FINITE GAP JACOBI MATRICES, III. BEYOND THE SZEGŐ CLASS

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Abstract. Let $\mathfrak{e} \subset \mathbb{R}$ be a finite union of $\ell + 1$ disjoint closed intervals and denote by ω_j the harmonic measure of the j leftmost bands. The frequency module for $\mathfrak{e}$ is the set of all integral combinations of $\omega_1, \dots, \omega_\ell$. Let $\{\tilde{a}_n, \tilde{b}_n\}_{n=1}^\infty$ be a point in the isospectral torus for $\mathfrak{e}$ and $\tilde{p}_n$ its orthogonal polynomials. Let $\{a_n, b_n\}_{n=1}^\infty$ be a half-line Jacobi matrix with $a_n = \tilde{a}_n + \delta a_n$, $b_n = \tilde{b}_n + \delta b_n$. Suppose

$$\sum_{n=1}^\infty |\delta a_n|^2 + |\delta b_n|^2 < \infty$$

and $\sum_{n=1}^N e^{2\pi i\omega n}\delta a_n$, $\sum_{n=1}^N e^{2\pi i\omega n}\delta b_n$ have finite limits as $N \to \infty$ for all ω in the frequency module. If, in addition, these partial sums grow at most subexponentially with respect to ω, then for $z \in \mathbb{C} \setminus \mathbb{R}$, $p_n(z)/\tilde{p}_n(z)$ has a limit as $n \to \infty$. Moreover, we show that there are non-Szegő class J's for which this holds.

1. Introduction

Let $\mathfrak{e}$ in $\mathbb{R}$ be a compact set with $\ell + 1$ intervals, that is,

$$\mathfrak{e} = \bigcup_{j=1}^{\ell+1} [\alpha_j, \beta_j], \qquad \alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_{\ell+1} \tag{1.1}$$

ℓ counts the number of gaps, that is, bounded intervals in $\mathbb{R} \setminus \mathfrak{e}$. In this paper, we continue our study of Jacobi matrices, J, and the asymptotics

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of the associated orthogonal polynomials on the real line (OPRL) when the essential support of the spectral measure is $\mathfrak{e}$. In paper I [1], we discussed the isospectral torus, $\mathcal{T}_\mathfrak{e}$, associated to $\mathfrak{e}$, a family of two-sided almost periodic Jacobi matrices associated to $\mathfrak{e}$. In paper II [2], we found equivalences among spectral conditions, one of which was a Szegő condition, and proved Szegő asymptotics in this case. To explain our goal in this paper, let us recall the case $\ell = 0$, that is, a single band.

We thus consider Jacobi parameters $\{a_n, b_n\}_{n=1}^\infty$ and define

$$\delta b_n = b_n, \qquad \delta a_n = a_n - 1 \tag{1.2}$$

(our free Jacobi matrix in this case has $\tilde{a}_n \equiv 1$, $\tilde{b}_n \equiv 0$). Recall that for this $\{\tilde{a}_n, \tilde{b}_n\}_{n=1}^\infty$ and $n \geq 0$,

$$\widetilde{P}_n(x) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}\,, \qquad z(x) = \frac{x + \sqrt{x^2 - 4}}{2} \tag{1.3}$$

The following is a result of Damanik–Simon [5].

Theorem 1.1. *Let $p_n(x)$ be the orthogonal polynomials associated to Jacobi parameters $\{a_n, b_n\}_{n=1}^\infty$ with $a_n \to 1$, $b_n \to 0$. Then the following are equivalent:*

(i) $\lim_{n\to\infty} p_n(x)/\tilde{p}_n(x)$ *exists for all $x \in \mathbb{C} \setminus \mathbb{R}$ with convergence uniform on compact subsets.*

(ii) *The Jacobi parameters obey*
  (a) $\sum_{n=1}^\infty |a_n - 1|^2 + |b_n|^2 < \infty$
  (b) $\sum_{n=1}^N (a_n - 1)$ *and* $\sum_{n=1}^N b_n$ *have limits in* $(-\infty, \infty)$.

Damanik–Simon [5] also have examples where (i) and (ii) hold, but the Szegő condition fails. They call (i) $\Rightarrow$ (ii) "the easy direction" since it follows in a few lines from an analysis of the first few Taylor coefficients at infinity for the m-functions. A key is the same miracle that makes Killip–Simon [9] work—that for reasons we don't understand, a combination of the zeroth and second Taylor coefficients is nonnegative. (ii) $\Rightarrow$ (i) is called "the hard direction" and [5] provides three proofs.

