

EIGENVALUE BOUNDS IN THE GAPS OF SCHRÖDINGER OPERATORS AND JACOBI MATRICES

DIRK HUNDERTMARK¹ AND BARRY SIMON²

ABSTRACT. We consider $C = A + B$ where A is selfadjoint with a gap (a, b) in its spectrum and B is (relatively) compact. We prove a general result allowing B of indefinite sign and apply it to obtain a $(\delta V)^{d/2}$ bound for perturbations of suitable periodic Schrödinger operators and a (not quite) Lieb–Thirring bound for perturbations of algebro-geometric almost periodic Jacobi matrices.

1. INTRODUCTION

The study of the eigenvalues of Schrödinger operators below the essential spectrum goes back over fifty years to Bargmann [5], Birman [6], and Schwinger [43], and of power bounds on the eigenvalues to Lieb–Thirring [35, 36].

There has been considerably less work on eigenvalues in gaps—much of what has been studied followed up on seminal work by Deift and Hempel [23]; see [1, 2, 25, 26, 27, 28, 29, 32, 33, 40, 41, 42] and especially work by Birman and collaborators [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Following Deift–Hempel, this work has mainly focused on the set of λ 's so that some given fixed e in a gap of $\sigma(A)$ is an eigenvalue of $A + \lambda B$ and the growth of the number of eigenvalues as $\lambda \rightarrow \infty$ most often for closed intervals strictly inside the gap. Most, but not all, of this work has focused on B 's of a definite sign. Our goal in this note is to make an elementary observation that, as regards behavior at an edge

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¹ School of Mathematics, Watson Hall, University of Birmingham, Birmingham, B15 2TT, UK. On leave from Department of Mathematics, Altgeld Hall, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA. Email: hundertd@for.mat.bham.ac.uk. Supported in part by NSF grant DMS-0400940.

² Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125, USA. E-mail: bsimon@caltech.edu. Supported in part by NSF grant DMS-0140592 and U.S.–Israel Binational Science Foundation (BSF) Grant No. 2002068.

for fixed λ , allows perturbations of either sign. The decoupling in steps we use does not work for the question raised by Deift–Hempel, which may be why it does not seem to be in the literature.

We will present two applications: a Cwikel–Lieb–Rozenblum-type finiteness result [20, 34, 39] for suitable gaps in $d \geq 3$ periodic Schrödinger operators and a critical power estimate on eigenvalues in some one-dimensional almost periodic problems.

To state our results precisely, we need some notation. For any self-adjoint operator C , $E_\Omega(C)$ will denote the spectral projections for C . We define

$$\#(C \in \Omega) = \dim(E_\Omega(C)) \quad (1.1)$$

and

$$\#(C > \alpha) = \dim(E_{(\alpha, \infty)}(C)) \quad (1.2)$$

and similarly for $\#(C \geq \alpha)$, $\#(C < \alpha)$, $\#(C \leq \alpha)$.

We will write

$$B = B_+ - B_- \quad (1.3)$$

with $B_\pm \geq 0$. While often we will take $B_\pm = \max(\pm B, 0)$, we do not require $B_+B_- = 0$ or $[B_+, B] = 0$. Our main technical result, which we will prove in Section 2, is

Theorem 1.1. *Let A be a selfadjoint operator and $x, y \in \mathbb{R}$ so $(x, y) \cap \sigma(A) = \emptyset$. Let B be given by (1.3) with B_+, B_- both compact. Let $C = A + B$. Let $x < e_0 < e_1 = \frac{1}{2}(x + y)$, then*

$$\#(C \in (e_0, e_1)) \leq \#(B_+^{1/2}(e_0 - A)^{-1}B_+^{1/2} \geq 1) + \#(B_- \geq \frac{1}{2}(y - x)) \quad (1.4)$$

In Section 3, we discuss an analog when A is unbounded but bounded below and B_\pm are only relatively compact.

