SCHRÖDINGER OPERATORS WITH PURELY DISCRETE SPECTRUM

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Dedicated to A. Ya. Povzner

ABSTRACT. We prove $-\Delta+V$ has purely discrete spectrum if $V \geq 0$ and, for all M, $|\{x \mid V(x) < M\}| < \infty$ and various extensions.

1. Introduction

Our main goal in this note is to explore one aspect of the study of Schrödinger operators

$$H = -\Delta + V \tag{1.1}$$

which we'll suppose have V's which are nonnegative and in $L^1_{\text{loc}}(\mathbb{R}^\nu)$, in which case (see, e.g., Simon [15]) H can be defined as a form sum. We're interested here in criteria under which H has purely discrete spectrum, that is, $\sigma_{\text{ess}}(H)$ is empty. This is well known to be equivalent to proving $(H+1)^{-1}$ or e^{-sH} for any (and so all) $s > 0$ is compact (see [9, Thm. XIII.16]). One of the most celebrated elementary results on Schrödinger operators is that this is true if

$$\lim_{|x|\to\infty} V(x) = \infty \tag{1.2}$$

But (1.2) is not necessary. Simple examples where (1.2) fails but H still has compact resolvent were noted first by Rellich [10]—one of the most celebrated examples is in $\nu = 2$, $x = (x_1, x_2)$, and

$$V(x_1, x_2) = x_1^2 x_2^2 \tag{1.3}$$

where (1.2) fails in a neighborhood of the axes. For proof of this and discussions of eigenvalue asymptotics, see [11, 16, 17, 20, 21].

Date: September 4, 2008.

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To appear in the journal, *Methods of Functional Analysis and Topology*, volume in memory of A. Ya. Povzner.

There are known necessary and sufficient conditions on V for discrete spectrum in terms of capacities of certain sets (see, e.g., Maz'ya [6]), but the criteria are not always so easy to check. Thus, I was struck by the following simple and elegant theorem:

Theorem 1. *Define*

$$\Omega_M(V) = \{x \mid 0 \leq V(x) < M\} \tag{1.4}$$

If (*with* $|\cdot|$ *Lebesgue measure*)

$$|\Omega_M(V)| < \infty \tag{1.5}$$

for all M*, then* H *has purely discrete spectrum.*

I learned of this result from Wang–Wu [25], but there is much related work. I found an elementary proof of Theorem 1 and decided to write it up as a suitable tribute and appreciation of A. Ya. Povzner, whose work on continuum eigenfunction expansions for Schrödinger operators in scattering situation [7] was seminal and inspired me as a graduate student forty years ago!

The proof has a natural abstraction:

Theorem 2. *Let* μ *be a measure on a locally compact space,* X *with* $L^2(X, d\mu)$ *separable. Let* L_0 *be a selfadjoint operator on* $L^2(X, d\mu)$ *so that its semigroup is ultracontractive* ([1])*: For some* $s > 0$*,* e^{-sL_0} *maps* L^2 *to* $L^\infty(X, d\mu)$*. Suppose* V *is a nonnegative multiplication operator so that*

$$\mu(\{x \mid 0 \leq V(x) < M\}) < \infty \tag{1.6}$$

for all M*. Then* $L = L_0 + V$ *has purely discrete spectrum.*

Remark. By $L_0 + V$, we mean the operator obtained by applying the monotone convergence theorem for forms (see, e.g., [13, 14]) to $L_0 + \min(V(x), k)$ as $k \to \infty$.

The reader may have noticed that (1.3) does not obey Theorem 1 (but, e.g., $V(x_1, x_2) = x_1^2 x_2^4 + x_1^4 x_2^2$ does). But out proof can be modified to a result that does include (1.3). Given a set Ω in $\mathbb{R}^\nu$, define for any x and any $\ell > 0$,

$$\omega_x^\ell(\Omega) = |\Omega \cap \{y \mid |y - x| \leq \ell\}| \tag{1.7}$$

For example, for (1.3), for $x \in \Omega_M$,

$$\omega_x^\ell(\Omega_M) \leq \frac{C_\ell}{|x| + 1} \tag{1.8}$$

We will say a set Ω is r-polynomially thin if

$$\int_{x \in \Omega} \omega_x^\ell(\Omega)^r \, d^\nu x < \infty \tag{1.9}$$

for all ℓ. For the example in (1.3), Ω_M is r-polynomially thin for any M and any $r > 0$. We'll prove