In this paper, our main goal is to prove a result akin to the hard direction of Theorem 1.1 for perturbations of elements of $\mathcal{T}_\mathfrak{e}$ for some finite gap set, $\mathfrak{e}$. Given $\mathfrak{e}$, let $d\rho_\mathfrak{e}$ be the potential theoretic equilibrium measure for $\mathfrak{e}$. Let $\omega_j = \rho_\mathfrak{e}([\alpha_1, \beta_j])$ for $j = 1, \dots, \ell$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_\ell)$. The frequency module, $\mathcal{M}(\mathfrak{e})$, for $\mathfrak{e}$ is the set of all numbers of the form

$$\mathbf{k} \cdot \boldsymbol{\omega} = \sum_{j=1}^{\ell} k_j \omega_j \tag{1.4}$$

for $\mathbf{k} = (k_1, \dots, k_\ell) \in \mathbb{Z}^\ell$. We'll only care about $e^{2\pi i(\mathbf{k}\cdot\boldsymbol{\omega})n}$, $n \in \mathbb{Z}$, so only about $(\mathbf{k}\cdot\boldsymbol{\omega})$ mod 1. Thus, the frequency module is essentially a subgroup of $\mathbb{R}/\mathbb{Z}$ with at most ℓ generators. In the periodic case, each ω_j is of the form m_j/p, where p is the period and there is no simpler common denominator. In that case, in $\mathbb{R}/\mathbb{Z}$, $\mathcal{M}(\mathfrak{e})$ has p elements $\{m/p\}_{m=0}^{p-1}$. If some ω_j is irrational, $\mathcal{M}(\mathfrak{e})$ is infinite. Here is our main theorem in this paper:

Theorem 1.2. *Let $\{\tilde a_n, \tilde b_n\}_{n=-\infty}^\infty$ be an element of the isospectral torus, $\mathcal{T}_\mathfrak{e}$, of a finite gap set, $\mathfrak{e}$. Let $\{a_n, b_n\}_{n=1}^\infty$ be another set of Jacobi parameters and $\delta a_n, \delta b_n$ given by*

$$\delta a_n = a_n - \tilde a_n, \qquad \delta b_n = b_n - \tilde b_n \tag{1.5}$$

Suppose that

(a)

$$\sum_{n=1}^\infty |\delta a_n|^2 + |\delta b_n|^2 < \infty \tag{1.6}$$

(b) *For any $\mathbf{k} \in \mathbb{Z}^\ell$,*

$$\sum_{n=1}^N e^{2\pi i(\mathbf{k}\cdot\boldsymbol{\omega})n}\delta a_n \quad \text{and} \quad \sum_{n=1}^N e^{2\pi i(\mathbf{k}\cdot\boldsymbol{\omega})n}\delta b_n \tag{1.7}$$

have (finite) limits as $N \to \infty$.

(c) *For every $\varepsilon > 0$,*

$$\sup_N \left\{ \left| \sum_{n=1}^N e^{2\pi i(\mathbf{k}\cdot\boldsymbol{\omega})n}\delta a_n \right| + \left| \sum_{n=1}^N e^{2\pi i(\mathbf{k}\cdot\boldsymbol{\omega})n}\delta b_n \right| \right\} \le C_\varepsilon \exp(\varepsilon|\mathbf{k}|) \tag{1.8}$$

Let p_n (resp. $\tilde p_n$) be the orthonormal polynomials for $\{a_n, b_n\}_{n=1}^\infty$ (resp. $\{\tilde a_n, \tilde b_n\}_{n=1}^\infty$). Then for any $x \in \mathbb{C}\setminus\mathbb{R}$,

$$\lim_{n\to\infty} \frac{p_n(x)}{\tilde p_n(x)} \tag{1.9}$$

exists and is finite and nonzero.

Thus, we have Szegő asymptotics. We do not have the converse result (i.e., what [5] calls the easy direction) in part because one does not even have an analog of the Killip–Simon theorem [9] except in the case where the frequency module is $\{0, 1/p, 2/p, \dots, (p-1)/p\}$ [4]. We suspect the converse is true and think that even Szegő asymptotics $\Rightarrow$ condition (b) would be interesting.

Another interesting object is the absolutely continuous spectrum. We believe that under the assumptions of Theorem 1.2 it fills out the entire set $\mathfrak{e}$. In the single interval case, it was a consequence of the Killip–Simon theorem [9]. Using the ideas of Damanik–Simon [5], one may attempt to control the Jost solution on the spectrum to show that the a.c. part of the spectral measure is supported on the finite gap set $\mathfrak{e}$. We will explore this idea elsewhere.

An example where the hypotheses of Theorem 1.2 hold but the Szegő condition fails is

$$\delta a_n = 0 \quad \text{and} \quad \delta b_n = \frac{1}{n^\alpha} \cos(2\pi\sqrt{n}) \tag{1.10}$$

with $\alpha \in (\frac{3}{4}, 1)$. We refer to Section 3 for details as well as an additional example.

Remark. In the periodic case the assumptions of Theorem 1.2 become simpler. In particular, (c) is automatic for it follows from (b) since the frequency module has only finitely many elements. Moreover, in the period p case (b) can be replaced by requiring

$$\sum_{n=0}^{N} \delta a_{np+j} \quad \text{and} \quad \sum_{n=0}^{N} \delta b_{np+j} \tag{1.11}$$

to have (finite) limits as $N \to \infty$ for each $j = 1, \dots, p$.

