n \rightarrow \infty} |b^{-n} \varphi_n - \tilde{Q}_n(z)| = 0$$

where

$$\tilde{Q}_n(z) = \left[ \sum_{\ell=1}^L \bar{C}_\ell \omega_\ell^n (\bar{b}_\ell - z)^{-1} \right] D(z)^{-1} \quad (2.12)$$

and

$$\omega_\ell = \frac{\bar{b}_\ell}{b} \quad (2.13)$$

*Remarks.* 1. In (2.11), the  $b^{2n}$  is actually  $b^{(3-\delta)n}$  for any  $\delta > 0$ .

2.  $D(z)\tilde{Q}_n(z) \prod_{\ell=1}^L (\bar{b}_\ell - z)$  is a polynomial of degree  $L - 1$  with almost periodic coefficients (periodic if  $b_\ell^p = 1$  for some  $p$  and all  $\ell$ ).

*Proof.* We begin by noting that

$$\begin{aligned} \sum_{j=1}^{\infty} \bar{b}_\ell^{n-j} z^{j-1} &= \bar{b}_\ell^{n-1} (1 - z\bar{b}_\ell^{-1})^{-1} \\ &= \bar{b}_\ell^n (\bar{b}_\ell - z)^{-1} \end{aligned} \quad (2.14)$$

and by an identical calculation,

$$\sum_{j=n+1}^{\infty} \bar{b}_\ell^{n-j} z^{j-1} = z^n (\bar{b}_\ell - z)^{-1} \quad (2.15)$$

It follows by (2.10) that if  $|z| < b - \varepsilon$ , then

$$\begin{aligned} &\left| \Phi_n(z) - D(z)^{-1} D(0) \sum_{\ell=1}^L \bar{c}_\ell \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1} \right| \\ &\leq C_\varepsilon |z|^n + \sum_{j=1}^n |z|^{j-1} \left| \bar{\alpha}_{n-j} \Phi_{n-j}^*(z) - \sum_{\ell=1}^L \bar{C}_\ell \bar{b}_\ell^n D(0) D(z)^{-1} \right| \end{aligned} \quad (2.16)$$

The  $|z|^n$  term is  $O(b^n(1 - \frac{\varepsilon}{b})^n)$ . By (1.5) and (2.5), the term in  $|\cdot|$  in (2.16) is bounded by

$$C[b^{2(n-j)} + (b\Delta)^{n-j}] \quad (2.17)$$

where the first term comes from  $|\Phi_n^* - D(0)D(z)^{-1}]\alpha_n|$  and the second from  $|D(z)^{-1}D(0)(\alpha_n - \sum_{\ell=1}^L C_\ell b_\ell^n)|$ . Since  $|z| < b$ , the sum in (2.16) is thus bounded by  $nC''[b^{2n} + \max(b\Delta, z)^n]$ . It follows that

$$D(0)^{-1}\Phi_n(z) - \left[ \sum_{\ell=1}^L \bar{C}_\ell \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1} \right] D(z) = \text{RHS of (2.11)}$$

(2.11) then follows from

$$\begin{aligned} D(0)^{-1}\Phi_n(z) &= \varphi_n(z) + O\left(b^n \left| \prod_{j=1}^{n-1} (1 - |\alpha_j|^2)^{1/2} - D(0) \right| \right) \\ &= \varphi_n(z) + O(b^{2n}) \quad \square \end{aligned}$$

From (2.10), we also get a result that only depends on ratio asymptotics:

**Theorem 2.3.** *Suppose that  $\alpha_n$  is a sequence of Verblunsky coefficients and  $n_j$  a subsequence so that*

$$(i) \quad \limsup_{n \rightarrow \infty} |\alpha_n|^{1/n} = b \in (0, 1) \quad (2.18)$$

$$(ii) \quad \liminf_{j \rightarrow \infty} |\alpha_{n_j}|^{1/n_j} = b \quad (2.19)$$

(iii) *For all  $k = 0, 1, 2, \dots$  and suitable  $\beta_k \in \mathbb{C}$ ,*

$$\lim_{j \rightarrow \infty} \frac{\bar{\alpha}_{n_j - k - 1}}{\bar{\alpha}_{n_j - 1}} = \beta_k \quad (2.20)$$

