MONOTONE JACOBI PARAMETERS AND NON-SZEGŐ WEIGHTS

YURY KREIMER[1], YORAM LAST[1,3], AND BARRY SIMON[2,3]

Abstract. We relate asymptotics of Jacobi parameters to asymptotics of the spectral weights near the edges. Typical of our results is that for $a_n \equiv 1$, $b_n = -Cn^{-\beta}$ $(0 < \beta < \frac{2}{3})$, one has $d\mu(x) = w(x)\,dx$ on $(-2,2)$, and near $x = 2$, $w(x) = e^{-2Q(x)}$ where

$$Q(x) = \beta^{-1} C^{\frac{1}{\beta}} \, \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{\beta}-\frac{1}{2})(2-x)^{\frac{1}{2}-\frac{1}{\beta}}}{\Gamma(\frac{1}{\beta}+1)} \, (1+O((2-x)))$$

1. Introduction

Since the earliest days of the general theory of orthogonal polynomials on the real line (OPRL), it has been known that a key role is played by the Szegő condition [38] that if

$$d\mu(x) = w(x)\,dx + d\mu_{\rm s} \tag{1.1}$$

where w is supported on $[-2,2]$ (we follow the spectral theorists' convention related to $a_n \to 1$, $b_n \to 0$ rather than the $[-1,1]$ tradition in the OP literature), then

$$\int \log(w(x))(4-x^2)^{-\frac{1}{2}}\,dx > -\infty \tag{1.2}$$

In this paper, we will examine asymptotics of $\log(w(x))$ for typical cases where (1.2) fails. Recall [39, 5, 2, 31, 34] that, given μ, one

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[1] Institute of Mathematics, The Hebrew University, 91904 Jerusalem, Israel. E-mail: yuryk@math.huji.ac.il; ylast@math.huji.ac.il. Supported in part by The Israel Science Foundation (grant no. 1169/06).

[2] Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125, USA. E-mail: bsimon@caltech.edu. Supported in part by NSF grant DMS-0140592.

[3] Research supported in part by Grants No. 2002068 and No. 2006483 from the United States-Israel Binational Science Foundation (BSF), Jerusalem, Israel.

can define monic orthogonal and orthonormal polynomials $P_n(x, d\mu)$, $p_n(x, d\mu)$ and Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$ by (b_n real, $a_n > 0$)

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x) \tag{1.3}$$

and

$$\|P_n\| = a_1 \cdots a_n \tag{1.4}$$

Favard's theorem (see, e.g., [31, 34]) asserts a one-one correspondence between μ's of compact but infinite support and bounded sets of a_n's and b_n's. Moreover, by Weyl's theorem, if $a_n \to 1$, $b_n \to 0$, then the essential support of $d\mu$ is $[-2, 2]$.

Roughly speaking, the boundary for (1.2) to hold is $a_n - 1$, b_n decaying faster than $O(n^{-1})$. Explicitly, Killip and Simon [11] proved a conjecture of Nevai [24] that $\sum_{n=1}^{\infty}(|a_n - 1| + |b_n|) < \infty \Rightarrow$ (1.2), and there are examples of Pollaczek [25, 26, 27] where (1.2) fails because $\log(w(x)) \sim (4 - x^2)^{-\frac{1}{2}}$ near $x = \pm 2$ and $b_n = 0$, $a_n = 1 - Cn^{-1} + O(n^{-2})$.

Killip–Simon [11] discovered a relevant weaker condition than (1.2) they called the quasi-Szegő condition:

$$\int \log(w(x))(4 - x^2)^{\frac{1}{2}}\, dx > -\infty \tag{1.5}$$

and they proved that

$$(1.5) + \sum_{x \in \operatorname{supp}(\mu) \setminus [-2,2]} (|x| - 2)^{\frac{3}{2}} < \infty \Leftrightarrow \sum_{n=1}^{\infty} |a_n - 1|^2 + |b_n|^2 < \infty \tag{1.6}$$

Our cases will include situations where (1.5) and (1.6) fail.

It is known (see [10, 20, 21, 22, 29, 40]) that when $\sum_{n=1}^{\infty}|a_n - 1|^2 + |b_n|^2 = \infty$, $d\mu$ can stop having an a.c. component, so we will need an additional condition. What we will use is

Theorem 1.1. *If $a_n \to 1$, $b_n \to 0$, and*

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| + |b_{n+1} - b_n| < \infty \tag{1.7}$$

then (1.1) *holds where $w(x)$ is continuous on $(-2, 2)$ and strictly positive there. Moreover, $d\mu_{\rm s}$ is supported on $\mathbb{R} \setminus (-2, 2)$.*

The continuum Schrödinger analog of this is a theorem of Weidmann [41]; for OPRL, it is due to Dombrowski–Nevai [4] (see also [12, 8, 32]). Most references do not discuss continuity of w but it holds; for example, it follows immediately from Theorem 1 of [4], since w can be obtained as a uniform limit of continuous functions on any closed subinterval of $(-2, 2)$.

In fact, we will focus on cases where $\{a_n\}$ and $\{b_n\}$ are monotone, so (1.7) is automatic. Typical is

$$a_n \equiv 1 \qquad b_n = -Cn^{-\beta} \tag{1.8}$$

where, roughly speaking, we will prove $w(x)$ is singular at $x = 2$ (i.e., the integral in (1.5) diverges there) with

$$w(x) = e^{-2Q(x)} \tag{1.9}$$

$$Q(x) \sim C_1(2-x)^{\frac{1}{2}-\frac{1}{\beta}} \tag{1.10}$$

Indeed, in Section 5, we will obtain for (1.8) an asymptotic series for $Q(x)$ near $x = 2$ up to terms of $O(\log(2-x))$; see (5.32).

Our interest in these problems was stimulated by a recent paper of Levin–Lubinsky [18] and their related earlier works on non-Szegő weights [16, 17]. They study the problem inverse to ours, namely, going from w (or Q) to a_n, b_n (which they call A_n, B_n). Unfortunately, they do not obtain even leading order asymptotics for a_n, b_n if $Q(x)$ has the form (1.10) but instead require

$$Q(x) \sim \exp_k(1-x^2)^{-\alpha} \tag{1.11}$$

with $\exp_k(x) = \exp(\exp_{k-1}(x))$ and $\exp_1(x) = e^x$. We will obtain inverse results to theirs in Section 5. We note that [16] does have asymptotics on the Rakhmanov–Mhaskar–Saff numbers when (1.10) holds and that their asymptotics should be connected to asymptotics of a_n, b_n.

It is hard to imagine strict if and only if results on $Q(x)$ to a_n, b_n since there will typically be side conditions (a_n, b_n monotone and/or convex in n or $Q(x)$ convex) that may not strictly carry over, but it is comforting (even with side conditions) to get results in both directions. It would be interesting to show that (1.9) and (1.10) (with extra conditions) lead to estimates on a_n, b_n with $|a_n - 1| + |b_n| = O(n^{-\beta})$. We suspect, with analyticity assumptions on Q, that this might be accessible with Riemann–Hilbert techniques.

Our key to going from (a_n, b_n) to (w, Q) is Carmona's formula that relates $d\mu$ to the growth of $p_n(x)$, namely,

Theorem 1.2. *If p_n are the orthonormal polynomials for a measure $d\mu$, a measure with finite moments for which the moment problem is determinate, then $d\nu^{(n)} \xrightarrow{w} d\mu$ where*

$$d\nu^{(n)}(x) = \frac{dx}{\pi(a_n^2 p_n(x)^2 + p_{n-1}^2(x))} \tag{1.12}$$

The continuum analog of this result is due to Carmona [1]. This theorem when $a_n = 1$ is stated without proof in Last–Simon [14] and later (with proof) in Krutikov–Remling [13] and Simon [33]. It implies:

Corollary 1.3. *Suppose uniformly on some interval $[\alpha, \beta]$, we have for strictly positive continuous functions $f_\pm(x)$ that*

$$\begin{aligned} \pi^{-1} f_-(x) &\leq \liminf (a_n^2 p_n(x)^2 + p_{n-1}(x)^2) \\ &\leq \limsup (a_n^2 p_n(x)^2 + p_{n-1}(x)^2) \leq \pi^{-1} f_+(x) \end{aligned} \tag{1.13}$$

Then $d\mu$ is purely absolutely continuous on (α, β) and

$$\frac{1}{f_+(x)} \leq w(x) \leq \frac{1}{f_-(x)} \tag{1.14}$$

there. In particular, if (1.13) *holds for each compact interval $[\alpha, \beta]$ in $(x_0, 2)$,*

$$f_\pm(x) = \exp(2(g(x) \pm h(x))) \tag{1.15}$$

then (1.9) *holds with*

$$|Q(x) - g(x)| \leq h(x) \tag{1.16}$$

Proof. By Theorem 1.1, for any positive continuous function, $\eta(x)$, on $[\alpha, \beta]$ supported on (α, β), we have

$$\int \frac{\eta(x)}{\pi f_+(x)}\, dx \leq \int \eta(x)\, d\mu(x) \leq \int \frac{\eta(x)}{\pi f_-(x)}\, dx \tag{1.17}$$

from which absolute continuity of $\mu \restriction (\alpha, \beta)$ and (1.14) are immediate. This in turn implies (1.15) and (1.16). □