2. Szegő Asymptotics

In this section we present a proof of Theorem 1.2 using a transfer matrix approach combined with a theorem of Coffman [3]. The latter is a discrete version of an ODE result of Hartman–Wintner [8] and reads:

Theorem 2.1 ([3]). *Let Λ be a $d \times d$ diagonal invertible matrix with entries $\lambda_1, \dots, \lambda_d$ along the diagonal and let $\{A_n\}$ be a sequence of $d \times d$ matrices such that*

$$\sum_{n=1}^{\infty} \|A_n\|^2 < \infty \tag{2.1}$$

$$\Lambda + A_n \quad \textit{is invertible for all } n \tag{2.2}$$

Consider solutions $\vec{y}_n = (y_{n,1}, \dots, y_{n,d}) \in \mathbb{C}^d$ of

$$\vec{y}_{n+1} = (\Lambda + A_n)\vec{y}_n \tag{2.3}$$

with some initial condition $\vec{y}_1$. Suppose λ_j is a simple eigenvalue of Λ with $|\lambda_k| \neq |\lambda_j|$ for all $k \neq j$ and let

$$\gamma_j(n) = \prod_{k=n_0}^{n-1} (\lambda_j + (A_k)_{j,j}) \tag{2.4}$$

where $n_0 \geq 1$ *is so large that* $\gamma_j(n) \neq 0$ *for all* $n > n_0$. *Then there exists an initial condition* $\vec{y}_1$ *so that*

$$\lim_{n\to\infty} \frac{y_{n,j}}{\gamma_j(n)} = 1 \tag{2.5}$$

while for $k \neq j$,

$$\lim_{n\to\infty} \frac{y_{n,k}}{\gamma_j(n)} = 0 \tag{2.6}$$

Under the additional assumption of conditional summability of the perturbation, we get the following corollary of Coffman's result:

Corollary 2.2. *With* Λ *and* A_n *as above, suppose that* $|\lambda_1| > \cdots > |\lambda_d| > 0$ *and*

$$\lim_{N\to\infty} \sum_{n=1}^{N} (A_n)_{j,j} \tag{2.7}$$

exists for every j. *Then for any initial condition* $\vec{y}_1 \neq \mathbf{0}$, *there exists* j *such that the solution of* (2.3) *obeys*

$$\lim_{n\to\infty} \frac{y_{n,j}}{\lambda_j^n} = c_j \neq 0 \tag{2.8}$$

while for $k \neq j$,

$$\lim_{n\to\infty} \frac{y_{n,k}}{\lambda_j^n} = 0 \tag{2.9}$$

Proof. Recall that if $\sum_k |a_k|^2 < \infty$ and $\sum_k a_k$ is conditionally convergent then

$$\sum_k |\log(1+a_k) - a_k| < \infty \tag{2.10}$$

and hence $\prod_k (1+a_k)$ converges. Thus, it follows from (2.1), (2.7), and (2.4) that $\gamma_j(n)/\lambda_j^n$ has a finite nonzero limit, say c_j, as $n \to \infty$.

By Theorem 2.1, there is for every j a solution $\vec{y}_n(j)$ of (2.4) such that

$$\frac{\vec{y}_n(j)}{\gamma_j(n)} \to \vec{e}_j \ \text{ as } \ n \to \infty \tag{2.11}$$

where $\{\vec{e}_1, \dots, \vec{e}_d\}$ is the standard basis for $\mathbb{C}^d$. Hence, $\vec{y}_n(j)/\lambda_j^n$ converges to $c_j \vec{e}_j$ as $n \to \infty$.

Note that the vectors $\vec{y}_n(1), \dots, \vec{y}_n(d)$ are linearly independent for all n and form a basis of $\mathbb{C}^d$. Since $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_d| > 0$, it follows that (2.8) holds for any solution $\vec{y}_n$ of (2.3). For if $\vec{y}_n = \sum_{j=1}^d r_j \vec{y}_n(j)$, then

$$\frac{\vec{y}_n}{\lambda_k^n} = \sum_{j=1}^d \frac{r_j \vec{y}_n(j)}{\lambda_j^n} \frac{\lambda_j^n}{\lambda_k^n} \to r_k c_k \vec{e}_k \text{ as } n \to \infty \tag{2.12}$$

where k is the smallest value of j for which $r_j \neq 0$. □

In what follows, $\mathbf{x}(z)$ will denote the universal covering map of $\mathbb{D}$ onto $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ as introduced in [1, Sect. 2]. It is the unique meromorphic map which is locally one-to-one with

$$\mathbf{x}(z) = \frac{x_\infty}{z} + \mathcal{O}(1) \tag{2.13}$$

near $z = 0$ and $x_\infty > 0$. Following [1, Sect. 4], we let $B(z)$ be the Blaschke product with zeros at the poles of $\mathbf{x}$. For every measure in the Szegő class for $\mathfrak{e}$, one can define a Jost solution (cf. [1, Sect. 9]). While the parameters $\{a_n, b_n\}_{n=1}^\infty$ from Theorem 1.2 may not correspond to a measure in the Szegő class, every point $\{\tilde{a}_n, \tilde{b}_n\}_{n=-\infty}^\infty$ in the isospectral torus does and its Jost solution, $\tilde{u}_n(z)$, satisfies the same three-term recurrence relation as $\tilde{p}_{n-1}(\mathbf{x}(z))$. We shall henceforth fix $z \in \mathbb{D}$ such that $\mathbf{x}(z) \in \mathbb{C} \setminus \mathbb{R}$.