*exists.*

(iv) *For every  $\varepsilon$ , there is  $C_\varepsilon$  so that for all  $j$  and  $k = 1, 2, \dots$ , we have*

$$|\alpha_{n_j - k - 1}|(b - \varepsilon)^k \leq C_\varepsilon |\alpha_{n_j - 1}| \quad (2.21)$$

*Then for  $|z| < b$ ,  $\sum_{j=0}^{\infty} \beta_j z^j$  converges absolutely and uniformly on each disk  $|z| < b - \varepsilon$  and*

$$\lim_{j \rightarrow \infty} \frac{\Phi_{n_j}(z)}{\bar{\alpha}_{n_j - 1}} = -D(z)^{-1} \sum_{k=0}^{\infty} \beta_k z^k \quad (2.22)$$

*Proof.* By (2.21), we have

$$|\beta_k| \leq C_\varepsilon (b - \varepsilon)^{-k} \quad (2.23)$$

proving that  $\sum_{j=0}^{\infty} \beta_j z^j$  has radius of convergence at least  $b$ .

In (2.10) divided by  $\alpha_{n_j - 1}$ , the summand is bounded by

$$|z|^{j-1} (b - \varepsilon)^{-j} \sup_{m, |z| \leq b} |\Phi_m^*(z)|$$

which is summable for  $|z| < b - \varepsilon$ , so by the dominated convergence theorem, (2.22) holds.  $\square$

**Corollary 2.4.** *Let  $\alpha_n$  be a sequence of Verblunsky coefficients so that*

$$\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n-1}} = b \in (0, 1) \quad (2.24)$$

*Then uniformly for  $|z| < b - \varepsilon$ ,*

$$\lim_{n \rightarrow \infty} \frac{\Phi_n(z)}{\bar{\alpha}_{n-1}} = -D(z)^{-1} \left(1 - \frac{z}{b}\right)^{-1} \quad (2.25)$$

*Remarks.* 1. By rotational covariance, if (2.24) holds for some  $b \in \mathbb{D}$ , we can find a rotated problem with the ratio in  $(0, 1)$ , so this implies a result whenever ratio asymptotics holds.

2. This is related to results of BLS [1]; see the discussion in Section 6.

*Proof.* Clearly, (2.20) holds with  $\beta_k = b^{-k}$  so we need only prove (2.21). For any  $\delta$ , we have

$$\left| \frac{\alpha_{m-1} b}{\alpha_m} \right| \leq C_m^{(\delta)} (1 + \delta)$$

where  $C_m^{(\delta)} = 1$  for  $m \geq M_\delta$  for some  $M_\delta$ . It follows that

$$\left| \frac{\alpha_{m-k} b^k}{\alpha_m} \right| \leq \left[ \prod_{m=1}^{M_\delta} C_m^{(\delta)} \right] (1 + \delta)^k$$

which implies (2.21).  $\square$

Similarly, we obtain

**Corollary 2.5.** *Let  $\alpha_n$  be a sequence of Verblunsky coefficients,  $b \in (0, 1)$ , and  $c_1, c_2, \dots, c_p$  a sequence so that*

$$\prod_{j=1}^p c_j = 1 \quad (2.26)$$

*and*

$$\lim_{m \rightarrow \infty} \frac{\alpha_{mp+\ell}}{\alpha_{mp+\ell-1}} = bc_\ell \quad \ell = 1, 2, \dots, p \quad (2.27)$$

*Then uniformly for  $|z| < b - \varepsilon$ ,*

$$\lim_{m \rightarrow \infty} \frac{\Phi_{mp+\ell}(z)}{\bar{\alpha}_{mp+\ell-1}} = -D(z)^{-1} G_\ell(z)$$



where

$$G_\ell(z) \left(1 - \frac{z^\ell}{b^\ell}\right) = 1 + (bc_{\ell-1})^{-1}z + (bc_{\ell-1})^{-1}(bc_{\ell-2})^{-1}z^2 + \cdots + \prod_{j=1}^{p-1} (bc_{\ell-j})^{-1}z^{p-1} \quad (2.28)$$

One can also say something when  $b = 1$  if we also have

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad (2.29)$$

A key issue is that it may not be true that  $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$ , so  $D(z)$  may not exist.