Thus, we need to show $a_n^2 p_n^2 + p_{n-1}^2$ is bounded as $n \to \infty$, but with bounds that diverge as $x \uparrow 2$. The difference equation is

$$\begin{aligned} \begin{pmatrix} p_{n+1} \\ a_{n+1} p_n \end{pmatrix} &= \frac{1}{a_{n+1}} \begin{pmatrix} x - b_{n+1} & -1 \\ a_{n+1}^2 & 0 \end{pmatrix} \begin{pmatrix} p_n \\ a_n p_{n-1} \end{pmatrix} \\ &\equiv A_{n+1}(x) \begin{pmatrix} p_n \\ a_n p_{n-1} \end{pmatrix} \end{aligned} \tag{1.18}$$

Here

$$\det(A_n) = 1 \qquad \operatorname{tr}(A_n) = x - b_n \tag{1.19}$$

In a case like (1.8) where b_n is negative and monotone increasing, a fundamental object is the turning point, the integer, $N(x)$, with

$$x - b_n \geq 2 \qquad \text{if } n \leq N(x) \tag{1.20}$$

$$x - b_n < 2 \qquad \text{if } n > N(x) \tag{1.21}$$

If $\gamma_n(x)$ is defined by $\gamma_n \geq 0$ and

$$x - b_n = 2\cosh(\gamma_n(x)) \qquad (n \leq N(x)) \tag{1.22}$$

then one expects some kind of exponential growth as $\exp(\sum_{j=1}^n \gamma_j(x))$, and we will prove that

$$\exp\biggl(\sum_{j=1}^N \gamma_j(x)\biggr) \le p_N(x) \le (N+1)\exp\biggl(\sum_{j=1}^N \gamma_j(x)\biggr) \tag{1.23}$$

As one expects, there is an intermediate region $N(x) \le n \le N_1(x)$ and an oscillatory region $n \ge N_1(x)$. We will see that so long as one is willing to accept $O((b_{N+2} - b_{N+1})^{-1})$ errors (and they will typically be very small compared to $\exp(\sum_{j=1}^N \gamma_j(x)))$, one can actually take $N_1 = N+2$ (!) and use the method of proof for Theorem 1.1 to control the region $n \ge N_1$. Thus, the key will be (1.23) and we will get (1.16) where

$$g(x) = \sum_{j=1}^N \gamma_j(x) \tag{1.24}$$

and

$$h(x) = O(\max(\log(N), \log((b_{N+2} - b_{N+1})^{-1}))) \tag{1.25}$$

The discussion of turning points sounds like WKB—and the reader might wonder if one can't obtain our result via standard WKB techniques. There is some literature on discrete WKB [6, 35, 36, 37], but we have not seen how to apply them to this situation (for a different application to OPRL, see [7]) or, because of a double $n \to \infty$, $x \to 2$ limit, how to use the continuum WKB theory (on which there is much more extensive literature) to the continuum analog of our problem here. That said, the current paper should be regarded as a WKB-like analysis.

In Section 2, we discuss the case $a_n \equiv 1$, $b_n < b_{n+1} < 0$. In Section 3, we discuss $b_n \equiv 0$, $a_n < a_{n+1} < 1$. It is likely one could handle mixed a_n, b_n cases with more effort. In Section 4, we discuss some Schrödinger operators. Finally, in Section 5, we discuss examples including (1.8) and (1.11).

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2. Monotone b_n

In this section, we will prove:

Theorem 2.1. *Let $d\mu$ be the spectral measure associated with a Jacobi matrix having $a_n \equiv 1$ and*

$$b_n \le b_{n+1} < 0 \qquad b_n \to 0 \text{ as } n \to \infty$$

Define $N(x)$ for x in $(0,2)$ and near 2 by (1.20)/(1.21) and $\gamma_n(x)$ by (1.22). Then $d\mu$ is purely absolutely continuous on $(-2,2)$, where $w = \frac{d\mu}{dx}$ is continuous and nonvanishing on $(-2,2)$,

$$C_1(x+2) \le w(x) \le C_2(x+2)^{-1} \quad \text{for } x \in (-2,0] \tag{2.1}$$

and on $(0,2)$,

$$w(x) = e^{-2Q(x)} \tag{2.2}$$

where

$$|Q(x) - g(x)| \le h(x) \tag{2.3}$$

where

$$g(x) = \sum_{j=1}^{N(x)} \gamma_j(x) \tag{2.4}$$

and $h(x)$ is given by

$$e^{h(x)} = CN(x)(b_{N(x)+2} - b_{N(x)+1})^{-1}(2-x)^{\frac{1}{2}} \tag{2.5}$$

for an explicit constant C (dependent on $\sup|b_n|$ but not on x).

Remark. Typically, h is much smaller than g. For example, if b_n is given by (1.8), $g(x) = O((2-x)^{\frac{1}{2}-\frac{1}{\beta}})$ and $e^{h(x)} = O(N(x)^{2+\beta}(2-x)^{\frac{1}{2}}) = O((2-x)^{-(\frac{1}{2}+\frac{2}{\beta})})$, so $h(x) = O(\log(2-x)^{-1})$.

As we explained in the introduction, we need to study the asymptotics of $p_n(x)$ as $x \uparrow 2$ with some uniformity in n. Given that $a_n \equiv 1$,

$$p_{n+1}(x) = (e^{\gamma_{n+1}} + e^{-\gamma_{n+1}})p_n(x) - p_{n-1}(x) \tag{2.6}$$

$$p_{-1}(x) = 0 \qquad p_0(x) = 1 \tag{2.7}$$

which suggests we define for $n \le N(x)$,

$$\psi_n(x) = e^{-\sum_{j=1}^n \gamma_j} p_n(x) \tag{2.8}$$

so ψ_n obeys

$$\psi_{n+1}(x) = (1 + e^{-2\gamma_{n+1}})\psi_n - e^{-(\gamma_n+\gamma_{n+1})}\psi_{n-1} \tag{2.9}$$

$$\psi_{-1}(x) = 0 \qquad \psi_0(x) = 1 \tag{2.10}$$

Lemma 2.2. *For* $0 \leq n < N(x)$,

$$\psi_{n+1} \geq \psi_n \tag{2.11}$$

In particular,

$$\psi_n(x) \geq 1 \tag{2.12}$$

Proof. As a preliminary, we note that $b_n \leq b_{n+1}$ implies $x - b_n \geq x - b_{n+1}$, so

$$0 \leq \gamma_{n+1} \leq \gamma_n \tag{2.13}$$

By (2.9),

$$\begin{aligned}(\psi_{n+1} - \psi_n) &= e^{-2\gamma_{n+1}}\psi_n - e^{-(\gamma_n+\gamma_{n+1})}\psi_{n-1} \\ &= e^{-2\gamma_{n+1}}(\psi_n - \psi_{n-1}) + e^{-\gamma_{n+1}}(e^{-\gamma_{n+1}} - e^{-\gamma_n})\psi_{n-1}\end{aligned} \tag{2.14}$$

For $n = 0$, $\psi_n - \psi_{n-1} = 1 \geq 0$ and $\psi_{n-1} = 0 \geq 0$. By (2.14) and (2.13) (which implies $e^{-\gamma_{n+1}} - e^{-\gamma_n} \geq 0$), we see inductively that $\psi_{n+1} - \psi_n \geq 0$, and so, $\psi_{n+1} \geq \psi_n \geq 0$, proving (2.11). □

Lemma 2.3. *Define for* $n = 0, 1, 2, \dots, N(x) - 1$,

$$W_n = e^{\gamma_{n+1}}\psi_n - e^{-\gamma_n}\psi_{n-1} \tag{2.15}$$

Then

$$W_n \leq e^{\gamma_{n+1}} \tag{2.16}$$

Proof. $W_0 = e^{\gamma_1} \leq e^{\gamma_1}$, starting an inductive proof of (2.16). By (2.9),

$$\psi_{n+1} = e^{-\gamma_{n+1}}W_n + e^{-2\gamma_{n+1}}\psi_n$$

so

$$\begin{aligned}W_{n+1} &= e^{(\gamma_{n+2}-\gamma_{n+1})}(W_n + e^{-\gamma_{n+1}}\psi_n) - e^{-\gamma_{n+1}}\psi_n \\ &= e^{(\gamma_{n+2}-\gamma_{n+1})}W_n + e^{-\gamma_{n+1}}(e^{(\gamma_{n+2}-\gamma_{n+1})} - 1)\psi_n && (2.17) \\ &\leq e^{(\gamma_{n+2}-\gamma_{n+1})}W_n && (2.18)\end{aligned}$$

since (2.13) implies $e^{\gamma_{n+2}} \leq e^{\gamma_{n+1}}$ and $\psi_n \geq 0$, $(e^{\gamma_{n+2}-\gamma_{n+1}} - 1)\psi_n \leq 0$. Thus, $W_n \leq e^{\gamma_{n+1}}$ implies $W_{n+1} \leq e^{\gamma_{n+2}}$ and (2.16) holds inductively. □