Let us begin by writing the three-term recurrence relation for the orthonormal polynomials p_n in matrix form

$$\begin{pmatrix} p_n(x) \\ a_n p_{n-1}(x) \end{pmatrix} = \frac{1}{a_n} \begin{pmatrix} x - b_n & -1 \\ a_n^2 & 0 \end{pmatrix} \begin{pmatrix} p_{n-1}(x) \\ a_{n-1} p_{n-2}(x) \end{pmatrix} \tag{2.14}$$

Similarly,

$$\begin{aligned} &\begin{pmatrix} \tilde{p}_n(\mathbf{x}(z)) & \tilde{u}_{n+1}(z) \\ \tilde{a}_n \tilde{p}_{n-1}(\mathbf{x}(z)) & \tilde{a}_n \tilde{u}_n(z) \end{pmatrix} \\ &\quad = \frac{1}{\tilde{a}_n} \begin{pmatrix} \mathbf{x}(z) - \tilde{b}_n & -1 \\ \tilde{a}_n^2 & 0 \end{pmatrix} \begin{pmatrix} \tilde{p}_{n-1}(\mathbf{x}(z)) & \tilde{u}_n(z) \\ \tilde{a}_{n-1} \tilde{p}_{n-2}(\mathbf{x}(z)) & \tilde{a}_{n-1} \tilde{u}_{n-1}(z) \end{pmatrix} \end{aligned} \tag{2.15}$$

To be dealing with bounded entries (for fixed z), we introduce

$$R_n(z) = \begin{pmatrix} \tilde{p}_n(\mathbf{x}(z)) & \tilde{u}_{n+1}(z) \\ \tilde{a}_n \tilde{p}_{n-1}(\mathbf{x}(z)) & \tilde{a}_n \tilde{u}_n(z) \end{pmatrix} \begin{pmatrix} B(z)^n & 0 \\ 0 & B(z)^{-n} \end{pmatrix} \tag{2.16}$$

Indeed, $B^{-n} \tilde{u}_n$ is almost periodic [1, Thm. 9.2] while $B^n \tilde{p}_{n-1}$ is almost periodic up to an exponentially small error [2, Thm. 7.3]. By (2.15), $\det R_n$ is n-independent. Hence,

$$\tilde{a}_n \big(\tilde{u}_n \tilde{p}_n - \tilde{u}_{n+1} \tilde{p}_{n-1} \big) = \det R_n = \det R_0 = \tilde{a}_0 \tilde{u}_0 \neq 0 \tag{2.17}$$

and so all R_n are invertible. Changing the variables in (2.14) to (ϕ_n, ψ_n) defined by

$$\begin{pmatrix} \phi_n(z) \\ \psi_n(z) \end{pmatrix} = R_n(z)^{-1} \begin{pmatrix} p_n(\mathbf{x}(z)) \\ a_n p_{n-1}(\mathbf{x}(z)) \end{pmatrix} \tag{2.18}$$

we get the recursion

$$\begin{pmatrix} \phi_n(z) \\ \psi_n(z) \end{pmatrix} = \frac{R_n(z)^{-1}}{a_n} \begin{pmatrix} \mathbf{x}(z) - b_n & -1 \\ a_n^2 & 0 \end{pmatrix} R_{n-1}(z) \begin{pmatrix} \phi_{n-1}(z) \\ \psi_{n-1}(z) \end{pmatrix} \tag{2.19}$$

It follows from (2.15)–(2.16) that

$$R_n(z)^{-1} = \begin{pmatrix} B(z)^{-1} & 0 \\ 0 & B(z) \end{pmatrix} \frac{R_{n-1}(z)^{-1}}{\tilde{a}_n} \begin{pmatrix} 0 & 1 \\ -\tilde{a}_n^2 & \mathbf{x}(z) - \tilde{b}_n \end{pmatrix} \tag{2.20}$$

and substitution of (2.20) into (2.19) leads to

$$\begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \big(R_{n-1}\big)^{-1} Q_n R_{n-1} \right] \begin{pmatrix} \phi_{n-1} \\ \psi_{n-1} \end{pmatrix} \tag{2.21}$$

where

$$\begin{aligned} Q_n(z) &= \frac{1}{a_n \tilde{a}_n} \begin{pmatrix} 0 & 1 \\ -\tilde{a}_n^2 & \mathbf{x}(z) - \tilde{b}_n \end{pmatrix} \begin{pmatrix} \mathbf{x}(z) - b_n & -1 \\ a_n^2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{a_n \tilde{a}_n} \begin{pmatrix} a_n \delta a_n & 0 \\ (a_n + \tilde{a}_n)(\mathbf{x}(z) - \tilde{b}_n)\delta a_n + \tilde{a}_n^2 \delta b_n & -\tilde{a}_n \delta a_n \end{pmatrix} \end{aligned} \tag{2.22}$$

We wish to apply Corollary 2.2 with

$$\Lambda = \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad A_n = \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} (R_{n-1})^{-1} Q_n R_{n-1} \tag{2.23}$$