**Theorem 2.6.** *Let  $\alpha_n$  be a sequence of Verblunsky coefficients and  $c_1, c_2, \dots, c_p$  is a sequence so that (2.26) holds and*

$$\lim_{m \rightarrow \infty} \frac{\alpha_{mp+\ell}}{\alpha_{mp+\ell-1}} = c_\ell \quad \ell = 1, 2, \dots, p \quad (2.30)$$

Then, uniformly in  $|z| < 1 - \varepsilon$ ,

$$\lim_{m \rightarrow \infty} \frac{\varphi_{mp+\ell}(z)}{\bar{\alpha}_{mp+\ell-1} \varphi_{mp+\ell}^*(z)} = -G_\ell(z) \quad (2.31)$$

where  $G$  is given by (2.28) with  $b = 1$ .

*Proof.* (2.29) and (2.9), together with  $|\Phi_n(z)| \leq |\Phi_n^*(z)|$  on  $\bar{\mathbb{D}}$ , implies that on  $\bar{\mathbb{D}}$ ,

$$\lim_{n \rightarrow \infty} \frac{\Phi_n^*(z)}{\Phi_{n+1}^*(z)} = 1$$

Dividing (2.10) by  $\bar{\alpha}_{mp+\ell-1} \Phi_{mp+\ell}^*$ , we obtain the result by the same argument that led to Corollary 2.5.  $\square$

### 3. ASYMPTOTICS IN THE CRITICAL REGION

In this section, we will determine asymptotics of  $\Phi_n(z)$  in an annulus about  $|z| = b$  when (1.5) holds. The idea will be to view (1.1) as an inhomogeneous equation, so we first look at some solutions with particular inhomogeneities. Define for  $z \neq \bar{b}_\ell$  and  $n = 0, 1, 2, \dots$ ,

$$u_n^{(\ell)} = \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1} \quad (3.1)$$

**Proposition 3.1.**  $u_n^{(\ell)}$  obeys

$$u_{n+1}^{(\ell)} = zu_n^{(\ell)} - \bar{b}_\ell^n \quad (3.2)$$

for all  $z \in \mathbb{C}$ ,  $z \neq \bar{b}_\ell$ , and all  $n = 0, 1, 2, \dots$

*Proof.*  $u_{n+1}^{(\ell)} - zu_n^{(\ell)} = (\bar{b}_\ell^n - z)u_n^{(\ell)} = -\bar{b}_\ell^n$ .  $\square$

Next, define

$$R_n(z) = \bar{\alpha}_n \Phi_n^*(z) - \sum_{\ell=1}^L \bar{C}_\ell \bar{b}_\ell^n D(z)^{-1} D(0) \quad (3.3)$$

and also define

$$s_n(z) = \sum_{j=0}^{\infty} z^{-j-1} R_{n+j}(z) \quad (3.4)$$

We have

**Proposition 3.2.** *Let  $\alpha_n$  obey (1.5). Then there is  $\Delta_1 < 1$ ,*

(i)

$$\sup_{|z| \leq 1} |R_n(z)| \leq C(b\Delta_1)^n \quad (3.5)$$

(ii) *The sum in (3.4) converges uniformly in*

$$\mathbb{A} = \{z \mid 1 > |z| > b\Delta_1\} \quad (3.6)$$

*and  $s_n(z)$  is analytic there.*

(iii) *We have in  $\mathbb{A}$  that*

$$|s_n(z)| \leq C(b\Delta_1)^n (|z| - b\Delta_1)^{-1} \quad (3.7)$$

(iv)  *$s_n$  obeys*

$$s_{n+1}(z) = z s_n(z) - R_n(z) \quad (3.8)$$

*Proof.* (i) follows from (1.5) and (2.6).