Lemma 2.4. *For* $n = 0, 1, 2, \dots, N(x) - 2$,

$$\psi_{n+1} \leq 1 + \psi_n \tag{2.19}$$

So, in particular, for $0 \leq n < N(x)$,

$$\psi_n \leq n + 1 \tag{2.20}$$

Proof. By (2.15),

$$\begin{aligned}\psi_{n+1} &= e^{-\gamma_{n+2}}W_{n+1} + e^{-(\gamma_{n+1}+\gamma_{n+2})}\psi_n \\ &\leq 1 + \psi_n\end{aligned}$$

since $e^{-\gamma_{n+2}}W_{n+1} \leq 1$ by (2.16) and $\gamma_j \geq 0$ implies $e^{-(\gamma_{n+1}+\gamma_{n+2})} \leq 1$. This proves (2.19), which inductively implies (2.20). □

We summarize with:

Proposition 2.5. *For any n with $1 \leq n < N(x)$,*

$$e^{\sum_{j=1}^n \gamma_j(x)} \leq p_n(x) \leq (n+1)e^{\sum_{j=1}^n \gamma_j(x)} \tag{2.21}$$

In particular, if

$$\eta_n(x) = p_{n-1}(x)^2 + p_n(x)^2 \tag{2.22}$$

then

$$e^{2\sum_{j=1}^n \gamma_j(x)} \leq \eta_n(x) \leq 2(n+1)^2 e^{2\sum_{j=1}^n \gamma_j(x)} \tag{2.23}$$

Proof. (2.21) is an immediate consequence of (2.8), (2.12) and (2.20). □

Suppose $x \in (0,2)$. For $n > N(x)$, define $\kappa_n(x)$ by $0 \leq \kappa_n < \frac{\pi}{2}$ and

$$x - b_n = 2\cos\kappa_n(x) \tag{2.24}$$

so $0 > b_{n+1} \geq b_n$ implies

$$0 \leq \kappa_n \leq \kappa_{n+1}$$

and $b_n \to 0$ implies

$$\kappa_n \to \kappa_\infty = \cos^{-1}(\tfrac{x}{2}) \tag{2.25}$$

For later reference, we note

$$\sin(\kappa_\infty) = (1 - (\tfrac{x}{2})^2)^{\frac{1}{2}} = \tfrac{1}{2}\,(4 - x^2)^{\frac{1}{2}} \tag{2.26}$$

So as $x \uparrow 2$,

$$\kappa_\infty = (2-x)^{\frac{1}{2}} + O((2-x)^{\frac{3}{2}}) \tag{2.27}$$

We first present a matrix method following Kooman [12] to control the region $[N(x)+2, \infty)$. At the end, we will discuss an alternate method using scalar Prüfer-like variables.

By (1.18), for $n > N$, A_n has eigenvalues $e^{\pm i\kappa_n}$. In fact,

$$\begin{pmatrix} 2\cos\kappa & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ e^{\mp i\kappa} \end{pmatrix} = e^{\pm i\kappa} \begin{pmatrix} 1 \\ e^{\mp i\kappa} \end{pmatrix} \tag{2.28}$$

so if

$$Y(\kappa) = \begin{pmatrix} 1 & 1 \\ e^{-i\kappa} & e^{i\kappa} \end{pmatrix} \tag{2.29}$$

and

$$V(\kappa) = \begin{pmatrix} e^{i\kappa} & 0 \\ 0 & e^{-i\kappa} \end{pmatrix} \tag{2.30}$$

then

$$A_n(x) = Y(\kappa_n)V(\kappa_n)Y(\kappa_n)^{-1} \tag{2.31}$$

Next, notice that

$$Y(\kappa)^{-1} = \frac{1}{2i\sin\kappa} \begin{pmatrix} e^{i\kappa} & -1 \\ -e^{-i\kappa} & -1 \end{pmatrix} \tag{2.32}$$

Following Kooman [12], we write for $n > \ell > N(x)$,

$$\begin{aligned} T_n(x) &\equiv A_n \cdots A_{\ell+1} \\ &= Y(\kappa_n)V_nY(\kappa_n)^{-1}Y(\kappa_{n-1})V_{n-1}\cdots Y(\kappa_{\ell+1})^{-1} \end{aligned} \tag{2.33}$$

and since $\|V_n(\kappa)\| = 1$,

$$\|T_n\| \le \|Y(\kappa_n)\|\,\|Y(\kappa_{\ell+1})^{-1}\| \prod_{j=\ell+1}^{n-1} \|Y(\kappa_{j+1})^{-1}Y(\kappa_j)\| \tag{2.34}$$

This prepares us for two critical estimates:

Lemma 2.6. *We have*

$$\|Y(\kappa_{j+1})^{-1}Y(\kappa_j)\| \le 1 + \frac{|e^{i\kappa_{j+1}} - e^{i\kappa_j}|}{\sin(\kappa_{j+1})} \tag{2.35}$$

so, in particular,

$$\|Y(\kappa_{j+1})^{-1}Y(\kappa_j)\| \le 1 + \frac{|\kappa_{j+1} - \kappa_j|}{\sin(\kappa_j)} \tag{2.36}$$

Proof. By (2.29) and (2.32),

$$Y(\kappa_{j+1})^{-1}Y(\kappa_j) - \mathbf{1} = \frac{1}{2\sin(\kappa_{j+1})} \begin{pmatrix} e^{-i\kappa_{j+1}} - e^{-i\kappa_j} & e^{i\kappa_{j+1}} - e^{i\kappa_j} \\ e^{-i\kappa_j} - e^{-i\kappa_{j+1}} & e^{i\kappa_j} - e^{i\kappa_{j+1}} \end{pmatrix} \tag{2.37}$$

If $A = (a_{ij})$ is a 2×2 matrix,

$$\begin{aligned} |\langle \varphi, A\psi\rangle| &\le \max(|a_{ij}|)(|\varphi_1| + |\varphi_2|)(|\psi_1| + |\psi_2|) \\ &\le 2\max(|a_{ij}|)(|\varphi_1|^2) + |\varphi_2|^2)^{\frac{1}{2}}(|\psi_1|^2 + |\psi_2|^2)^{\frac{1}{2}} \end{aligned}$$

since $(|x| + |y|) \le \sqrt{2}(|x|^2 + |y|^2)^{\frac{1}{2}}$, so

$$\|Y(\kappa_{j+1})^{-1}Y(\kappa_j) - \mathbf{1}\| \le \frac{1}{\sin(\kappa_{j+1})}\, |e^{i\kappa_{j+1}} - e^{i\kappa_j}|$$

which implies (2.35).

(2.35) implies (2.36) since $\frac{\pi}{2} > \kappa_{j+1} \ge \kappa_j$ implies $\sin(\kappa_{j+1}) \ge \sin(\kappa_j)$. □

Remark. That (2.36) holds with a 1 in front of $|\kappa_{j+1} - \kappa_j| / \sin(\kappa_j)$ is critical. Lest it seem a miracle of Kooman's method, we give an alternate calculation at the end of this section.

Lemma 2.7. *We have that*

$$\prod_{j=\ell+1}^{\infty} \left(1 + \frac{|\kappa_{j+1} - \kappa_j|}{\sin(\kappa_j)}\right) \leq \frac{\kappa_\infty}{\kappa_{\ell+1}} \exp(\kappa_\infty e(\kappa_\infty)) \tag{2.38}$$

where

$$e(y) = \sup_{0<x\leq y} \left(\frac{1}{\sin(x)} - \frac{1}{x}\right) \tag{2.39}$$

Remark. Since $\sin(x) = x - \frac{x^3}{6} + O(x^5)$, $\frac{1}{\sin(x)} = \frac{1}{x} + \frac{x}{6} + O(x^3)$ and since $\sin(x) < x$, we see $e(y)$ is finite and

$$e(y) = O(\tfrac{y}{6}) \qquad \text{as } y \downarrow 0 \tag{2.40}$$

Proof. We have

$$\frac{1}{\sin(\kappa_j)} \leq \frac{1}{\kappa_j} + e(\kappa_\infty) \tag{2.41}$$

so, since $\kappa_{j+1} \geq \kappa_j$,

$$1 + \frac{|\kappa_{j+1} - \kappa_j|}{\sin(\kappa_j)} \leq \frac{\kappa_{j+1}}{\kappa_j} + (\kappa_{j+1} - \kappa_j)e(\kappa_\infty) \tag{2.42}$$

$$\leq \frac{\kappa_{j+1}}{\kappa_j} (1 + (\kappa_{j+1} - \kappa_j)e(\kappa_\infty)) \tag{2.43}$$

$$\leq \frac{\kappa_{j+1}}{\kappa_j} \exp((\kappa_{j+1} - \kappa_j)e(\kappa_\infty)) \tag{2.44}$$

from which (2.38) is immediate if we note that $\kappa_\infty - \kappa_\ell \leq \kappa_\infty$. □

Proof of Theorem 2.1. By (2.34) and Lemmas 2.6 and 2.7, if $T_k(x)$ is the transfer matrix from $N(x) + 2$ to $k > N(x) + 2$, then uniformly in k,

$$\|T_n\| \leq 2(\sin(\kappa_{N(x)+2}))^{-1} \frac{\kappa_\infty}{\kappa_{N(x)+2}} \exp(\kappa_\infty e(\kappa_\infty)) \tag{2.45}$$

where we also used $\|Y(\kappa_k)\| \leq 2$ and $\|Y(\kappa_{N(x)+2})^{-1}\| \leq 2/2\sin(\kappa_{N(x)+2})$.