A straightforward computation shows that $B \det(R_{n-1})(A_n)_{1,1}$ can be written as

$$\begin{aligned} &\frac{\tilde{p}_{n-1}}{\tilde{a}_n} \Big(\tilde{a}_{n-1} \tilde{u}_{n-1} - \tilde{u}_n \big(\mathbf{x}(z) - \tilde{b}_n \big) \Big) \delta a_n \\ &\qquad + \frac{\tilde{u}_n}{a_n} \Big(\tilde{a}_{n-1} \tilde{p}_{n-2} - \tilde{p}_{n-1} \big(\mathbf{x}(z) - \tilde{b}_n \big) \Big) \delta a_n - \frac{\tilde{a}_n}{a_n} \tilde{u}_n \tilde{p}_{n-1} \delta b_n \\ &\qquad = -\Big(\tilde{u}_{n+1} \tilde{p}_{n-1} + \frac{\tilde{a}_n}{a_n} \tilde{u}_n \tilde{p}_n \Big) \delta a_n - \frac{\tilde{a}_n}{a_n} \tilde{u}_n \tilde{p}_{n-1} \delta b_n \end{aligned} \tag{2.24}$$

so that

$$\begin{aligned} B(A_n)_{1,1} = &\left(\frac{1/\tilde{a}_n}{1 - \tilde{u}_n \tilde{p}_n / (\tilde{u}_{n+1} \tilde{p}_{n-1})} - \frac{1/a_n}{1 - \tilde{u}_{n+1} \tilde{p}_{n-1} / (\tilde{u}_n \tilde{p}_n)} \right) \delta a_n \\ &+ \frac{1/a_n}{\tilde{u}_{n+1}/\tilde{u}_n - \tilde{p}_n / \tilde{p}_{n-1}} \delta b_n \end{aligned} \tag{2.25}$$

Moreover,

$$B^{-1}(A_n)_{2,2} = \frac{(\delta a_n)^2}{a_n \tilde{a}_n} - B(A_n)_{1,1} \tag{2.26}$$

To prove that (2.7) holds, we need the following lemma.

Lemma 2.3. *For any real analytic almost periodic sequence* $\{f_n\}_{n=1}^\infty$ *with frequency module contained in* $\mathcal{M}(\mathfrak{e})$,

$$\sum_{n=1}^N f_n \, \delta a_n \quad \textit{and} \quad \sum_{n=1}^N f_n \, \delta b_n \tag{2.27}$$

have limits as $N \to \infty$.

Proof. For simplicity, we only consider δa_n. By assumption,

$$f_n = \sum_{\mathbf{k}} c_{\mathbf{k}} e^{2\pi i (\mathbf{k} \cdot \boldsymbol{\omega}) n} \tag{2.28}$$

where the Fourier coefficients $c_{\mathbf{k}}$ obey

$$|c_{\mathbf{k}}| \leq C e^{-D|\mathbf{k}|} \tag{2.29}$$

for some $C, D > 0$. Given $\varepsilon > 0$, we will show that

$$\left| \sum_{n=1}^N f_n \, \delta a_n - \sum_{n=1}^M f_n \, \delta a_n \right| < \varepsilon \tag{2.30}$$

for N, M sufficiently large. Assume $N > M$ and start by taking K so large that

$$\sum_{|\mathbf{k}| > K} |c_{\mathbf{k}}| \left| \sum_{n=M+1}^N e^{2\pi i (\mathbf{k} \cdot \boldsymbol{\omega}) n} \delta a_n \right| < \varepsilon/2 \tag{2.31}$$

for all N, M. We can do so using (1.8) and (2.29). Next, take N, M so large that

$$|c_{\mathbf{k}}| \left| \sum_{n=M+1}^N e^{2\pi i (\mathbf{k} \cdot \boldsymbol{\omega}) n} \delta a_n \right| < \frac{\varepsilon/2}{\#\{\mathbf{k} : |\mathbf{k}| \leq K\}} \tag{2.32}$$

for all $\mathbf{k}$ with $|\mathbf{k}| \leq K$. Since only finitely many $\mathbf{k}$'s occur here, this can be done by (1.7). Combining (2.31) and (2.32) leads to (2.30) and the result follows. ☐

By [2, Thm. 7.3], the sequence of polynomials $\tilde{p}_{n-1}$ can be written as a linear combination,

$$\tilde{p}_{n-1} = r_1 B^{-n} \tilde{v}_n + r_2 \tilde{e}_n \tag{2.33}$$

with $r_j \neq 0$, $j = 1, 2$, where $\tilde{e}_n$ is exponentially decaying and $\tilde{v}_n$ is almost periodic. In fact, $\tilde{a}_{n+1} \tilde{v}_n$ is given by the Jost function (cf. [1, Sect. 8]) sampled along an equally spaced orbit on the isospectral torus $\mathcal{T}_{\mathfrak{e}}$, hence $|\tilde{v}_n| \geq c > 0$. Thus, (2.25) can be rewritten as

$$B(A_n)_{1,1} = \left(\frac{1/\tilde{a}_n}{1 - \tilde{u}_n\tilde{v}_{n+1}/(B\tilde{u}_{n+1}\tilde{v}_n)} - \frac{1/a_n}{1 - B\tilde{u}_{n+1}\tilde{v}_n/(\tilde{u}_n\tilde{v}_{n+1})} \right)\delta a_n + \frac{1/a_n}{\tilde{u}_{n+1}/\tilde{u}_n - \tilde{v}_{n+1}/(B\tilde{v}_n)}\delta b_n + e_n \tag{2.34}$$