Theorem 3. *Let V be a nonnegative potential so that for any M, there is an $r > 0$ so that Ω_M is r-polynomially thin. Then H has purely discrete spectrum.*

As mentioned, this covers the example in (1.3). It is not hard to see that if $P(x)$ is any polynomial in $x_1, \dots, x_\nu$ so that for no $v \in \mathbb{R}^\nu$ is $\vec{v} \cdot \vec{\nabla} P \equiv 0$ (i.e., P isn't a function of fewer than ν linear variables), then $V(x) = P(x)^2$ obeys the hypotheses of Theorem 3.

In Section 2, we'll present a simple compactness criterion on which all theorems rely. In Section 3, we'll prove Theorems 1 and 2. In Section 4, we'll prove Theorem 3.

It is a pleasure to thank Peter Stollmann for useful correspondence and Ehud de Shalit for the hospitality of Hebrew University where some of the work presented here was done.

2. Segal's Lemma

Segal [12] proved the following result, sometimes called Segal's lemma:

Proposition 2.1. *For A, B positive selfadjoint operators,*

$$\|e^{-(A+B)}\| \le \|e^{-A} e^{-B}\| \tag{2.1}$$

Remarks. 1. $A + B$ can always be defined as a closed quadratic form on $Q(A) \cap Q(B)$. That defines $e^{-(A+B)}$ on $\overline{Q(A) \cap Q(B)}$ and we set it to 0 on the orthogonal complement. Since the Trotter product formula is known in this generality (see Kato [5]), (2.1) holds in that generality.

2. Since $\|C^*C\| = \|C\|^2$, $\|e^{-A/2} e^{-B/2}\|^2 = \|e^{-B/2} e^{-A} e^{-B/2}\|$, and since $\|e^{-(A+B)/2}\|^2 = \|e^{-(A+B)}\|$, (2.1) is equivalent to

$$\|e^{-A+B}\| \le \|e^{-B/2} e^{-A} e^{-B/2}\| \tag{2.2}$$

which is the way Segal [12] stated it.

3. Somewhat earlier, Golden [4] and Thompson [22] proved

$$\mathrm{Tr}(e^{-(A+B)}) \le \mathrm{Tr}(e^{-A} e^{-B}) \tag{2.3}$$

and Thompson [23] later extended this to any symmetrically normed operator ideal.

Proof. There are many; see, for example, Simon [18, 19]. Here is the simplest, due to Deift [2, 3]: If σ is the spectrum of an operator

$$\sigma(CD) \setminus \{0\} = \sigma(DC) \setminus \{0\} \tag{2.4}$$

so with σ_r the spectral radius,

$$\sigma_r(CD) = \sigma_r(DC) \le \|DC\| \tag{2.5}$$

If CD is selfadjoint, $\sigma_r(CD) = \|CD\|$, so

$$CD \text{ selfadjoint} \Rightarrow \|CD\| \le \|DC\| \tag{2.6}$$

Thus,

$$\|e^{-A/2}e^{-B/2}\|^2 = \|e^{-B/2}e^{-A}e^{-B/2}\| \le \|e^{-A}e^{-B}\| \tag{2.7}$$

By induction,

$$\|(e^{-A/2^n}e^{-B/2^n})^{2^n}\| \le \|e^{-A/2^n}e^{-B/2^n}\|^{2n} \le \|e^{-A}e^{-B}\| \tag{2.8}$$

Take $n \to \infty$ and use the Trotter product formula to get (2.1). □

In [18], I noted that this implies for any symmetrically normed trace ideal, $\mathcal{I}_\Phi$, that

$$e^{-A/2}e^{-B}e^{-A/2} \in \mathcal{I}_\Phi \Rightarrow e^{-(A+B)} \in \mathcal{I}_\Phi \tag{2.9}$$

I explicitly excluded the case $\mathcal{I}_\Phi = \mathcal{I}_\infty$ (the compact operators) because the argument there doesn't show that, but it is true—and the key to this paper!

Since $C \in \mathcal{I}_\infty \Leftrightarrow C^*C \in \mathcal{I}_\infty$ and $e^{-(A+B)} \in \mathcal{I}_\infty$ if and only if $e^{-\frac{1}{2}(A+B)} \in \mathcal{I}_\infty$, it doesn't matter if we use the symmetric form (2.2) or the following asymmetric form which is more convenient in applications.