If V is a periodic locally $L^{d/2}$ function on \mathbb{R}^d ($d \geq 3$), then $A = -\Delta + V$ can be written as a direct integral of operators, $A(k)$, with compact resolvent, with the integral over the fundamental cell of a dual lattice (see [38]). If $\varepsilon_1(k) \leq \varepsilon_2(k) \leq \dots$ are the eigenvalues of $A(k)$, then (x, y) is a gap in $\sigma(A)$ (i.e., connected component of $\mathbb{R} \setminus \sigma(A)$) if and only if there is ℓ with

$$\max_k \varepsilon_{\ell-1}(k) = x < y = \min_k \varepsilon_\ell(k) \quad (1.5)$$

We say y is a nondegenerate gap edge if and only if

$$\min_k \varepsilon_{\ell+1}(k) > y \quad (1.6)$$

and $\varepsilon_\ell(k) = y$ at a finite number of points $\{k_j\}_{j=1}^N$ in the unit cell so that for some C and all k in the unit cell,

$$\varepsilon_\ell(k) - y \geq C \min |k - k_j|^2 \quad (1.7)$$

There is a similar definition at the bottom edge if $x > -\infty$. It is a general theorem [31] that the bottom edge is always nondegenerate. In Section 4, we will prove

Theorem 1.2. *Let $d \geq 3$. Let $V \in L_{\text{loc}}^{d/2}(\mathbb{R}^d)$ be periodic and let $W \in L^{d/2}(\mathbb{R}^d)$. Let (x, y) be a gap in the spectrum $A = -\Delta + V$ which is nondegenerate at both ends, and let $N_{(x,y)}(W) = \#(-\Delta + V + W \in (x, y))$. Then $N_{(x,y)}(W) < \infty$.*

This will be a simple extension of the result of Birman [11] who proved this if W has a fixed sign. Note we have not stated a bound by $\|W\|_{d/2}^{d/2}$. This is discussed further in Section 4.

In the final section, Section 5, we will consider certain two-sided Jacobi matrices, J , on $\ell^2(\mathbb{Z})$ with

$$J_{k\ell} = \begin{cases} b_k & k = \ell \\ a_k & \ell = k + 1 \\ a_{k-1} & \ell = k - 1 \\ 0 & |\ell - k| \geq 2 \end{cases} \quad (1.8)$$

If $E = \cup_{j=1}^{\ell+1} E_j$ is a finite union of bounded closed disjoint intervals, there is an isospectral torus \mathcal{T}_E associated to E of almost periodic J 's with $\sigma(J) = E$ (see [3, 4, 18, 19, 24, 37, 46, 47]). We conjecture the following:

Conjecture. *Let J_0 lie in some \mathcal{T}_E . Let $J = J_0 + \delta J$ be a Jacobi matrix for which δJ is trace class, that is,*

$$\sum_n |\delta a_n| + |\delta b_n| < \infty \quad (1.9)$$

Then

$$\sum_{\lambda \in \sigma(J) \setminus E} \text{dist}(\lambda, E)^{1/2} < \infty \quad (1.10)$$

For $e = [-2, 2]$ so J_0 is the free Jacobi matrix with $a_n \equiv 1$, $b_n \equiv 0$, this is a result of Hundertmark–Simon [30]. It has recently been proven [21] for the case where J_0 is periodic, and it has recently been proven [45] that (1.10) holds for the sum over λ 's above the top of the spectrum or below the bottom. In Section 5, we will prove

Theorem 1.3. *If (1.9) holds, then (1.10) holds if $\frac{1}{2}$ is replaced by any $\alpha > \frac{1}{2}$.*

Theorem 1.4. *If*

$$\sum_n [\log(|n| + 1)]^{1+\varepsilon} [|\delta a_n| + |\delta b_n|] < \infty \quad (1.11)$$

for some $\varepsilon > 0$, then (1.10) holds.

Both the conjecture and Theorem 1.4 are interesting because they imply that the spectral measure obeys a Szegő condition. This is discussed in [18].

2. ABSTRACT BOUNDS IN GAPS (COMPACT CASE)

Our goal here is to prove Theorem 1.1. We begin by recalling the version of the Birman–Schwinger principle for points in gaps, which is essentially the key to [1, 2, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 23, 25, 26, 27, 28, 29, 32, 33, 40, 41, 42]:

Proposition 2.1. *Let A be a bounded selfadjoint operator with $(x, y) \cap \sigma(A) = \emptyset$. Let B be compact with $B \geq 0$. Let $e \in (x, y)$. Then*

$$e \in \sigma(A + \mu B) \Leftrightarrow \mu^{-1} \in \sigma(B^{1/2}(e - A)^{-1}B^{1/2}) \quad (2.1)$$

with equal multiplicity. In particular,

$$\#(A + B \in (e, y)) \leq \#(B^{1/2}(e - A)^{-1}B^{1/2} \geq 1) \quad (2.2)$$

Proof. This is so elementary that we sketch the proof. If for $\varphi \neq 0$,

$$(A + \mu B)\varphi = e\varphi \quad (2.3)$$

then

$$B\varphi \neq 0 \quad (2.4)$$

since $e \notin \sigma(A)$. Moreover,

$$(e - A)^{-1}B\varphi = \mu^{-1}\varphi \quad (2.5)$$

and (2.5) implies (2.3). Thus

$$e \in \sigma(A + \mu B) \Leftrightarrow \mu^{-1} \in \sigma((e - A)^{-1}B) \quad (2.6)$$

and (2.1) follows by $\sigma(CD) \setminus \{0\} = \sigma(DC) \setminus \{0\}$ (see, e.g., Deift [22]).