As $x \uparrow 2$, $\kappa_\infty \to 0$. Indeed, by (2.27), $\kappa_\infty = (2-x)^{\frac{1}{2}} + O((2-x)^{\frac{3}{2}})$. Moreover, by the definition of $N(x)$,

$$x - b_{N+1} < 2 \tag{2.46}$$

while

$$x - b_{N+2} = 2\cos(\kappa_{N+2}) \tag{2.47}$$

so

$$2(1-\cos(\kappa_{N+2})) > b_{N+2} - b_{N+1} \tag{2.48}$$

Since $N(x) \to \infty$, $b_{N(x)+2} \to 0$ so $\kappa_{N+2}(x) \to 0$ and (2.48) implies

$$\kappa_{N+2}(x)^2 > (1+o(1))(b_{N+2} - b_{N+1}) \tag{2.49}$$

Thus, in (2.45), $[\kappa_{N(x)+2}\sin(\kappa_{N+2})]^{-1} \leq (1+o(1))(b_{N+2}-b_{N+1})$ and (2.45) becomes

$$\sup_{n\geq N(x)+2} \|\tilde T_n\| \leq C(2-x)^{\frac12}(b_{N+2}-b_{N+1})^{-1} \equiv A(x) \tag{2.50}$$

where now $\tilde T_n$ transfers from $N-1$ to n and we use the boundedness from $N-1$ to $N+2$. Using

$$\|\tilde T_n\|^{-2}(|p_{n+1}|^2+|p_n|^2) \leq |p_N|^2 + |p_{N-1}|^2 \leq \|\tilde T_n^{-1}\|^2(|p_{n+1}|^2+|p_n|^2) \tag{2.51}$$

and (2.23), we obtain for all $n > N$,

$$C_1 A(x)^{-2} e^{2\sum_1^N \gamma_j(x)} \leq (|p_n|^2 + |p_{n+1}|^2) \leq CA(x)^2 N(x)^2 e^{2\sum_1^N \gamma_j(x)} \tag{2.52}$$

which, given Corollary 1.3, implies (2.2)–(2.5).

In going from (2.51) to (2.52), we used

$$\det(\tilde T_n) = 1 \Rightarrow \|\tilde T_n^{-1}\| = \|\tilde T_n\|$$

We also need to control the region $x > -2$ with $2-x$ small. By replacing x by $-x$ (and $p_n(x)$ by $(-1)^n p_n(-x)$), this is the same as looking at $x + b_n$ with still $b_n < b_{n+1} < 0$. We define $\theta_n(x)$ by

$$2\cos(\theta_n(x)) = x + b_n \tag{2.53}$$

so

$$\theta_1 \geq \theta_2 \geq \cdots \geq \theta_\infty = \kappa_\infty = (2-x)^{\frac12} + O((2-x)^{\frac32}) \tag{2.54}$$

As above, we have (2.35), so

$$\|Y(\theta_{j+1})^{-1}Y(\theta_j)\| \leq 1 + \frac{|\theta_{j+1}-\theta_j|}{\sin(\theta_{j+1})} \tag{2.55}$$

but since $\theta_{j+1} < \theta_j$, we have

$$1 + \frac{|\theta_{j+1}-\theta_j|}{|\theta_{j+1}|} = \frac{\theta_{j+1} + (\theta_j - \theta_{j+1})}{\theta_{j+1}} = \frac{\theta_j}{\theta_{j+1}} \tag{2.56}$$

and we find that, with T_n being the transfer matrix from 1 to n,

$$\|T_n\| \leq \frac{\theta_1}{\theta_\infty}\, 2\, \frac{2}{2\sin(\theta_1)} \leq \frac{C}{\theta_\infty} \leq C(2-x)^{\frac12}(1+o(1)) \tag{2.57}$$

This bound on the transfer matrix and Corollary 1.3 yield (2.1). □

Remark. It might be surprising that (2.1) has $(x+2), (x+2)^{-1}$ rather than $(x+2)^{\frac{1}{2}}, (x+2)^{-\frac{1}{2}}$ (because Carmona's formula (1.12) relates $w(x)$ to $\|T_n\|^2$ and $\sup \|T_n\|$ goes like $(2-x)^{\frac{1}{2}}$). Even in the free case, bounds from Carmona's formula give the wrong behavior: $\sin^2(n\theta) + \sin^2((n+1)\theta)$ have oscillations that cause the actual square root behavior in the free case, and bounds based only on $\|T_n\|$ lose that.

That completes the proof of Theorem 2.1, the main result of this paper. Here is an alternate approach to controlling p_n for $n > N$, using the complex quantities:

$$\Phi_n = p_n - e^{-i\kappa_n} p_{n-1} \tag{2.58}$$

so, since p_j is real,

$$\begin{aligned} \sin(\kappa_n)|p_{n-1}| &= |\mathrm{Im}(-\Phi_n)| \\ &\leq |\Phi_n| \end{aligned} \tag{2.59}$$

By (2.24), we have

$$p_{n+1} = (e^{i\kappa_{n+1}} + e^{-i\kappa_{n+1}})p_n - p_{n-1} \tag{2.60}$$

so

$$\begin{aligned} \Phi_{n+1} &= e^{i\kappa_{n+1}}[p_n - e^{-i\kappa_{n+1}} p_{n-1}] \\ &= e^{i\kappa_{n+1}} \Phi_n + e^{i\kappa_{n+1}}(e^{-i\kappa_n} - e^{-i\kappa_{n+1}})p_{n-1} \end{aligned} \tag{2.61}$$

Using (2.59),

$$|\Phi_{n+1}| \leq |\Phi_n| + \frac{|\kappa_n - \kappa_{n+1}|}{\sin(\kappa_n)} |\Phi_n| \tag{2.62}$$

and similarly,

$$|\Phi_{n+1}| \geq |\Phi_n| - \frac{|\kappa_n - \kappa_{n+1}|}{\sin(\kappa_n)} |\Phi_n| \tag{2.63}$$

These replace (2.36) and imply, via Lemma 2.7 and the analysis in (2.46), that

$$C_1(2-x)^{-\frac{1}{2}}(b_{N+2} - b_{N+1}) \leq \frac{|\Phi_n|}{|\Phi_{N+2}|} \leq C(2-x)^{\frac{1}{2}}(b_{N+2} - b_{N+1})^{-1}$$

Since

$$|\Phi_n|^2 \leq |p_n|^2 + |p_{n-1}|^2$$

and

$$2|\Phi_n|^2 \geq \sin^2(\kappa_{n+1})(|p_n|^2 + |p_{n-1}|^2)$$

we can go from this to Theorem 2.1.

3. Monotone a_n

In this section, we will consider

$$b_n \equiv 0 \qquad a_{n+1} \le a_n \le 1 \qquad a_n \to 1 \tag{3.1}$$

The weight will be symmetric, the measure purely absolutely continuous (i.e., no eigenvalues outside $[-2,2]$), and so for non-Szegő weights, the integral will diverge at both ends. Here is the main result:

Theorem 3.1. *Let $d\mu(x) = w(x)\,dx$ be the measure associated with Jacobi parameters obeying* (3.1)*. For any $x \in (-2,2)$, define $N(x)$ by*

$$2a_n \le |x| \quad \text{for } n \le N(x) \qquad 2a_n > |x| \quad \text{for } n > N(x) \tag{3.2}$$

and $\gamma_n(x)$ for $n \le N(x)$ by

$$\frac{|x|}{a_n} = 2\cosh(\gamma_n(x)) \tag{3.3}$$

Then

$$w(x) = e^{-2Q(x)} \tag{3.4}$$

where

$$|Q(x) - g(x)| \le h(x)$$

$$g(x) = \sum_{j=1}^{N(x)} \gamma_j(x)$$

and $h(x)$ is given by

$$e^{h(x)} = CN(x)(a_{N(x)+2} - a_{N(x)+1})^{-1} \tag{3.5}$$

The proof will closely mimic the proof of Theorem 2.1, so we will only indicate the changes. By symmetry, without loss, we can suppose $x > 0$. The recursion relation becomes

$$p_{n+1}(x) = (e^{\gamma_{n+1}(x)} + e^{-\gamma_{n+1}(x)})p_n(x) - \frac{a_n}{a_{n+1}}\,p_{n-1}(x) \tag{3.6}$$

where we note, by (3.3), that

$$\frac{a_n}{a_{n+1}} = \frac{\cosh(\gamma_{n+1}(x))}{\cosh(\gamma_n(x))} \tag{3.7}$$

Define $\psi_n(x)$ by (2.8), so (2.9) becomes

$$\psi_{n+1}(x) = (1 + e^{-2\gamma_{n+1}(x)})\psi_n(x) - \frac{a_n}{a_{n+1}}\,e^{-(\gamma_n(x)+\gamma_{n+1}(x))}\psi_{n-1}(x) \tag{3.8}$$

(2.10) still holds.