Theorem 2.2. *Let $\mathcal{I}_\infty$ be the ideal of compact operators on some Hilbert space, $\mathcal{H}$. Let A, B be nonnegative selfadjoint operators. Then*

$$e^{-A}e^{-B} \in \mathcal{I}_\infty \Rightarrow e^{-(A+B)} \in \mathcal{I}_\infty \tag{2.10}$$

Proof. For any bounded operator, C, define $\mu_n(C)$ by

$$\mu_n(C) = \min_{\psi_1 \dots \psi_{n-1}} \sup_{\substack{\|\varphi\|=1 \\ \varphi \perp \psi_1, \dots, \psi_{n-1}}} \|C\varphi\| \tag{2.11}$$

By the min-max principle (see [9, Sect. XIII.1]),

$$\lim_{n\to\infty} \mu_n(C) = \sup(\sigma_{\text{ess}}(|C|)) \tag{2.12}$$

and $\mu_n(C)$ are the singular values if $C \in \mathcal{I}_\infty$. In particular,

$$C \in \mathcal{I}_\infty \Leftrightarrow \lim_{n\to\infty} \mu_n(C) = 0 \tag{2.13}$$

Let $\wedge^\ell(\mathcal{H})$ be the antisymmetric tensor product (see [8, Sects. II.4, VIII.10], [9, Sect. XIII.17], and [18, Sect. 1.5]). As usual (see [18, eqn. (1.14)]),

$$\|\wedge^m(C)\| = \prod_{j=1}^m \mu_j(C) \tag{2.14}$$

Since $\mu_1 \geq \mu_2 \geq \cdots \geq 0$, we have

$$\lim_{n\to\infty} \mu_n(C) = \lim_{n\to\infty} (\mu_1(C)\dots\mu_n(C))^{1/n} \tag{2.15}$$

(2.13)–(2.15) imply

$$C \in \mathcal{I}_\infty \Leftrightarrow \lim_{n\to\infty} \|\wedge^n(C)\|^{1/n} = 0 \tag{2.16}$$

As usual, there is a selfadjoint operator, $d\wedge^n(A)$ on $\wedge^n(\mathcal{H})$ so

$$\wedge^n(e^{-tA}) = e^{-t\,d\wedge^n(A)} \tag{2.17}$$

so Segal's lemma implies that

$$\begin{aligned}\|\wedge^n(e^{-(A+B)})\| &\leq \|\wedge^n(e^{-A})\wedge^n(e^{-B})\| \\ &= \|\wedge^n(e^{-A}e^{-B})\|\end{aligned} \tag{2.18}$$

Thus,

$$\lim_{n\to\infty} \|\wedge^n(e^{-(A+B)})\|^{1/n} \leq \lim_{n\to\infty} \|\wedge^n(e^{-A}e^{-B})\|^{1/n} \tag{2.19}$$

By (2.16), we obtain (2.10). □

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. By Theorem 2.2, we need only show $C = e^{\Delta}e^{-V}$ is compact. Write

$$C = C_m + D_m \tag{3.1}$$

where

$$C_m = C\chi_{\Omega_m} \qquad D_m = C\chi_{\Omega_m^c} \tag{3.2}$$

with χ_S the operator of multiplication by the characteristic function of a set $S \subset \mathbb{R}^\nu$.

$$\|e^{-V}\chi_{\Omega_m^c}\|_\infty \leq e^{-m}$$

and $\|e^{\Delta}\| = 1$, so

$$\|D_m\| \leq e^{-m} \tag{3.3}$$

and thus,

$$\lim_{m\to\infty} \|C - C_m\| = 0 \tag{3.4}$$

If we show each C_m is compact, we are done. We know e^Δ has integral kernel $f(x-y)$ with f a Gaussian, so in L^2. Clearly, since V is positive, C_m has an integral kernel $C_m(x,y)$ dominated by

$$|C_m(x,y)| \le f(x-y)\chi_{\Omega_m}(y) \tag{3.5}$$

Thus,

$$\int |C_m(x,y)|^2\, d^\nu x d^\nu y \le \|f\|^2_{L^2(\mathbb{R}^\nu)} \|\chi_{\Omega_m}\|_{L^2(\mathbb{R}_\nu)} < \infty$$

since $|\Omega_m| < \infty$. Thus, C_m is Hilbert–Schmidt, so compact. □

Proof of Theorem 2. We can follow the proof of Theorem 1. It suffices to prove that $e^{-sL_0}e^{-sV}$ is compact, and so, that $e^{-sL_0}\chi_{\Omega_m}$ is Hilbert–Schmidt.