where e_n is again an exponentially decaying sequence. Using the identity $1/a_n = 1/\tilde{a}_n - \delta a_n/(a_n\tilde{a}_n)$, we can rewrite (2.34) once more as

$$B(A_n)_{1,1} = \left(\frac{1/\tilde{a}_n}{1 - \tilde{u}_n\tilde{v}_{n+1}/(B\tilde{u}_{n+1}\tilde{v}_n)} - \frac{1/\tilde{a}_n}{1 - B\tilde{u}_{n+1}\tilde{v}_n/(\tilde{u}_n\tilde{v}_{n+1})} \right)\delta a_n + \frac{1/\tilde{a}_n}{\tilde{u}_{n+1}/\tilde{u}_n - \tilde{v}_{n+1}/(B\tilde{v}_n)}\delta b_n + e_n + s_n \tag{2.35}$$

Here s_n involves two terms, each given by a product of either $(\delta a_n)^2$ or $\delta a_n \delta b_n$ and a bounded factor. Hence the sequence s_n is absolutely summable by (1.6).

Now the definitions of the Jost function in [1, Sect. 8] and the Jost solution in [1, Sect. 9] combined with [1, Cor. 6.4] show that $1/\tilde{a}_n$, $\tilde{u}_{n+1}/\tilde{u}_n$, and $\tilde{v}_{n+1}/\tilde{v}_n$ are real analytic quasiperiodic sequences. So (2.26), (2.35), absolute summability of e_n and s_n, and Lemma 2.3 imply that

$$\sum_{n=1}^{N} (A_n)_{1,1} \quad \text{and} \quad \sum_{n=1}^{N} (A_n)_{2,2} \tag{2.36}$$

have limits as $N \to \infty$. Moreover, (2.22)–(2.23) combined with (1.6) imply that

$$\sum_{n=1}^{\infty} \|A_n\|^2 < \infty \tag{2.37}$$

Thus, the recursion (2.21) satisfies the conditions of Corollary 2.2 and hence we either have

$$\lim_{n\to\infty} B^n \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.38}$$

or

$$\lim_{n\to\infty} B^{-n} \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix} = c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.39}$$

for some nonzero constants c_1, c_2. The latter, however, is impossible. For since $B^n\tilde{p}_{n-1}$ and $B^{-n}\tilde{u}_n$ are bounded, (2.39) would, by (2.16) and

(2.18), imply that

$$p_n(\mathbf{x}(z)) = \phi_n(z)B(z)^n\tilde{p}_n(\mathbf{x}(z)) + \psi_n(z)B(z)^{-n}\tilde{u}_{n+1}(z) \tag{2.40}$$

decays exponentially as $n \to \infty$. Hence, the associated Jacobi matrix has an eigenvalue at $\mathbf{x}(z)$ which by assumption is a point in $\mathbb{C} \setminus \mathbb{R}$. This cannot be true and the first alternative (2.38) must therefore hold. Rewriting (2.40) as

$$\frac{p_n(\mathbf{x}(z))}{\tilde{p}_n(\mathbf{x}(z))} = B(z)^n\phi_n(z) + \frac{\tilde{u}_{n+1}(z)B(z)^{-n}}{\tilde{p}_n(\mathbf{x}(z))B(z)^n}B(z)^n\psi_n(z) \tag{2.41}$$

then leads to the conclusion of Theorem 1.2.

3. Examples

In this section we supplement Theorem 1.2 with examples for which (1.6)–(1.8) hold. In particular, we give an example showing that the Szegő condition is not necessary for Szegő asymptotics to hold. To put our results in the right perspective, recall that, by [2, Thm. 7.4], Szegő asymptotics (1.9) holds when the measure of orthogonality $d\mu(x) = w(x)dx + d\mu_{\mathrm{s}}(x)$ has essential support $\mathfrak{e}$ and belongs to the Szegő class for $\mathfrak{e}$, that is,

$$\int_{\mathfrak{e}} \frac{\log(w(x))}{\operatorname{dist}(x, \mathbb{R}\setminus\mathfrak{e})^{1/2}}\,dx > -\infty \qquad \text{(Szegő condition)} \tag{3.1}$$

and

$$\sum_k \operatorname{dist}(x_k, \mathfrak{e})^{1/2} < \infty \qquad \text{(Blaschke condition)} \tag{3.2}$$

where $\{x_k\}$ denote the mass points of $d\mu$ in $\mathbb{R}\setminus\mathfrak{e}$.

In continuation of [6] and [2], it was shown in [7, Thm. 1.3] that the generalized Nevai conjecture holds, that is, the measure of orthogonality is in the Szegő class if the sequence of recurrence coefficients $\{a_n, b_n\}_{n=1}^\infty$ is an ℓ^1-perturbation of some element $\{\tilde{a}_n, \tilde{b}_n\}_{n=-\infty}^\infty$ in $\mathcal{T}_{\mathfrak{e}}$. Thus,

$$\sum_{n=1}^\infty |a_n - \tilde{a}_n| + |b_n - \tilde{b}_n| < \infty \tag{3.3}$$

is a sufficient condition for Szegő asymptotics. Our main result, Theorem 1.2, shows that (3.3) can be relaxed and the weaker set of conditions (1.6)–(1.8) is also sufficient for Szegő asymptotics to hold.