Lemma 3.2. *$\psi_{n+1} \ge \psi_n$, so $\psi_n(x) \ge 1$ for $n \ge 0$.*

Proof. We still have (2.13), and (2.14) becomes

$$\psi_{n+1}-\psi_n=e^{-2\gamma_{n+1}}(\psi_n-\psi_{n-1})+e^{-\gamma_{n+1}}\left(e^{-\gamma_{n+1}}-\frac{a_n}{a_{n+1}}\,e^{-\gamma_n}\right)\psi_{n-1} \tag{3.9}$$

Since $a_n \le a_{n+1}$, $\frac{a_n}{a_{n+1}}<1$, and so

$$\frac{a_n}{a_{n+1}}\,e^{-\gamma_n}\le e^{-\gamma_n}\le e^{-\gamma_{n+1}}$$

Thus, by (3.9), $\psi_{n+1}-\psi_n\ge 0$ and $\psi_{n+1}\ge 0$ inductively. □

Lemma 3.3.

$$e^{\gamma_{n+2}}\le e^{\gamma_{n+1}}\,\frac{\cosh(\gamma_{n+2})}{\cosh(\gamma_{n+1})} \tag{3.10}$$

Proof. This is equivalent to

$$e^{\gamma_{n+2}+\gamma_{n+1}}+e^{\gamma_{n+2}-\gamma_{n+1}}\le e^{\gamma_{n+2}+\gamma_{n+1}}+e^{\gamma_{n+1}-\gamma_{n+2}} \tag{3.11}$$

so to $\gamma_{n+2}-\gamma_{n+1}\le 0$, so to (2.13). □

Lemma 3.4. *Define*

$$W_n=e^{\gamma_{n+1}}\psi_n-\frac{a_n}{a_{n+1}}\,e^{-\gamma_n}\psi_{n-1} \tag{3.12}$$

Then

$$W_n\le e^{\gamma_{n+1}} \tag{3.13}$$

Proof. (3.13) holds for $n=0$ by (3.12) for $n=0$, so we can try an inductive proof. The analog of (2.17) is

$$W_{n+1}=e^{(\gamma_{n+2}-\gamma_{n+1})}W_n+e^{-\gamma_{n+1}}\left(e^{(\gamma_{n+2}-\gamma_{n+1})}-\frac{a_{n+1}}{a_{n+2}}\right)\psi_n \tag{3.14}$$

By (3.7) and (3.10),

$$e^{(\gamma_{n+2}-\gamma_{n+1})}-\frac{a_{n+1}}{a_{n+2}}\le 0$$

so (3.14) says

$$W_{n+1}\le e^{(\gamma_{n+2}-\gamma_{n+1})}W_n\le e^{\gamma_{n+2}}$$

by induction. □

Lemma 3.5. $\psi_{n+1}\le 1+\psi_n$ *so inductively,* $\psi_n\le n+1$.

Proof. By (3.12) and (3.13),

$$\begin{aligned}\psi_{n+1}&=e^{-\gamma_{n+2}}W_{n+1}+\frac{a_{n+1}}{a_{n+2}}\,e^{-\gamma_{n+2}-\gamma_{n+1}}\psi_n\\&\le 1+\psi_n\end{aligned}$$

since $\frac{a_{n+1}}{a_{n+2}}\le 1$. □

If now

$$\eta_n(x) = p_{n-1}(x)^2 + a_n^2 p_n(x)^2 \tag{3.15}$$

then we have proven (2.23) for large n.

To control the region $n \geq N(x) + 2$, we use the scalar variable technique from the end of Section 2. Define κ_n for $n \geq N(x) + 1$ by (recall $x > 0$)

$$\frac{x}{a_n} = 2\cos(\kappa_n(x)) \tag{3.16}$$

so $a_{n+1} \geq a_n$ implies

$$\kappa_n(x) \leq \kappa_{n+1}(x) \tag{3.17}$$

Define

$$\Phi_n = p_n - e^{-i\kappa_n} p_{n-1} \tag{3.18}$$

Then

Lemma 3.6. (i)

$$|p_{n-1}| \leq \frac{|\Phi_n|}{\sin(\kappa_n)} \tag{3.19}$$

(ii)

$$\frac{|\Phi_{n+1}|}{|\Phi_n|} \leq 1 + \frac{|e^{i\kappa_n}\cos(\kappa_n) - e^{i\kappa_{n+1}}\cos(\kappa_{n+1})|}{\cos(\kappa_n)\sin(\kappa_n)} \tag{3.20}$$

$$\leq 1 + \frac{|\kappa_{n+1} - \kappa_n|}{\frac{1}{2}\sin(2\kappa_n)} \tag{3.21}$$

Proof. (i) This comes from $|\operatorname{Im}\Phi_n| = \sin(\kappa_n)(p_{n-1})$.

(ii) From

$$p_{n+1} = (e^{i\kappa_{n+1}} + e^{-i\kappa_{n+1}})p_n - \frac{a_n}{a_{n+1}}\, p_{n-1}$$

we obtain

$$|\Phi_{n+1} - e^{i\kappa_{n+1}}\Phi_n| = \left| e^{i\kappa_n} - \frac{a_n}{a_{n+1}}\, e^{i\kappa_{n+1}} \right| p_{n-1} \tag{3.22}$$

By (3.16),

$$\frac{a_n}{a_{n+1}} = \frac{\cos(\kappa_{n+1})}{\cos(\kappa_n)} \tag{3.23}$$

so (3.22) and (3.19) imply (3.20). This in turn implies (3.21) since

$$e^{i\kappa_n}\cos(\kappa_n) - e^{i\kappa_{n+1}}\cos(\kappa_{n+1}) = \tfrac{1}{2}\left(e^{2i\kappa_n} - e^{2i\kappa_{n+1}}\right) \tag{3.24}$$

□

With this formula, we can mimic the proof of Theorem 2.1 to complete the proof of Theorem 3.1.

4. Schrödinger Operators

In this section, we consider Schrödinger operators $H = -\frac{d^2}{dx^2} + V(x)$ on $L^2([0,\infty))$ where one places $u(0) = 0$ boundary conditions. H is unitarily equivalent to multiplication by E on $L^2(\mathbb{R}, d\mu(E))$, where $d\mu$ is the conventional spectral measure (see [3, 19, 23]). If $u(x,E)$ obeys

$$-u'' + Vu = Eu \qquad u(0,E) = 0,\ u'(0,E) = 1 \tag{4.1}$$

then Carmona's formula [1] takes the form

$$\frac{\pi^{-1}dE}{(|u(x,E)|^2 + |u'(x,E)|^2)} \xrightarrow{w} d\mu(E) \tag{4.2}$$

In particular, if uniformly in compact subsets of $E \in (0,\infty)$,

$$\begin{aligned} \exp(2(g(E) - h(E))) &\le \liminf_{x\to\infty} (|u(x,E)|^2 + |u'(x,E)|^2) \\ &\le \limsup_{x\to\infty} (|u(x,E)|^2 + |u'(x,E)|^2) \\ &\le \exp(2(g(E) + h(E))) \end{aligned} \tag{4.3}$$

then $d\mu$ is purely absolutely continuous on $(0,\infty)$, $d\mu(E) = e^{-2Q(E)}\,dE$, and

$$|Q(E) - g(E)| \le h(E) \tag{4.4}$$

We want to assume the following conditions on V:

(a) V is C^1 on $[0,\infty)$.

(b) V is positive and strictly monotone decreasing on $[0,\infty)$. Indeed,

$$V'(x) < 0 \tag{4.5}$$

(c)

$$\lim_{x\to\infty} V(x) = 0 \tag{4.6}$$

Of course, the canonical example is

$$V(x) = (x + x_0)^{-\beta} \tag{4.7}$$

Our main result in this section is:

Theorem 4.1. *Let V obey* (a), (b), (c) *so $d\mu(E) = e^{-2Q(E)}\,dE$. Define for $E < V(0)$,*

$$N(E) = V^{-1}(E)$$

so

$$\begin{aligned} V(x) &> E \qquad \text{if } x < N(E) \\ V(x) &< E \qquad \text{if } x > N(E) \end{aligned} \tag{4.8}$$

For $x < N(E)$, define

$$\gamma(x,E) = (V(x) - E)^{\frac{1}{2}} \tag{4.9}$$

Then (4.4) *holds where for* $E < V(0)$,

$$g(E) = \int_0^{N(E)} \gamma(x, E)\, dx \tag{4.10}$$

and for $E < V(0)$,

$$e^{h(E)} = CN(E)\,(V(N(E)) - V(N(E)+1))^{-1} E^{\frac{1}{2}} \tag{4.11}$$

This proof will illuminate the proofs of the previous two sections. We begin with an analysis of the region $x < N(E)$. We define

$$\psi(x) = u(x, E) \exp\biggl(-\int_0^x \gamma(y, E)\, dy\biggr) \tag{4.12}$$

and are heading towards

$$0 \le \psi'(x) \le 1 \tag{4.13}$$

Lemma 4.2. *For* $0 < E < V(0)$ *and* $x < N(E)$, *we have*

$$\text{(a)} \qquad u'(x) \ge 1 \tag{4.14}$$

$$\text{(b)} \qquad u(x) \ge x \tag{4.15}$$

Proof. $u'' = \gamma^2 u$, so $u'' > 0$. This implies $u'(x) \ge u'(0) = 1$, and then $u(x) = \int_0^x u'(y)\, dy \ge x$. □