Since $\sigma(A + \mu B) \subset \sigma(A) + [-\mu\|B\|, \mu\|B\|]$ and discrete eigenvalues are continuous in μ and strictly monotone by (2.4) and (see [38])

$$\frac{de(\mu)}{d\mu} = \langle \varphi, B\varphi \rangle \quad (2.7)$$

eigenvalues of $A + B$ in (x, y) must pass through e as μ goes from 0 to 1 and (2.2) follows from (2.1). We only have inequality in (2.2) since eigenvalues can get reabsorbed at y . \square

Proof of Theorem 1.1. Let $C_+ = A + B_+$ so $C = C_+ - B_-$. By Proposition 2.1, if

$$n_1 = \#(C_+ \in (e_0, e_1)) \quad n_2 = \#(C_+ \in (e_1, y)) \quad (2.8)$$

then

$$n_1 + n_2 \leq \#(B_+^{1/2}(e_0 - A)^{-1}B_+^{1/2} \geq 1) \quad (2.9)$$

By a limiting argument, we can suppose that e_1 is not an eigenvalue of C_+ . Since eigenvalues of $C_+ - \mu B_-$ are strictly monotone decreasing in μ , the number of eigenvalues of C in (e_0, e_1) can only increase by passing through e_1 . By repeating the argument in Proposition 2.1,

$$\#(C \in (e_0, e_1)) \leq n_1 + \#(B_-^{1/2}(C_+ - e_1)^{-1}B_-^{1/2} \geq 1) \quad (2.10)$$

Now write

$$B_-^{1/2}(C_+ - e_1)^{-1}B_-^{1/2} = D_1 + D_2 + D_3 \quad (2.11)$$

where D_1 has $E_{(-\infty, e_1)}(C_+)$ inserted in the middle, D_2 an $E_{(e_1, y)}(C_+)$, and D_3 an $E_{[y, \infty)}(C_+)$. Since $D_1 \leq 0$ and $\text{rank}(D_2) \leq n_2$, we see

$$\#(B_-^{1/2}(C_+ - e_1)^{-1}B_-^{1/2} \geq 1) \leq n_2 + \#(D_3 \geq 1) \quad (2.12)$$

Since $(C_+ - e_1)^{-1}E_{[y, \infty)}(C_+) \leq (y - e_1)^{-1} = [\frac{1}{2}(y - x)]^{-1}$, we have

$$D_3 \leq [\frac{1}{2}(y - x)]^{-1}B_- \quad (2.13)$$

and thus

$$\begin{aligned} \#(D_3 \geq 1) &\leq \#([\frac{1}{2}(y - x)]^{-1}B_- \geq 1) \\ &= \#(B_- \geq \frac{1}{2}(y - x)) \end{aligned} \quad (2.14)$$

(2.9), (2.10), (2.12), and (2.14) imply (1.4). \square

3. ABSTRACT BOUNDS IN GAPS (RELATIVELY COMPACT CASE)

In this section, we suppose A is a semibounded selfadjoint operator with

$$q = \inf \sigma(A) \quad (3.1)$$

We will suppose B is a form-compact perturbation, which is a difference of two positive form-compact perturbations. We abuse notation and write compact operators

$$B_{\pm}^{1/2}(A - e)^{-1}B_{\pm}^{1/2} \quad (3.2)$$

for $e \notin \sigma(A)$ even though B_{\pm} need not be operators — (3.2) can be defined via forms in a standard way.