In the following example we use Theorem 1.2 to show that (1.9) may hold for a non ℓ^1-perturbation of an element in $\mathcal{T}_{\mathfrak{e}}$; hence, the ℓ^1-condition (3.3) is not necessary for Szegő asymptotics.

Example 3.1. Let $\{\tilde{a}_n, \tilde{b}_n\}_{n=-\infty}^{\infty}$ be an arbitrary element of the isospectral torus $\mathcal{T}_{\mathfrak{e}}$. Pick $\alpha \in (\frac{1}{2}, 1)$, $\omega \in (0,1)$, and let

$$a_n = \tilde{a}_n + \delta a_n, \quad b_n = \tilde{b}_n + \delta b_n \tag{3.4}$$

with

$$\delta a_n = 0, \quad \delta b_n = \frac{1}{n^\alpha} \cos(2\pi\omega n) \tag{3.5}$$

In addition, assume that the almost periods $\omega_j = \rho_{\mathfrak{e}}([\alpha_1, \beta_j])$ obey a Diophantine condition relative to the frequency ω, that is, there exist a constant $C > 0$ and an integer s such that for all $\boldsymbol{k} = (k_1, \dots, k_\ell) \in \mathbb{Z}^\ell$,

$$\biggl\{ \pm\, \omega + \sum_{j=1}^{\ell} k_j \omega_j \biggr\} \geq \frac{C}{(1 + |\boldsymbol{k}|)^s} \tag{3.6}$$

where $\{x\} = x \bmod 1$ denotes the fractional part of x. It is known that the Diophantine condition is satisfied for Lebesgue a.e. $\boldsymbol{\omega}$. Hence, by a theorem of Totik [10], Lebesgue a.e. $\{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$ lead to the condition (3.6).

We start by noting that (1.6) is trivially satisfied since $\alpha > \frac{1}{2}$. On the other hand, it is easy to see that (3.3) fails to hold since $\alpha < 1$. Next, we verify (1.7)–(1.8). It follows from Theorem 2.6 in [11, Chap. I] that

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i x n}}{n^\alpha} \tag{3.7}$$

converges uniformly for x in $[\varepsilon, 1-\varepsilon]$ for every $\varepsilon > 0$. Moreover, by (2.26)–(2.27) in [11, Chap. V] there exists a constant $D > 0$ so that for all $x \in [\varepsilon, 1-\varepsilon]$,

$$\sup_N \biggl| \sum_{n=1}^{N} \frac{e^{2\pi i x n}}{n^\alpha} \biggr| < \frac{D}{\varepsilon} \tag{3.8}$$

So taking $x = \pm\omega + \boldsymbol{k} \cdot \boldsymbol{\omega}$ in (3.7) yields convergence of the series

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi\omega n)}{n^\alpha}\, e^{2\pi i (\boldsymbol{k}\cdot\boldsymbol{\omega}) n} \tag{3.9}$$

for all $\boldsymbol{k} \in \mathbb{Z}^\ell$ since by (3.6), $x \neq 0 \bmod 1$. Hence, (1.7) holds. Similarly, taking $x = \pm\omega + \boldsymbol{k} \cdot \boldsymbol{\omega}$ in (3.8) and utilizing (3.6) gives

$$\sup_{N \in \mathbb{N}} \biggl| \sum_{n=1}^{N} \frac{\cos(2\pi\omega n)}{n^\alpha}\, e^{2\pi i (\boldsymbol{k}\cdot\boldsymbol{\omega}) n} \biggr| \leq \frac{D}{C} (1 + |\boldsymbol{k}|)^s \tag{3.10}$$

Hence, (1.8) is also satisfied. Thus we have Szegő asymptotics by Theorem 1.2.

Our second example illustrates that neither the ℓ^1-condition (3.3) nor belonging to the Szegő class is necessary for Szegő asymptotics to hold.

Example 3.2. Let $\{\tilde{a}_n, \tilde{b}_n\}_{n=-\infty}^{\infty}$ be an arbitrary element of the isospectral torus $\mathcal{T}_{\mathfrak{e}}$. Pick $\alpha \in (\frac{3}{4}, 1)$ and let a_n, b_n be given as in (3.4) but now with

$$\delta a_n = 0, \quad \delta b_n = \frac{1}{n^\alpha} \cos(2\pi\sqrt{n}) \tag{3.11}$$

Then (1.6) is trivially satisfied since $\alpha > \frac{1}{2}$ and (3.3) fails to hold since $\alpha < 1$. To show that (1.7)–(1.8) are fulfilled, we note that Theorem 5.2(i) in [11, Chap. V] with 2π adjusted in both exponents implies that

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i\sqrt{n}}}{n^\alpha}\, e^{2\pi inx} \tag{3.12}$$

converges uniformly in x to a continuous function on $[-\frac{1}{2}, \frac{1}{2}]$. Combining real and imaginary parts of this series at $x = \pm\omega$ yields convergence of the series

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi\sqrt{n})}{n^\alpha}\, e^{2\pi in\omega} \tag{3.13}$$

for all $\omega \in [0, 1]$. Moreover, the partial sums are uniformly bounded

$$\sup_{\omega\in[0,1]} \sup_{N\in\mathbb{N}} \left| \sum_{n=1}^{N} \frac{\cos(2\pi\sqrt{n})}{n^\alpha}\, e^{2\pi in\omega} \right| \leq C \tag{3.14}$$

Since (3.13)–(3.14) imply (1.7)–(1.8), we have Szegő asymptotics by Theorem 1.2.