(ii), (iii) Since

$$|z^{j-1} R_{n+j}(z)| \leq |z|^{-1} (b\Delta_1)^n (|z|^{-1} b\Delta_1)^j$$

we have a geometric series which yields (ii) and (iii).

(iv) Since the sum converges absolutely,

$$\begin{aligned} s_{n+1}(z) - z s_n(z) &= \sum_{j=1}^{\infty} z^{-j} R_{n+j}(z) - \sum_{j=0}^{\infty} z^{-j} R_{n+j} \\ &= -R_n(z) \end{aligned} \quad \square$$

The main result of this section is

**Theorem 3.3.** *Let  $\alpha_n$  obey (1.5). Then for some  $\Delta_1 < 1$  and  $z \in \mathbb{A}$  given by (3.7), we have that*

$$\Phi_n(z) = s_n(z) + \left[ \sum_{\ell=1}^L \bar{C}_\ell \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1} \right] D(0) D(z)^{-1} + z^n \overline{D(1/\bar{z})}^{-1} \quad (3.9)$$

*Remarks.* 1. Since  $\varphi_n = \kappa_n \Phi_n(z)$  and  $\kappa_n = D(0)^{-1}(1 + O(b^n))$ , this also gives us asymptotics for  $\varphi_n$ .

2. Since  $\Phi_n$  is analytic in  $\mathbb{A}$ , the poles at  $\bar{b}_\ell$  in the second and third terms of (3.9) must cancel.

3. In  $\mathbb{A}$ , (3.7) implies  $s_n$  is small compared to both  $z^n$  and  $b^n$ , so the asymptotics of  $\Phi_n$  comes from the competition between the second and third terms in (3.9).

*Proof.* Let

$$Q_n(z) = \Phi_n(z) - s_n(z) - \left[ \sum_{\ell=1}^L \bar{C}_\ell \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1} \right] D(0)D(z)^{-1}$$

By (1.1), (2.2), and (3.8), we have

$$Q_{n+1}(z) = zQ_n(z)$$

so

$$Q_n(z) = f(z)z^n$$

Since  $Q_n$  is analytic in  $\mathbb{A} \setminus \{\bar{b}_\ell\}_{\ell=0}^L$ ,  $f(z)$  is analytic there.

By (3.7),

$$\lim_{n \rightarrow \infty} |z|^{-n} |s_n(z)| = 0$$

in  $\mathbb{A}$ , and if  $|z| > b$ ,  $|z|^{-n} \sum_{\ell=1}^L \bar{C}_\ell \bar{b}_\ell^n (z - \bar{b}_\ell)^{-1} \rightarrow 0$ , so for  $|z| > b$ ,

$$\begin{aligned} f(z) &= \lim_{n \rightarrow \infty} z^{-n} Q_n(z) = \lim_{n \rightarrow \infty} z^{-n} \Phi_n(z) \\ &= D(0) \overline{D(1/\bar{z})}^{-1} \end{aligned}$$

by (2.8). □

#### 4. ZEROS IN $|z| < b - \varepsilon$

In this section, we use the asymptotic result from Section 2 to analyze zeros of  $\varphi_n$  in the region where  $|z| < b$ . We initially focus on the case where (1.5) holds. A key role is played by the polynomials

$$P_n(z) = \sum_{\ell=1}^L \bar{C}_\ell \omega_\ell^n \prod_{k \neq \ell} (z - \bar{b}_k) \tag{4.1}$$

of degree at most  $L - 1$ . Here  $\omega_\ell = \bar{b}_\ell/b$ .

The  $P_n$  are almost periodic in  $n$  and, in particular, for any sequence  $n_j$ , there is a subsequence  $n_{j(k)}$  so  $P_\infty \equiv \lim P_{n_{j(k)}}$  exists and is a nonzero polynomial (since  $P_\infty / \prod_\ell (\bar{b}_\ell - z)$  has poles at each  $\bar{b}_\ell$ ).