In the bounded case, we only considered intervals in the lower half of a gap since $A \rightarrow -A$, $B \rightarrow -B$ flips half-intervals. But, as has been noted in the unbounded case (see, e.g., [11, 40]), there is now an asymmetry, so we will state separate results. We start with the bottom half case:

Theorem 3.1. *Let A be a semibounded selfadjoint operator and $x, y \in \mathbb{R}$ so $(x, y) \cap \sigma(A) = \emptyset$. Let $B = B_+ - B_-$ with B_+ form-compact positive perturbations of A . Let $C = A + B$ and $x < e_0 < e_1 = \frac{1}{2}(x + y)$. Then*

$$\begin{aligned} \#(C \in (e_0, e_1)) &\leq \#(B_+^{1/2}(e_0 - A)^{-1}B_+^{1/2} \geq 1) \\ &\quad + \# \left(B_-^{1/2}(A - q + 1)^{-1}B_-^{1/2} \geq \frac{1}{2} \left[\frac{y - x}{y - q + 1} \right] \right) \end{aligned} \quad (3.3)$$

Proof. We follow the proof of Theorem 1.1 without change until (2.13) noting that instead

$$(C_+ - e_1)^{-1}E_{[y, \infty)}(C_+) \leq \frac{y - q + 1}{y - e_1} (C_+ + q + 1)^{-1} \quad (3.4)$$

$$\leq \frac{y - q + 1}{y - e_1} (A - q + 1)^{-1} \quad (3.5)$$

since $q \leq A \leq C_+$ and

$$\sup_{x \geq y} \frac{x - q + 1}{x - e_1}$$

is taken at $x = y$ since $q - 1 < e_1$. By (3.5),

$$\#(D_3 \geq 1) \leq \# \left(B_-^{1/2}(A - q + 1)^{-1}B_-^{1/2} \geq \frac{y - e_1}{y - q + 1} \right) \quad \square$$

Theorem 3.2. *Let A be a semibounded selfadjoint operator and $(x, y) \in \mathbb{R}$ so $(x, y) \cap \sigma(A) = \emptyset$. Let $B = B_+ - B_-$ with B_{\pm} form-compact positive perturbations of A . Let $C = A + B$ and $e_1 = \frac{1}{2}(x + y) < e_0 < y$. Then*

$$\begin{aligned} \#(C \in (e_1, e_0)) &\leq \#(B_-^{1/2}(A - e_0)^{-1}B_-^{1/2} \geq 1) \\ &\quad + \#(B_+^{1/2}(A - B_- - e_1)^{-1}E_{(-\infty, x)}(A - B_-)B_+^{1/2} \geq 1) \end{aligned} \quad (3.6)$$

Proof. Identical to the proof of Theorem 1.1 through (2.13). \square

The second term in (3.6) is easily seen to be finite since the operator is compact. However, any bound depends on both B_+ and B_- .

4. $L^{n/2}$ BOUNDS IN GAPS FOR PERIODIC SCHRÖDINGER OPERATORS

Birman [11] proved for V , as in Theorem 1.2, and any W that uniformly in any gap (x, y) , $\sup_{\lambda \in (x, y)} \| |W|^{1/2} (-\Delta + V - \lambda)^{-1} |W|^{1/2} \|_{\mathcal{I}_{d/2}^w} \leq c \|W\|_{d/2}$ where $\|\cdot\|_{\mathcal{I}_{d/2}^w}$ is a weak \mathcal{I}_d trace class norm [44]. To be precise, in his Proposition 3.1, he proved $\| |W|^{1/2} (-\Delta + V - \lambda_0)^{-1} |W|^{1/2} \|_{\mathcal{I}_{d/2}}$ is finite away from x and y , and then in (3.15), he proved the weak estimate at the end points. He used this to prove for W of a definite sign

$$N_{(x,y)}(W) \leq c \int_{\mathbb{R}^d} |W(z)|^{d/2} dz \quad (4.1)$$

It implies relative compactness, and given Theorems 3.1 and 3.2, proves Theorem 1.2.

Note that, by Theorem 3.1, we get for any $x' > x$,

$$N_{(x',y)}(W) \leq c_{x'} \int_{\mathbb{R}^d} |W(z)|^{d/2} dz \quad (4.2)$$

but we do not get such a bound for $x' = x$ since there is a W_-, W_+ cross term in (3.6).

5. GAPS FOR PERTURBATIONS OF FINITE GAP ALMOST PERIODIC JACOBI MATRICES

Our goal here is to prove Theorems 1.3 and 1.4. Let

$$G_0(n, m; \lambda) = \langle \delta_n, (J_0 - \lambda)^{-1} \delta_m \rangle \quad (5.1)$$

and let (λ_0, λ_1) be a gap in $\sigma(J_0)$. As input, we need two estimates for G_0 proven in [18]. First we have

$$|G_0(n, m; \lambda)| \leq C \text{dist}(\lambda, \sigma(J_0))^{-1/2} \quad (5.2)$$

uniformly in real $\lambda \notin \sigma(J_0)$ and n and m .