Next, we show that the eigenvalues $\{x_k\}$ in $\mathbb{R}\setminus\mathfrak{e}$ of the Jacobi matrix J with parameters $\{a_n, b_n\}_{n=1}^{\infty}$ satisfy

$$\sum_k \operatorname{dist}(x_k, \mathfrak{e})^q = \infty \tag{3.15}$$

for any $q < \frac{1}{2\alpha}$, in particular, for $q = \frac{1}{2}$. Hence, the Blaschke condition (3.2) fails and, by [2, Thm. 4.1], also the Szegő condition (3.1) fails to hold in the present example.

Indeed, let

$$B_m = \big[(8m)^2 - m, (8m)^2 + m\big], \quad m \geq 1 \tag{3.16}$$

denote disjoint subsets of $\mathbb{N}$. Recall that, by [1, Thm. 9.6], the unperturbed difference equation

$$\tilde{a}_n w_{n+1} + (\tilde{b}_n - \beta_{\ell+1}) w_n + \tilde{a}_{n-1} w_{n-1} = 0 \tag{3.17}$$

has a positive bounded solution u_n so that

$$c_1 \leq u_n \leq c_2 \tag{3.18}$$

uniformly in n with some fixed positive constants c_1, c_2. In addition, by [1, Thm. 9.7], (3.17) also has an increasing solution v_n which, by [1, Eq. (9.31)], is of the form

$$v_n = \kappa n u_n + s_n \tag{3.19}$$

where κ is a fixed positive constant and $\{s_n\}$ a bounded real-valued sequence. Let c_3 be an upper bound for $|s_n|$ and $\mu_\mp = (8m)^2 \mp m$ be the left (resp. right) endpoint of B_m. Then, by (3.18)–(3.19), the linear combinations

$$\pm(v_n - \kappa \mu_\mp u_n) + (c_3/c_1) u_n \tag{3.20}$$

are positive on B_m and of magnitude at least $m\kappa c_1$ at the center of B_m and at most $c_3(1 + c_2/c_1)$ at the left (resp. right) endpoint of B_m. Rescaling the expression in (3.20), we obtain solutions $v_\pm^{(m)}$ of (3.17) with the following properties:

- $v_\pm^{(m)}$ are positive on B_m and equal to 1 at the center of B_m,
- $v_+^{(m)}$ is of order $\mathcal{O}(1/m)$ at the left endpoint of B_m, and
- $v_-^{(m)}$ is of order $\mathcal{O}(1/m)$ at the right endpoint of B_m.

Now define $\phi^{(m)}$ by

$$\left(\phi^{(m)}\right)_n = \begin{cases} \left(v_+^{(m)}\right)_n & \text{if } n \in \left[(8m)^2 - m, (8m)^2\right] \\ \left(v_-^{(m)}\right)_n & \text{if } n \in \left[(8m)^2, (8m)^2 + m\right] \\ 0 & \text{if } n \notin B_m \end{cases}$$

Then there is a constant $C > 0$ (independent of m) such that

$$\|\phi^{(m)}\|^2 \geq Cm \tag{3.21}$$

As $\left|\sqrt{(8m)^2 \pm m} - 8m\right| < \frac{1}{8}$, we have $\delta b_n > C' m^{-2\alpha}$ for $n \in B_m$ so that

$$\left\langle \phi^{(m)}, (J - \tilde{J})\phi^{(m)} \right\rangle \geq \frac{C' \|\phi^{(m)}\|^2}{m^{2\alpha}} \tag{3.22}$$

Moreover, since $\phi^{(m)}$ satisfies (3.17) for all n except at the center and near the endpoints of B_m,

$$\big|\big\langle \phi^{(m)}, (\tilde J - \beta_{\ell+1})\phi^{(m)}\big\rangle\big| \leq \frac{C''}{m} \tag{3.23}$$

Combining the above inequalities gives

$$\frac{\big\langle \phi^{(m)}, (J - \beta_{\ell+1})\phi^{(m)}\big\rangle}{\|\phi^{(m)}\|^2} \geq \frac{C'}{m^{2\alpha}} - \frac{C''/C}{m^2} \tag{3.24}$$

Hence, for large m and some small constant $D > 0$, we have

$$\operatorname{dist}(x_m, \mathfrak{e}) \geq \frac{D}{m^{2\alpha}} \tag{3.25}$$

Thus, (3.15) holds for all $q < \frac{1}{2\alpha}$.

Remark. With the additional Diophantine assumption (3.6) and somewhat more delicate estimates as in [5, Sect. 9], one can show that there are examples where (3.15) holds with q arbitrarily close to $\frac{3}{2}$, $q < \frac{3}{2}$.

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