That e^{-sL_0} maps L^2 to L^∞ implies, by the Dunford–Pettis theorem (see [24, Thm. 46.1]), that there is, for each $x \in X$, a function $f_x(\,\cdot\,) \in L^2(X, d\mu)$ with

$$(e^{-sL_0}g)(x) = \langle f_x, g\rangle \tag{3.6}$$

and

$$\sup_x \|f_x\|_{L^2} = \|e^{-sL_0}\|_{L^2\to L^\infty} \equiv C < \infty \tag{3.7}$$

Thus, e^{-sL_0} has an integral kernel $K(x,y)$ with

$$\sup_x \int |K(x,y)|^2\, d\mu(y) = C < \infty \tag{3.8}$$

(for $K(x,y) = f_x(y)$). But e^{-sL_0} is selfadjoint, so its kernel is complex symmetric, so

$$\sup_y \int |K(x,y)|^2\, d\mu(x) = C < \infty \tag{3.9}$$

Thus,

$$\int |K(x,y)\chi_{\Omega_m}(y)|^2\, d\mu(x)d\mu(y) \le C\mu(\Omega_m) < \infty \tag{3.10}$$

and $e^{-sL_0}\chi_{\Omega_m}$ is Hilbert–Schmidt. □

4. Proof of Theorem 3

As with the proof of Theorem 1, it suffices to prove that for each M, $e^\Delta\chi_{\Omega_M}$ is compact. e^Δ is convolution with an L^1 function, f. Let Q_R be the characteristic function of $\{x \mid |x| < R\}$. Let F_R be convolution with fQ_R. Then

$$\|e^\Delta - F_R\| \le \|f(1-Q_R)\|_1 \to 0 \tag{4.1}$$

as $R \to \infty$, so

$$\|e^\Delta\chi_{\Omega_M} - F_R\chi_{\Omega_M}\| \to 0 \tag{4.2}$$

and it suffices to prove for each R, M,

$$C_{M,R} = F_R \chi_{\Omega_M} \tag{4.3}$$

is compact. Clearly, this works if we show for some k, $(C_{M,R}^* C_{M,R})^k$ is Hilbert–Schmidt.

Let D be the operator with integral kernel

$$D(x,y) = \chi_{\Omega_M}(x) Q_{2R}(x-y) \chi_{\Omega_M}(y) \tag{4.4}$$

Since f is bounded, it is easy to see that

$$(C_{M,R}^* C_{M,R})(x,y) \leq cD(x,y) \tag{4.5}$$

for some constant c, so it suffices to show D^k is Hilbert–Schmidt.

D^k has integral kernel

$$D^k(x,y) = \int D(x,x_1) D(x_1,x_2) \dots D(x_{k-1},y)\, dx_1 \dots, dx_{k-1} \tag{4.6}$$

Fix y. This integral is zero unless $|x - x_1| < 2R, \dots |x_{k-1} - y| < 2R$, so, in particular, unless $|x-y| \leq 2kR$. Moreover, the integrand can certainly be restricted to the regions $|x_j - y| \leq 2kR$. Thus,

$$D^k(x,y) \leq Q_{2kR}(x-y) \biggl(\int_{|x_j - y| \leq 2kR} \prod_{j=1}^{k-1} \chi_{\Omega_M}(x_j)\, dx_1 \dots dx_{k-1} \biggr) \chi_{\Omega_m}(y) \tag{4.7}$$

$$= Q_{2kR}(x-y) (\omega_y^{2kR}(\Omega_M)^{k-1}) \chi_{\Omega_M}(y) \tag{4.8}$$

by the definition of ω_x^ℓ in (1.7).

Thus,

$$\int |D^k(x,y)|^2\, d^\nu x d^\nu y \leq C(kR)^\nu \int_{x \in \Omega} [\omega_x^{2kR}(\Omega_M)]^{2k-2}\, d^\nu x$$

so if $2k - 2 > r$ and (1.9) holds, D^k is Hilbert–Schmidt. □

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