To describe the other estimate, we need some notions. At a band edge, λ_0 (here and below, we study λ_0 but there is also an analysis at λ_1), there is a unique almost periodic sequence $\{u_n(\lambda_0)\}_{n=-\infty}^{\infty}$ solving $(J_0 - \lambda_0)u_n = 0$. If $u_n = 0$, we say n is a resonance point. If $u_n \neq 0$, we have a nonresonance. Since $u_n = 0 \Rightarrow u_{n\pm 1} \neq 0$, we have lots of nonresonance points. Without loss, we will suppose henceforth that 0 is a nonresonance point. At a nonresonance point, $\lim_{\lambda \downarrow \lambda_0} \text{dist}(\lambda, \lambda_0)^{1/2} G_0(n, n; \lambda) \neq 0$.

The Dirichlet Green's function is defined by

$$G_0^D(n, m; \lambda) = G_0(n, m; \lambda) - G_0(0, 0; \lambda)^{-1} G_0(n, 0; \lambda) G_0(0, m; \lambda) \quad (5.3)$$

Then [18] proves that if 0 is a nonresonance at λ_0 , then for some small ε ,

$$\lambda \in (\lambda_0, \lambda_0 + \varepsilon) \Rightarrow |G_0^D(n, n; \lambda)| \leq Cn \quad (5.4)$$

$$\Rightarrow |G_0^D(n, n; \lambda)| \leq C|\lambda - \lambda_0|^{-1/2} \quad (5.5)$$

Following [30], we use (with $c_{\pm} = \max(\pm c, 0)$) with $a > 0$,

$$\begin{pmatrix} b & a \\ a & b \end{pmatrix} = \begin{pmatrix} b_+ + a & 0 \\ 0 & b_+ + a \end{pmatrix} - \begin{pmatrix} a + b_- & -a \\ -a & a + b_- \end{pmatrix} \quad (5.6)$$

to define $\delta J = \delta J_+ - \delta J_-$ where δJ_+ is diagonal and given by

$$(\delta J_+)_{nn} = (\delta b_n)_+ + \delta a_{n-1} + \delta a_n \quad (5.7)$$

and (δJ_-) is tridiagonal with

$$(\delta J_-)_{nn+1} = \delta a_n \quad (5.8)$$

$$(\delta J_-)_{nn-1} = \delta a_{n-1} \quad (5.9)$$

$$(\delta J_-)_{nn} = (\delta b_n)_- + \delta a_{n-1} + \delta a_n \quad (5.10)$$

We also use the fact obtained via an integration by parts that if $f(\lambda_0) = 0$, f continuous on $[\lambda_0, \lambda_0 + \varepsilon)$, and $C^1(\lambda_0, \lambda_0 + \varepsilon)$ with $f' > 0$, then

$$\sum_{\substack{\lambda \in (\lambda_0, \lambda_0 + \varepsilon) \\ \lambda \in \sigma(J)}} f(\lambda) = \int_{\lambda_0}^{\lambda_0 + \varepsilon} f'(\lambda) \#(J \in (\lambda, \lambda_0 + \varepsilon)) d\lambda \quad (5.11)$$

Since $f' \in L^1(\lambda_0, \lambda_0 + \varepsilon)$ and δJ_- is compact, Theorem 1.1 implies

$$\sum_{\substack{\lambda \in (\lambda_0, \lambda_0 + \varepsilon) \\ \lambda \in \sigma(J)}} f(\lambda) < \infty \Leftrightarrow \int_{\lambda_0}^{\lambda_0 + \varepsilon} \#((\delta J_+)^{1/2}(\lambda - J_0)^{-1}(\delta J_+)^{1/2} \geq 1) f'(\lambda) d\lambda < \infty \quad (5.12)$$

This leads to

Proposition 5.1. *If δJ_{\pm} are trace class and*

$$\int_{\lambda_0}^{\lambda_0 + \varepsilon} f'(\lambda) |\text{Tr}((\delta J_+)^{1/2} G_0^D(\cdot, \cdot; \lambda) (\delta J_+)^{1/2})| d\lambda < \infty \quad (5.13)$$

then

$$\sum_{\substack{\lambda \in (\lambda_0, \lambda_0 + \varepsilon) \\ \lambda \in \sigma(J)}} f(\lambda) < \infty \quad (5.14)$$