Lemma 4.3. *For* $E < V(0)$ *and* $x < N(E)$,

$$\psi'(x) \ge 0 \tag{4.16}$$

Proof. Let

$$f(x) = u'(x) - \gamma(x)u(x) \tag{4.17}$$

so

$$\psi'(x) = f(x) \exp\biggl(-\int_0^x \gamma(y, E)\, dy\biggr) \tag{4.18}$$

and (4.16) is equivalent to $f \ge 0$. Note that

$$\begin{aligned} f' + \gamma f &= u'' - \gamma u' - \gamma' u + \gamma u' - \gamma^2 u \\ &= -\gamma' u \end{aligned} \tag{4.19}$$

since (4.1) says

$$u'' = \gamma^2 u \tag{4.20}$$

(4.5) implies

$$\gamma'(y) \le 0 \tag{4.21}$$

so (4.19) says

$$\biggl(f \exp\biggl(\int_0^x \gamma(y)\, dy\biggr)\biggr)' \ge 0 \tag{4.22}$$

which, given $f(0) = 1$, implies $f \ge 0$ and so $\psi' \ge 0$. □

Lemma 4.4. *Let*

$$W(x) = \psi'(x) + 2\gamma(x)\psi(x) \tag{4.23}$$

Then $W'(x) \leq 0$ *and so*

$$\psi'(x) \leq 1 \tag{4.24}$$

Proof. By (4.18),

$$\psi' + 2\gamma(x)\psi = (u' + \gamma(x)u)e^{-\int_0^x \gamma(y)\,dy} \tag{4.25}$$

so

$$\begin{aligned} W'(x) &= (u'' + \gamma u' + \gamma' u - \gamma u' - \gamma^2 u)e^{-\int_0^x \gamma(y)\,dy} \\ &= \gamma' u e^{-\int_0^x \gamma(y)\,dy} \\ &\leq 0 \end{aligned} \tag{4.26}$$

by (4.21). But $W(x=0) = \psi'(0) = 1$, so

$$W(x) \leq 1 \tag{4.27}$$

and thus

$$\psi' = W - 2\gamma\psi \leq 1 \tag{4.28}$$

□

Proposition 4.5. *If* E *is such that* $N(E) > 1$*, then*

$$e^{-2V(0)}e^{2\int_0^{N(E)} \gamma(y)\,dy} \leq u(N(E))^2 + u'(N(E))^2 \leq (N(E)^2+1)e^{2\int_0^{N(E)} \gamma(y)\,dy} \tag{4.29}$$

Proof. Since $\gamma(N(E)) = 0$,

$$\psi'(N(E)) = u'(N(E))e^{-\int_0^{N(E)} \gamma(y)\,dy}$$

so $0 \leq \psi' \leq 1$ and $\psi(0) = 0$ yield the upper bound in (4.29).

For the lower bound, (4.15) implies $u(1) \geq 1$. So, since $\gamma(y) \leq \gamma(0) \leq V(0)$,

$$\psi(1) \geq e^{-V(0)} \tag{4.30}$$

which, given that $\psi' > 0$ and $N(E) > 1$, implies

$$u(N(E)) \geq e^{-V(0)}e^{\int_0^{N(E)} \gamma(y)\,dy}$$

□

In the region $[N(E), N(E)+1]$, we note that since

$$\left\| \begin{pmatrix} 1 & V(x) - E \\ 1 & 0 \end{pmatrix} \right\| \leq 1 + |E| + |V(0)|$$

the matrix form of the Schrödinger equation implies that if $C(x) = |u(x)|^2 + |u'(x)|^2$, then

$$e^{-2(1+|E|+V(0))|x-y|}C(y) \leq C(x) \leq e^{2(1+|E|+V(0))|x-y|}C(y)$$

giving a constant term in $e^{h(E)}$ in (4.11).

Finally, in the region $[N(E)+1,\infty)$, we use the method of Appendix 2 of Simon [30] (see also Hinton–Shaw [9]). Define for $x > N(E)$,

$$\kappa(x,E) = \sqrt{E - V(x)} \tag{4.31}$$

and define

$$u_\pm(x) = \exp\biggl(\pm i \int_{N(E)}^x \kappa(y)\, dy\biggr) \tag{4.32}$$

If

$$F(x) = \frac{i}{2}\, V'(x)(E-V(x))^{-\frac{1}{2}} \tag{4.33}$$

and if $a(x)$, $b(x)$ are defined by

$$u(x) = a(x)u_+(x) + b(x)u_-(x) \tag{4.34}$$

$$u'(x) = a(x)u'_+(x) + b(x)u'_-(x) \tag{4.35}$$

then $u'' = -\kappa^2 u$ is equivalent to (see Problem 98 on p. 395 of [28])

$$\begin{pmatrix} a(x) \\ b(x) \end{pmatrix}' = M(x) \begin{pmatrix} a(x) \\ b(x) \end{pmatrix} \tag{4.36}$$

where

$$M(x) = w(x)^{-1} \begin{pmatrix} -F(x) & u_-^2(x)F(x) \\ u_+^2(x)F(x) & -F(x) \end{pmatrix} \tag{4.37}$$

with

$$\begin{aligned} w(x) &= u'_+(x)u_-(x) - u'_-(x)u_+(x) \\ &= 2i\kappa(x) \end{aligned} \tag{4.38}$$

Proposition 4.6. *Let $M(x)$ be given by* (4.37)*. Then*

$$\int_{N(E)+1}^\infty \|M(x)\|\, dx \le \log\biggl(\frac{\kappa(\infty,E)}{\kappa(N(E)+1,E)}\biggr) \tag{4.39}$$

Proof. Since $|u_\pm| = 1$,

$$\begin{aligned} \|M(x)\| &\le |w(x)|^{-1} \left\| \begin{pmatrix} |F(x)| & |F(x)| \\ |F(x)| & |F(x)| \end{pmatrix} \right\| \\ &= 2|w(x)|^{-1}|F(x)| \\ &= -\tfrac{1}{2}\, V'(x)(E-V(x))^{-1} \\ &= \tfrac{d}{dx}\, \log((E-V(x))^{\frac{1}{2}}) \end{aligned} \tag{4.40}$$

from which (4.39) follows. $\square$

Proof of Theorem 4.1. Let

$$Y(x) = \begin{pmatrix} u_+(x) & u_-(x) \\ u_+'(x) & u_-'(x) \end{pmatrix} \tag{4.41}$$

Let $T(x,y)$ be the $\binom{u}{u'}$ transfer matrix from x to y and $\tilde T(x,y)$ be the $\binom{a}{b}$ transfer matrix. For $y > N(E)+1$, we have just seen

$$\begin{aligned} \|\tilde T(N(E)+1,y)\| &\leq \exp\biggl(\int_{N(E)+1}^\infty \|M(x)\|\, dx\biggr) \\ &= \frac{\kappa(\infty,E)}{\kappa(N(E)+1,E)} \end{aligned} \tag{4.42}$$

On the other hand,

$$\|Y(y)\| \leq 1+\kappa \leq 2 \tag{4.43}$$

for κ small while

$$\|Y(y)^{-1}\| = |\det(Y)^{-1}|\,\|Y\| \leq \kappa(y)^{-1} \tag{4.44}$$

and

$$T(x,y) = Y(y)\tilde T(x,y)Y(x)^{-1}$$

so

$$\|T(N(E)+1,y)\| \leq \frac{2\kappa(\infty,E)}{\kappa(N(E)+1,E)^2} \tag{4.45}$$

Since $E=V(N(E))$,

$$\kappa(N(E)+1,E)^2 = V(N(E)) - V(N(E)+1) \tag{4.46}$$

and we have the bound (4.4) with the error built from $e^{-V(0)}$, $N(E)$, (4.39), and (4.45). □

It is interesting that the differential equation methods of this section lead to terms that are identical to what we found in the discrete case.

5. Examples

We start with the continuum case.