**Theorem 4.1.** *Let (1.5) hold. Then for any  $\varepsilon > 0$ , there is an  $N$  so that for  $n \geq N$ ,  $\varphi_n(z)$  has at most  $L - 1$  zeros in  $\{z \mid |z| < b - \varepsilon\} \equiv \mathbb{S}$ .*

*Proof.* If not, we can find a sequence  $n(j) \rightarrow \infty$  so that  $P_{n(j)}(z)$  has at least  $L$  zeros in  $\bar{\mathbb{S}}$ . By passing to a further subsequence, we can suppose  $P_{n(j)} \rightarrow P_\infty$  and that the  $L$  zeros have limits  $z_1, \dots, z_L$  in  $\bar{\mathbb{S}}$  (maybe not distinct). By Theorem 2.2,

$$\lim_{j \rightarrow \infty} \varphi_{n(j)} D(z) \prod_{\ell=1}^L (z - \bar{b}_\ell) = P_\infty(z) \quad (4.2)$$

in a neighborhood of  $\bar{\mathbb{S}}$ , so by Hurwitz's theorem,  $P_\infty$  has  $L$  zeros (counting multiplicity). Since  $P_\infty$  has degree  $L-1$  and is not identically zero, we have a contradiction.  $\square$

Using Hurwitz's theorem and (4.2), we also have an existence result for zeros:

**Theorem 4.2.** *Let (1.5) hold and let  $\omega_\ell = \bar{b}_\ell/b$ . Suppose  $n(j)$  is a subsequence so that  $\lim \omega_\ell^{n(j)}$  exists, call it  $\omega_\ell^{(\infty)}$ . Let*

$$P_\infty(z) = \sum_{\ell=1}^L \bar{C}_\ell \omega_\ell^{(\infty)} \prod_{k \neq \ell} (z - \bar{b}_k) \quad (4.3)$$

and let  $\{w_j\}_{j=1}^J$  be its zeros in  $\{z \mid |z| < b\}$ . Then for all sufficiently small  $\delta$  and  $j \geq N_\delta$ ,  $\varphi_{n(j)}(z)$  has one zero within  $\delta$  of each  $w_j$  and no other zero in  $\{z \mid |z| < b - \delta\}$ .

*Remark.* By “one zero within  $\delta$  of  $w_j$ ,” we actually mean exactly  $k$  zeros if some  $w_j$  occurs  $k$  times in the list of zeros counting multiplicity.

Since the right side of (2.25) is nonvanishing on  $\{z \mid |z| < b\}$ , we recover a result of BLS [1] from Corollary 2.4, Theorem 2.6, and Hurwitz's theorem:

**Theorem 4.3.** *Let  $\alpha_n$  be a sequence of Verblunsky coefficients so that*

$$\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n-1}} = b \in (0, 1]$$

*Then for any  $\varepsilon > 0$ , there is an  $N_\varepsilon$  so  $\varphi_n(z)$  has no zeros in  $\{z \mid |z| < b - \varepsilon\}$  if  $n \geq N_\varepsilon$ .*

Finally, Corollary 2.5 and Hurwitz's theorem imply

**Theorem 4.4.** *Let  $\alpha_n$  be a sequence of Verblunsky coefficients,  $b \in (0, 1)$ , and  $c_1, c_2, \dots, c_p$  a sequence so that (2.26) and (2.27) hold. Let  $G_\ell$  be given by (2.25), let  $W_\ell = G_\ell(1 - z^\ell/b^\ell) = \text{RHS of (2.25)}$ , and let  $\{w_j^{(\ell)}\}_{j=1}^{N_\ell}$  be the zeros of  $W_\ell$  in  $\{z \mid |z| < b\}$ . Then for any sufficiently small  $\delta$ , there is an  $N$  so for  $mp + \ell \geq N$ , we have that the only zeros of  $\varphi_{mp+\ell}$  in  $\{z \mid |z| < b - \delta\}$  are one each within  $\delta$  of each  $w_j^{(\ell)}$ .*

*Remark.* As we will explain in Section 6, that the only possible limit points of zeros are the  $w_j^{(\ell)}$  is a result of BLS [1], but they do not prove there actually are zeros there.