Proof. $G_0 - G_0^D$ is rank one and $\#(C \geq 1) \leq \|C\|_1$, so

$$\#((\delta J_+)^{1/2} G_0(\cdot, \cdot; \lambda) (\delta J_+)^{1/2} \geq 1) \leq 1 + \|(\delta J_+)^{1/2} G_0^D(\cdot, \cdot; \lambda) (\delta J_+)^{1/2}\|_1$$

The negative part of $G_0^D(\cdot, \cdot; \lambda)$ is uniformly bounded in norm by $|a - \lambda|^{-1}$ where a is either λ_1 or the unique eigenvalue of the Dirichlet J_0 in $(\lambda_0 - \lambda_1)$ and

$$\begin{aligned} \|C\|_1 &\leq \text{Tr}(C_+) + \text{Tr}(C_-) \\ &\leq \text{Tr}(C) + 2\text{Tr}(C_-) \end{aligned}$$

Thus (5.14) is implied by (5.12) so long as (5.13) holds. \square

Proof of Theorem 1.3. By (5.5) and $\delta J_+ \in \mathcal{I}_1$, we have

$$|\text{Tr}((\delta J_+)^{1/2} G_0^D(\cdot, \cdot; \lambda) (\delta J_+)^{1/2})| \leq C |\lambda - \lambda_0|^{-1/2}$$

so the integral in (5.13) is bounded by

$$C \int_{\lambda_0}^{\lambda_0 + \varepsilon} |\lambda - \lambda_0|^{\alpha-1} |\lambda - \lambda_0|^{-1/2} d\lambda < \infty$$

so long as $\alpha - \frac{1}{2} > 0$. \square

Lemma 5.2. *For any $\alpha > 0$, there is a C so for all $x, y > 1$,*

$$\min(x, y) \leq C [\log(x+1)]^\alpha \frac{y}{[\log(y+1)]^\alpha} \quad (5.15)$$

Proof. Pick $d \geq 1$ (e.g., $d = e^\alpha$), so $[\log(x+d)]^\alpha x^{-1}$ is monotone decreasing on $[1, \infty)$. Then

$$\min(x, y) \leq [(\log(x+d))]^\alpha \frac{y}{[\log(y+d)]^\alpha} \quad (5.16)$$

If $y \leq x$, the right-hand side is bigger than y and so $\min(x, y)$. If $y \geq x$, the monotonicity shows

$$\text{RHS} \geq [\log(x+d)]^\alpha \frac{x}{[\log(x+d)]^\alpha} = x$$

(5.15) follows since on $[1, \infty)$, $\frac{\log(x+d)}{\log(x+1)}$ is bounded above and below. \square

Proof of Theorem 1.4. By (5.4), (5.5), and (5.15),

$$|G_0^D(n, n; \lambda)| \leq C \frac{[\log(1+|n|)]^\alpha}{|\lambda - \lambda_0|^{1/2}} [\log(\lambda - \lambda_0)^{-1/2}]^{-\alpha}$$

By (1.11), we see

$$|\text{Tr}[(\delta J)^{1/2} G_0^D(\delta J)^{1/2}]| \leq \frac{C [\log(\lambda - \lambda_0)^{1/2}]^{-(1+\varepsilon)}}{(\lambda - \lambda_0)^{1/2}}$$

Since

$$\int_{\lambda_0}^{\lambda_0+\varepsilon} (\lambda - \lambda_0)^{-1} [\log(\lambda - \lambda_0)^{-1/2}]^{-(1+\varepsilon)} d\lambda < \infty$$