Example 5.1.

$$V(x) = C_0 x^{-\beta} \qquad \beta < 2 \qquad C_0 > 0 \tag{5.1}$$

Technically this does not fit into Theorem 4.1 since $V(0)=\infty$, but when $\beta<2$, it is easy to extend the analysis. The spectral measure is $e^{-2Q(E)}\, dE$ where (4.4) holds.

$$N(E) = \biggl(\frac{E}{C_0}\biggr)^{-\frac{1}{\beta}} \tag{5.2}$$

$$\begin{aligned} V(N(E)) - V(N(E)+1) &\sim V'(N(E)) \\ &\sim N(E)^{-1} V(N(E)) \\ &= EN(E)^{-1} \end{aligned} \tag{5.3}$$

so $h(E) = O(\log(N(E)^2 E^{-\frac{1}{2}})) = O(\log(E))$. On the other hand, letting $y = x/N(E)$,

$$g(E) = \int_0^{N(E)} (V(x) - E)^{\frac{1}{2}}\, dx \tag{5.4}$$

$$= N(E) E^{\frac{1}{2}} \int_0^1 (y^{-\beta} - 1)^{\frac{1}{2}}\, dy \tag{5.5}$$

$$= E^{\frac{1}{2}} N(E) \beta^{-1} \int_0^1 (1-u)^{\frac{1}{2}} u^{\frac{1}{\beta} - \frac{3}{2}}\, du$$

$$= E^{\frac{1}{2}} N(E) \beta^{-1} \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{1}{\beta} - \frac{1}{2})}{\Gamma(\frac{1}{\beta} + 1)} \tag{5.6}$$

using a $u = y^\beta$ change of variables. Thus,

$$g(E) = c_1 C_0^{\frac{1}{\beta}} E^{\frac{1}{2} - \frac{1}{\beta}} \qquad c_1 = \beta^{-1} \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{1}{\beta} - \frac{1}{2})}{\Gamma(\frac{1}{\beta} + 1)} \tag{5.7}$$

Since $\beta < 2$, $g(E) \to \infty$ and is much larger than the $\log(E)$ error. $\beta = 1$, the Coulomb case, has $g(E) = C_0 c_1 E^{-\frac{1}{2}}$ and $\beta = \frac{1}{2}$, the quasi-Szegő borderline, has $g(E) = C_0^2 c_1 E^{-\frac{3}{2}}$. We emphasize that g occurs in an exponential, so w is very small near $E = 0$. □

Example 5.2.

$$V(x) = C_0 (x + x_0)^{-\beta} \qquad \beta < 2 \tag{5.8}$$

We claim that the changes from Example 5.1 are small compared to $\log(E)$ errors in h; explicitly,

$$g(E) = c_1 C_0^{\frac{1}{\beta}} E^{\frac{1}{2} - \frac{1}{\beta}} + O(1) + O(E^{\frac{1}{2}}) \tag{5.9}$$

For in this case,

$$N(E) = \left(\frac{E}{C_0} \right)^{-\frac{1}{\beta}} - x_0 \tag{5.10}$$

and one changes variables to $y = (x + x_0)/(N(E) + x_0)$, so (5.5) becomes

$$g(E) = N(E) E^{\frac{1}{2}} \int_{s(E)}^1 (y^{-\beta} - 1)^{\frac{1}{2}}\, dy \tag{5.11}$$

where

$$s(E) = y(x=0) = \frac{x_0}{N(E)+x_0} \tag{5.12}$$

Then

$$\begin{aligned} N(E)E^{\frac{1}{2}} \int_0^{s(E)} (y^{-\beta}-1)^{\frac{1}{2}}\,dy &= N(E)E^{\frac{1}{2}} O(s(E)^{1-\frac{\beta}{2}}) \\ &= O(1) \end{aligned} \tag{5.13}$$

by (5.10) and (5.12), so

$$\begin{aligned} g(E) &= c_1 N(E) E^{\frac{1}{2}} + O(1) \\ &= c_1 C_0^{\frac{1}{\beta}} E^{\frac{1}{2}-\frac{1}{\beta}} + O(1) + O(E^{\frac{1}{2}}) \end{aligned} \tag{5.14}$$

as claimed. □

Now we turn to the discrete case.

Example 5.3 (= (1.8))**.**

$$a_n \equiv 1 \qquad b_n = -Cn^{-\beta} \tag{5.15}$$

Define

$$\delta = 2-x \qquad \delta_n = Cn^{-\beta} - \delta \tag{5.16}$$

so

$$x - b_n = 2 + \delta_n \tag{5.17}$$

We have (with $[y]$ = maximal integer $\leq y$)

$$N(x) = [(C^{-1}\delta)^{-\frac{1}{\beta}}] \tag{5.18}$$

We have $b_{N+2} - b_{N+1} = O(N^{-\beta-1})$, so the RHS of (2.5) is of order $CN(x)^{\beta+2}\delta^{\frac{1}{2}} = O(\delta^{-\frac{1}{2}-\frac{2}{\beta}})$ and thus, $h(x) = O(\log(2-x))$ and we need to compute $g(x) = \sum_{j=1}^{N(x)} \gamma_j(x)$ up to $O(\log\delta)$ terms.

We will suppose below that $C \leq 1$ and explain at the end what to change if $C > 1$.

Define c_ℓ to be the Taylor coefficients in

$$\cosh^{-1}(1+\tfrac{z}{2}) = \sqrt{z} \sum_{\ell=0}^{\infty} c_\ell z^\ell \tag{5.19}$$

so, courtesy of Mathematica,

$$c_0 = 1 \qquad c_1 = -\frac{1}{24} \qquad c_2 = \frac{3}{640} \qquad c_3 = -\frac{5}{7168}$$

and, for example,

$$c_{20} = 34,461,632,205/12,391,489,651,049,749,040,738,304$$

(assuming that we managed to copy it without a typo). Thus,

$$g(x) = \sum_{\ell=0}^{\infty} c_\ell \sum_{j=1}^{N(x)} \delta_j^{\ell+\frac{1}{2}} \tag{5.20}$$

Notice that since $\delta > 0$,

$$\delta_j \le Cj^{-\beta} \tag{5.21}$$

so, if $\beta(\ell+\frac{1}{2}) > 1$, a crude δ-independent bound of $\sum_{j=1}^{N(x)} \delta_j^{\ell+\frac{1}{2}}$ can be summed independently of $N(x)$. Moreover, if F is the function in (5.19), then

$$2\sqrt{z}\,\frac{dF}{dz} = \frac{1}{\sqrt{1+\frac{z}{4}}} \tag{5.22}$$

so the c_ℓ power series has radius of convergence 4 and so $\sum |c_\ell| < \infty$. Thus, if

$$\ell_0 = [\tfrac{1}{\beta} - \tfrac{1}{2}] + 1 \tag{5.23}$$

then

$$\sum_{\ell=\ell_0}^{\infty} |c_\ell| \sum_{j=1}^{N(x)} \delta_j^{\ell+\frac{1}{2}} \le \biggl(\sum_{0}^{\infty} |c_\ell|\biggr) \sum_{j=1}^{\infty} j^{-\beta(\ell_0+1)} \tag{5.24}$$

(since $C \le 1$) so

$$\sum_{j=1}^{N} \gamma_j = \sum_{0\le \ell \le \frac{1}{\beta}-\frac{1}{2}} c_\ell \sum_{j=1}^{N} \delta_j^{\ell+\frac{1}{2}} + O(1) \tag{5.25}$$

If $\ell = \frac{1}{\beta} - \frac{1}{2}$ occurs, then

$$\begin{aligned} \sum_{j=1}^{N} \delta_j^{\frac{1}{\beta}-\frac{1}{2}+\frac{1}{2}} &= \sum_{j=1}^{N} \delta_j^{\frac{1}{\beta}} \\ &= \sum_{j=1}^{N} (C\,\tfrac{1}{j^\beta} - \delta)^{\frac{1}{\beta}} \\ &\le C^{\frac{1}{\beta}} \sum_{j=1}^{N} j^{-1} \\ &= O(\log N) \end{aligned} \tag{5.26}$$

On the other hand, if $\ell < \frac{1}{\beta} - \frac{1}{2}$, then

$$\sum_{j=1}^{N} \delta_j^{\ell+\frac{1}{2}} = \sum_{j=1}^{N} (Cj^{-\beta} - \delta)^{\ell+\frac{1}{2}}$$

$$= C^{\ell+\frac{1}{2}} \sum_{j=1}^{N} (\tfrac{1}{j^\beta} - \tfrac{1}{N^\beta})^{\ell+\frac{1}{2}} + O(1)$$

$$= C^{\ell+\frac{1}{2}} \sum_{j=1}^{N} j^{-\beta(\ell+\frac{1}{2})} (1 - (\tfrac{j}{N})^\beta)^{\ell+\frac{1}{2}} + O(1)$$

$$= C^{\ell+\frac{1}{2}} \int_1^N x^{-\beta(\ell+\frac{1}{2})} (1 - (\tfrac{x}{N})^\beta)^{\ell+\frac{1}{2}} + O(1) \tag{5.27}$$

$$= C^{\ell+\frac{1}{2}} \beta^{-1} N^{1-(\ell+\frac{1}{2})\beta} \int_{N^{-\beta}}^1 u^{(\frac{1}{\beta}-\ell-\frac{3}{2})} (1-u)^{\ell+\frac{1}{2}}\, du + O(1) \tag{5.28}$$

$$= C^{\ell+\frac{1}{2}} \beta^{-1} N^{1-(\ell+\frac{1}{2})\beta} \int_0^1 u^{(\frac{1}{\beta}-\ell-\frac{3}{2})} (1-u)^{\ell+\frac{1}{2}}\, du + O(1) \tag{5.29}$$

$$= C^{\ell+\frac{1}{2}} \beta^{-1} \frac{\Gamma(\ell+\frac{3}{2})\Gamma(\frac{1}{\beta}-\frac{1}{2}-\ell)}{\Gamma(\frac{1}{\beta}+1)} N^{1-(\ell+\frac{1}{2})\beta} + O(1)$$

In the above, (5.27) comes from the fact that the function in the integrand is monotone decreasing, and if $f(x)$ is monotone, then

$$f(j) \geq \int_j^{j+1} f(y)\, dy \geq f(j+1)$$

so

$$\sum_{j=1}^{N-1} f(j) \geq \int_1^N f(y)\, dy \geq \sum_{j=2}^{N} f(j)$$

and

$$\left| \int_1^N f(y)\, dy - \sum_{j=1}^{N} f(j) \right| \leq f(1) \tag{5.30}$$

(5.28) is the change of variables $u = (\frac{x}{N})^\beta$. Finally, (5.29) comes from the same cancellation that occurred in (5.13).