**Example 4.5.** Let  $\alpha_n$  be given by (1.7). We have  $b = \frac{1}{2}$ ,  $p = 4$ , and  $c_1 = -1$ ,  $c_2 = -1$ ,  $c_3 = 3$ ,  $c_4 = \frac{1}{3}$ . Thus

$$W_2(z) = 1 - 2z - 12z^2 - 8z^3$$

which has zeros at  $-\frac{1}{2}$  and at  $\frac{1}{2}(-1 \pm \sqrt{2})$ . Only  $(\sqrt{2} - 1)/2$  is in  $\{z \mid |z| < \frac{1}{2}\}$ . The comparison of the limit and the zeros of  $\Phi_{22}$  appears in Section 1 just after Figure 2. It is not coincidental that  $W_2$  has a zero at  $z = -\frac{1}{2}$ . In this case, the second term in (3.9) is, for  $n \equiv 2 \pmod{4}$ ,  $C(\frac{1}{2})^n W_2(z)/(z^4 - \frac{1}{16})$  with poles only at  $\frac{1}{2}, \pm \frac{1}{2}i$ . The potential pole at  $z = -\frac{1}{2}$  has to be cancelled by a zero in  $W_2$ .  $\square$

As in [7], one can analyze how close the zeros of  $\varphi_n$  are to the points  $w_j^{(\ell)}$ . In general, they are exponentially close. If the  $w_j^{(\ell)}$  are in the annulus where (3.9) holds, one can write down the leading asymptotic exactly. For example, if  $w_j^{(\ell)}$  is a  $k$ -fold zero and  $D(1/\bar{w}_j(z)) \neq 0$ , then the zeros have a clock structure as in Theorem 4.5 of [7].

By using Theorem 2.6, we see that Theorem 4.4 extends to the case  $b = 1$  if  $\lim \alpha_n = 0$ . In particular, if  $\lim \alpha_n = 0$  and  $\lim_{n \rightarrow \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1$  (e.g.,  $\alpha_n = (n+2)^{-\beta}$ ), then there are no zeros of  $\varphi_n$  in  $\{z \mid |z| < 1 - \delta\}$  for  $n$  large.

## 5. ZEROS IN THE CRITICAL REGION

Given Theorem 3.3 and the estimate (3.7), the analysis of zeros of  $\varphi_n$  in the region  $\{z \mid b\Delta_1 < |z| < b\Delta_1^{-1}\}$  is identical to the analysis in [7] of the zeros in case  $L = 1$ . The gap in that case if  $z = b$  comes from the analysis of  $\overline{D(1/\bar{z})} \varphi_n(z)$  which has zeros with no gap. The gap in zeros of  $\varphi_n(z)$  comes from the fact that  $\overline{D(1/\bar{z})}$  has zeros at  $z = b$ . In our case, when (1.5) holds,  $\overline{D(1/\bar{z})}$  has a zero at each  $\bar{b}_\ell$ , so there are gaps at all those points.

The following extends Theorem 4.3 of [7] and has the same proof:

**Theorem 5.1.** *Let  $\alpha_n$  be a sequence of Verblunsky coefficients obeying (1.5). Then for some  $\delta$ , all the zeros  $\{z_j^{(n)}\}_{j=1}^{N_n}$  of  $\varphi_n(z)$  with  $||z| - b| < \delta$  obey*

(1)

$$\sup_j ||z_j^{(n)}| - b| = O\left(\frac{\log n}{n}\right) \quad (5.1)$$



which implies (6.3) inductively once one notes that (6.3) holds for  $m = 1$  since  $\Delta_1(z) = z + x_1$ .

(iii) This follows by induction from (6.3).  $\square$

In [1], they consider sequences of Verblunsky coefficients where (2.26) and (2.27) hold to prove that the only accumulation points of zeros of  $\varphi_{mp+\ell}$  are given by zeros of a polynomial that has the form of (6.1). Using (6.4) and

$$(x_1 \dots x_m)^{-1} \Delta_m(z) = 1 + x_m^{-1}z + x_m^{-1}x_{m-1}^{-1}z^2 + \dots + (x_1 \dots x_m)^{-1}z^m$$

one sees their polynomials are up to a constant, our polynomial  $W_\ell$ . Thus our results extend theirs (in that we prove there are, in fact, always limit points).

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