the result follows. \square

REFERENCES

- [1] S. Alama, M. Avellaneda, P. A. Deift, and R. Hempel, *On the existence of eigenvalues of a divergence-form operator $A + \lambda B$ in a gap of $\sigma(A)$* , *Asymptotic Anal.* **8** (1994), 311–344.
- [2] S. Alama, P.A. Deift, and R. Hempel, *Eigenvalue branches of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$* , *Comm. Math. Phys.* **121** (1989), 291–321.
- [3] A. J. Antony and M. Krishna, *Almost periodicity of some Jacobi matrices*, *Proc. Indian Acad. Sci. Math. Sci.* **102** (1992), 175–188.
- [4] A. I. Aptekarev, *Asymptotic properties of polynomials orthogonal on a system of contours, and periodic motions of Toda chains*, *Mat. Sb. (N.S.)* **125(167)** (1984), 231–258 (Russian).
- [5] V. Bargmann, *On the number of bound states in a central field of force*, *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 961–966.
- [6] M. Sh. Birman, *On the spectrum of singular boundary-value problems*, *Mat. Sb. (N.S.)* **55(97)** (1961), 125–174; translated in *Amer. Math. Soc. Transl.* **53** (1966), 23–80.
- [7] M. Sh. Birman, *Discrete spectrum in the gaps of the continuous one in the large-coupling-constant limit*, in “Order, Disorder and Chaos in Quantum Systems” (Dubna, 1989), pp. 17–25, *Oper. Theory Adv. Appl.*, 46, Birkhäuser, Basel, 1990.
- [8] M. Sh. Birman, *Discrete spectrum in a gap of perturbed periodic operator at large coupling constants*, in “Rigorous Results in Quantum Dynamics” (Liblice, 1990), pp. 16–24, *World Sci. Publ.*, River Edge, NJ, 1991.
- [9] M. Sh. Birman, *Discrete spectrum in the gaps of a continuous one for perturbations with large coupling constant*, in “Estimates and Asymptotics for Discrete Spectra of Integral and Differential Equations” (Leningrad, 1989–90), pp. 57–73, *Adv. Soviet Math.*, 7, *Amer. Math. Soc.*, Providence, RI, 1991.
- [10] M. Sh. Birman, *On a discrete spectrum in gaps of a second-order perturbed periodic operator*, *Funct. Anal. Appl.* **25** (1991), 158–161; Russian original: *Funktsional. Anal. i Prilozhen.* **25** (1991), 89–92.
- [11] M. Sh. Birman, *The discrete spectrum in gaps of the perturbed periodic Schrödinger operator. I. Regular perturbations*, in “Boundary Value Problems, Schrödinger Operators, Deformation Quantization,” pp. 334–352, *Math. Top.*, 8, *Akademie Verlag*, Berlin, 1995.
- [12] M. Sh. Birman, *The discrete spectrum of the periodic Schrödinger operator perturbed by a decreasing potential*, *St. Petersburg Math. J.* **8** (1997), 1–14; Russian original: *Algebra i Analiz* **8** (1996), 3–20.
- [13] M. Sh. Birman, *The discrete spectrum in gaps of the perturbed periodic Schrödinger operator. II. Nonregular perturbations*, *St. Petersburg Math. J.* **9** (1998), 1073–1095; Russian original: *Algebra i Analiz* **9** (1997), 62–89.

- [14] M. Sh. Birman, A. Laptev, and T. A. Suslina, *The discrete spectrum of a two-dimensional second-order periodic elliptic operator perturbed by a decreasing potential. I. A semi-infinite gap*, St. Petersburg Math. J. **12** (2001), 535–567; Russian original: Algebra i Analiz **12** (2000), 36–78.
- [15] M. Sh. Birman and A. B. Pushnitskiĭ, *The discrete spectrum in the gaps of the perturbed pseudo-relativistic magnetic Hamiltonian*, J. Math. Sci. (New York) **101** (2000), 3437–3447; Russian original: Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 249 (1997), Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. **29**, 102–117, 315.
- [16] M. Sh. Birman and G. D. Raĭkov, *Discrete spectrum in the gaps for perturbations of the magnetic Schrödinger operator*, in “Estimates and Asymptotics for Discrete Spectra of Integral and Differential Equations” (Leningrad, 1989–90), pp. 75–84, Adv. Soviet Math., 7, Amer. Math. Soc., Providence, RI, 1991.
- [17] M. Sh. Birman and T. Weidl, *The discrete spectrum in a gap of the continuous one for compact supported perturbations*, in “Mathematical Results in Quantum Mechanics” (Blossin, 1993), pp. 9–12, Oper. Theory Adv. Appl., 70, Birkhäuser, Basel, 1994.
- [18] J. Christiansen, B. Simon, and M. Zinchenko, in preparation.
- [19] W. Craig, *The trace formula for Schrödinger operators on the line*, Comm. Math. Phys. **126** (1989), 379–407.
- [20] M. Cwikel, *Weak type estimates for singular values and the number of bound states of Schrödinger operators*, Ann. Math. (2) **106** (1977), 93–100.
- [21] D. Damanik, R. Killip, and B. Simon, *Perturbations of orthogonal polynomials with periodic recursion coefficients*, preprint.
- [22] P. A. Deift, *Applications of a commutation formula*, Duke Math. J. **45** (1978), 267–310.
- [23] P. A. Deift and R. Hempel, *On the existence of eigenvalues of the Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$* , Comm. Math. Phys. **103** (1986), 461–490.
- [24] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, *Nonlinear equations of Korteweg–de Vries type, finite-band linear operators and Abelian varieties*, Uspehi Mat. Nauk **31** (1976), no. 1(187), 55–136 (Russian).
- [25] F. Gesztesy, D. Gurarie, H. Holden, M. Klaus, L. Sadun, B. Simon, and P. Vogl, *Trapping and cascading of eigenvalues in the large coupling limit*, Comm. Math. Phys. **118** (1988), 597–634.
- [26] F. Gesztesy and B. Simon, *On a theorem of Deift and Hempel*, Comm. Math. Phys. **116** (1988), 503–505.
- [27] R. Hempel, *On the asymptotic distribution of the eigenvalue branches of the Schrödinger operator $H \pm \lambda W$ in a spectral gap of H* , J. Reine Angew. Math. **399** (1989), 38–59.
- [28] R. Hempel, *Eigenvalues in gaps and decoupling by Neumann boundary conditions*, J. Math. Anal. Appl. **169** (1992), 229–259.
- [29] R. Hempel, *On the asymptotic distribution of eigenvalues in gaps*, in “Quasiclassical Methods” (Minneapolis, MN, 1995), pp. 115–124, IMA Vol. Math. Appl., 95, Springer, New York, 1997.
- [30] D. Hundertmark and B. Simon, *Lieb–Thirring inequalities for Jacobi matrices*, J. Approx. Theory **118** (2002), 106–130.
- [31] W. Kirsch and B. Simon, *Comparison theorems for the gap of Schrödinger operators*, J. Funct. Anal. **75** (1987), 396–410.