Since $|N - C^{\frac{1}{\beta}} \delta^{-\frac{1}{\beta}}| \leq 1$ and $0 < 1 - (\ell + \frac{1}{2})\beta < 1$,

$$N^{1-(\ell+\frac{1}{2})\beta} = (C^{\frac{1}{\beta}} \delta^{-\frac{1}{\beta}})^{1-(\ell+\frac{1}{2})\beta} + o(1) \tag{5.31}$$

Thus, we find

$$Q(x) = \beta^{-1} C^{\frac{1}{\beta}} \sum_{0 \leq \ell < (\frac{1}{\beta}-\frac{1}{2})} c_\ell \frac{\Gamma(\ell+\frac{3}{2})\Gamma(\frac{1}{\beta}-\frac{1}{2}-\ell)}{\Gamma(\frac{1}{\beta}+1)} \delta^{-\frac{1}{\beta}+\ell+\frac{1}{2}} + O(\log \delta) \tag{5.32}$$

If $C > 1$, we should not expand the power series of $\cosh^{-1}$ for small j (actually, as noted, the power series has radius of convergence 4 so we need only worry if $C \geq 4$). Instead, we do not expand for those j with $Cj^{-\beta} > 1$. That is only finitely many terms, so it adds $O(1)$ errors to $\sum_1^N \gamma_j(x)$. We add back these small j terms to (5.25), again making $O(1)$ errors. The final result does not change. □

Finally, we will explore examples that lead to Q's roughly of the type (1.11) to link to work of Levin–Lubinsky [18]. We suppose

$$a_n = 1 - f(\log(n+1)) \tag{5.33}$$

where the f's we have in mind are typically

$$f(x) = (1+x)^{-\alpha} \tag{5.34}$$

or

$$f(x) = \log_k(x + c_k) \tag{5.35}$$

an iterated log (where c_k is chosen to keep all log's that enter positive). We will need

Proposition 5.4. *Let f be defined and C^2 on $[\log 2, \infty)$ and obey*

$$\text{(i)} \qquad f(x) > 0, \quad f'(x) < 0, \quad f''(x) > 0 \tag{5.36}$$

$$\text{(ii)} \qquad \lim_{n\to\infty} f(n) = 0 \tag{5.37}$$

$$\text{(iii)} \qquad \lim_{N\to\infty} N^{\varepsilon}(-f'(\log N))^{\frac{1}{2}} = \infty \tag{5.38}$$

$$\text{(iv)} \qquad \lim_{\varepsilon\downarrow 0}\left(\limsup_{k\to\infty} \frac{-f'((1-\varepsilon)k)}{-f'(k)}\right) = 1 \tag{5.39}$$

Let

$$S_N = \sum_{j=2}^{N} \sqrt{f(\log j) - f(\log N)} \tag{5.40}$$

Then

$$\lim_{N\to\infty} \frac{S_N}{N(-f'(\log N))^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{2} \tag{5.41}$$

Remark. It is easy to see that if $f(x) = e^{-kx}$ (i.e., $f(\log(n+1)) \sim (n+1)^{-k}$), then (5.41) fails. In this case, both (5.38) and (5.39) fail, but they hold for the f's of (5.34) and (5.35).

Proof. Since $(-f')' < 0$ and if $x < y$,

$$f(x) - f(y) = \int_x^y (-f'(s))\, ds \tag{5.42}$$

we have,

$$(y-x)(-f'(y)) \le f(x) - f(y) \le (y-x)(-f'(x)) \tag{5.43}$$

We thus get a lower bound

$$f(\log j) - f(\log N) \ge (-f'(\log N))(-\log(\tfrac{j}{N})) \tag{5.44}$$

so

$$S_N \ge N(-f'(\log N))^{\frac{1}{2}} \sum_{j=2}^{N} \tfrac{1}{N}(-\log(\tfrac{j}{N}))^{\frac{1}{2}} \tag{5.45}$$

As $N \to \infty$, the sum converges to $\int_0^1 (-\log(x))^{\frac{1}{2}}\, dx = \frac{\sqrt{\pi}}{2}$ (courtesy of Mathematica). Thus,

$$\liminf(\text{LHS of } (5.41)) \ge \frac{\sqrt{\pi}}{2} \tag{5.46}$$

For the upper bound, fix $\varepsilon > 0$ and break $S_N = S_N^{(1)} + S_N^{(2)}$ where $S_N^{(1)}$ has $j \le N^{1-\varepsilon}$ and $S_N^{(2)}$ has $j > N^{1-\varepsilon}$. Clearly,

$$S_N^{(1)} \le f(\log 2) N^{1-\varepsilon} \tag{5.47}$$

so, by hypothesis (5.38), it contributes 0 to the ratio in (5.41) as $N \to \infty$.

For $S_N^{(2)}$, we use the upper bound when $j > N^{1-\varepsilon}$

$$f(\log j) - f(\log N) \le -f'((1-\varepsilon)\log N)(-\log(\tfrac{j}{N}))$$

which yields (since the Riemann sum still converges to the integral)

$$\limsup(\text{LHS of } (5.41)) \le \frac{\sqrt{\pi}}{2} \limsup_{k\to\infty} \left(\frac{-f'((1-\varepsilon)k)}{-f'(k)} \right)^{\frac{1}{2}}$$

Since ε is arbitrary, we can use (5.39) to complete the proof of (5.41). □

Example 5.5. Let a_n have the form (5.33) where f obeys all the hypotheses of Proposition 5.4. By (3.2) and (3.3), $N(x)$ roughly solves

$$\frac{x}{1 - f(\log(N+1))} = 2 \tag{5.48}$$

namely,

$$N(x) = [\exp(f^{-1}(1 - \tfrac{x}{2}))] - 1 \tag{5.49}$$

For example, if f is (5.34), then

$$N(x) = [\exp((1 - \tfrac{x}{2})^{-\alpha} - 1)] - 1 \tag{5.50}$$

Next, define z by $\frac{x}{2a} = 1 + \frac{z}{2}$, namely,

$$z = \tfrac{x}{a} - 2 \tag{5.51}$$

where $\frac{x}{a} > 2$. Writing $x = 2 - \delta$ and $a = 1 - f$, we see

$$z = -\delta + 2f + O(f^2) + O(f\delta) \tag{5.52}$$

Taking into account that $N(x)$ is such that

$$2f(\log(N+2)) \leq \delta \leq 2f(\log(N+1))$$

and that (5.19) says

$$\cosh^{-1}(\tfrac{x}{2a}) = \sqrt{z} + O(z^{\frac{3}{2}})$$

we see that

$$\gamma_j(x) = \sqrt{2f(\log(j+1)) - \delta} + O(f^{\frac{3}{2}}) + O(f^{\frac{1}{2}}\delta)$$

and thus

$$g(x) = \sum_{j=1}^{N(x)} \gamma_j(x)$$

is asymptotically the same as $\sqrt{2}\,S_N$. Thus,

$$|Q(x) - g(x)| \leq h(x) \tag{5.53}$$

where

$$g(x) = \sqrt{\frac{\pi}{2}}\,N(x)(-f'(\log N(x)))^{\frac{1}{2}}(1+o(1)) \tag{5.54}$$

and

$$h(x) = O(\log N(x)) + O(\log(1 - \tfrac{2}{x}))$$

$N(x)$ is huge, so while $\log N(x) \sim (1 - \frac{x}{2})^{-\alpha}$ in case (5.34), it is still small relative to $g(x)$.

The reader may be puzzled in comparing our results with those of Levin–Lubinsky [18]. They have no $\sqrt{\frac{\pi}{2}}$ and their relations (after making the modifications from $[-1,1]$ to $[-2,2]$) suggest

$$1 - a_n = (\log n)^{-\frac{1}{2}}(1+o(1)) \tag{5.55}$$

should correspond to

$$Q(x) = \exp((1 - \tfrac{x}{2})^{-\alpha}) \tag{5.56}$$

so there is no sign of $(-f'(\log N(x)))^{\frac{1}{2}}$ either.

The mystery is solved by the fact that multiple Q's lead to the same leading asymptotics for a_n. In their scheme, after corrections to move to $[-2,2]$, leading asymptotics for f are given by

$$n = Q(1 - 2(f(n)(1+o(1)))) \tag{5.57}$$

If

$$Q(x) = e^{1/(1-\frac{x}{2})} \tag{5.58}$$

then

$$n = \exp((f(n))^{-1}) \tag{5.59}$$

solved by

$$f(n) = \tfrac{1}{\log n}\,(1 + o(1)) \tag{5.60}$$

Changing (5.58) to

$$Q(x) = \tfrac{\pi}{2}(1 - \tfrac{x}{2}) \exp((1 - \tfrac{x}{2})^{-1})$$

is solved by

$$f(n) = 1/(\log(\tfrac{2n}{\pi} \log n) + O(\log\log n))$$

Since

$$\log \tfrac{2n}{\pi}\, \log n = \log n + \log_2 n + \log(\tfrac{2}{\pi})$$

(5.60) still holds! □

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