- [32] M. Klaus, *Some applications of the Birman–Schwinger principle*, *Helv. Phys. Acta* **55** (1982/83), 49–68.
- [33] S. Z. Levendorskii, *Lower bounds for the number of eigenvalue branches for the Schrödinger operator $H - \lambda W$ in a gap of H : The case of indefinite W* , *Comm. Partial Differential Equations* **20** (1995), 827–854.
- [34] E. H. Lieb, *Bounds on the eigenvalues of the Laplace and Schrödinger operators*, *Bull. Amer. Math. Soc.* **82** (1976), 751–753.
- [35] E. H. Lieb and W. Thirring, *Bound for the kinetic energy of fermions which proves the stability of matter*, *Phys. Rev. Lett.* **35** (1975), 687–689. Errata **35** (1975), 1116.
- [36] E. H. Lieb and W. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, in “*Studies in Mathematical Physics. Essays in Honor of Valentine Bargmann*,” pp. 269–303, Princeton University Press, Princeton, NJ, 1976.
- [37] F. Peherstorfer and P. Yuditskii, *Asymptotic behavior of polynomials orthonormal on a homogeneous set*, *J. Anal. Math.* **89** (2003), 113–154.
- [38] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV: Analysis of Operators*, Academic Press, New York, 1978.
- [39] G. V. Rozenblum, *Distribution of the discrete spectrum of singular differential operators*, *Soviet Math. (Iz. VUZ)* **20** (1976), 63–71; Russian original in *Izv. Vysš. Učebn. Zaved. Matematika* **1(164)** (1976), 75–86.
- [40] O. L. Safronov, *The discrete spectrum in the gaps of the continuous one for non-signdefinite perturbations with a large coupling constant*, *Comm. Math. Phys.* **193** (1998), 233–243.
- [41] O. L. Safronov, *The discrete spectrum in the spectral gaps of semibounded operators with non-sign-definite perturbations*, *J. Math. Anal. Appl.* **260** (2001), 641–652.
- [42] O. L. Safronov, *The discrete spectrum of selfadjoint operators under perturbations of variable sign*, *Comm. Partial Differential Equations* **26** (2001), 629–649.
- [43] J. Schwinger, *On the bound states of a given potential*, *Proc. Nat. Acad. Sci. U.S.A.* **47** (1961), 122–129.
- [44] B. Simon, *Analysis with weak trace ideals and the number of bound states of Schrödinger operators*, *Trans. Amer. Math. Soc.* **224** (1976), 367–380.
- [45] B. Simon, *Critical Lieb–Thirring bounds for one-dimensional Schrödinger operators and Jacobi matrices with regular ground states*, preprint.
- [46] M. Sodin and P. Yuditskii, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite-dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions*, *J. Geom. Anal.* **7** (1997), 387–435.
- [47] H. Widom, *Extremal polynomials associated with a system of curves in the complex plane*, *Advances in Math.* **3** (1969), 127–232.