UNITARIES PERMUTING TWO ORTHOGONAL PROJECTIONS

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ABSTRACT. Let P and Q be two orthogonal projections on a separable Hilbert space, $\mathcal{H}$. Wang, Du and Dou proved that there exists a unitary, U, with $UPU^{-1} = Q$, $\quad UQU^{-1} = P$ if and only if $\dim(\ker P \cap \ker(1-Q)) = \dim(\ker Q \cap \ker(1-P))$ (both may be infinite). We provide a new proof using the supersymmetric machinery of Avron, Seiler and Simon.

Let P and Q be two orthogonal projections on a separable Hilbert space, $\mathcal{H}$. It is a basic result in eigenvalue perturbations theory that when

$$\|P - Q\| < 1 \tag{1}$$

there exists a unitary U so that

$$UP = QU \tag{2}$$

It is even known that there exist unitaries, U, so that

$$UPU^{-1} = Q, \quad UQU^{-1} = P \tag{3}$$

The simpler question involving (2) goes back to Sz-Nagy [13] and was further studied by Kato [9] who found a cleaner formula for U than Sz-Nagy, namely Kato used

$$U = [QP + (1-Q)(1-P)]\left[1 - (P-Q)^2\right]^{-1/2} \tag{4}$$

Using Nagy's formula, Wolf [15] had extended this to arbitrary pairs of projections on a Banach space (requiring only that U is invertible rather than unitary) so long as

$$\|P - Q\|\|P\|^2 < 1 \qquad \|P - Q\|\|Q\|^2 < 1 \tag{5}$$

For non–orthogonal projections and projections on a Banach space, in general, $\|P\| \geq 1$ with equality in the Hilbert space case only if P is

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orthogonal so (5) is strictly stronger than (1). One advantage of Kato's form (4), is that in the Banach space case where the square root can be defined by a power series, it only requires (1).

For the applications they had in mind, it is critical not only that U exist but that on the set of pairs that (1) holds, U is analytic in P and Q. For they considered an analytic family, $A(z)$, and λ_0 an isolated eigenvalue of $A(0)$ of finite algebraic multiplicity. Then one can define

$$P(z) = \frac{1}{2\pi i} \oint_{|\lambda-\lambda_0|=r} (\lambda - A(z))^{-1} d\lambda$$

for fixed small r and $|z|$ small. For $|z|$ very small, $\|P(z) - P(0)\| < 1$. If $U(z)$ is given by (4) with $Q = P(z)$, then $U(z)A(z)U(z)^{-1}$ leaves $\operatorname{ran} P(0)$ invariant and the study of eigenvalues of $A(z)$ near λ_0 is reduced to the finite dimensional problem $U(z)A(z)U(z)^{-1} \restriction \operatorname{ran} P(0)$. See the books of Kato [10], Baumgärtel [3] or Simon [12] for this subject.

There is a rich structure of pairs of orthogonal projections when (1) might fail using two approaches. One goes back to Krein et al. [11], Diximier [6], Davis [5] and Halmos [7]. Let

$$\mathcal{K}_{P,Q} = \operatorname{ran} P \cap \ker Q \tag{6}$$

The four mutually orthogonal spaces $\mathcal{K}_{P,Q}$, $\mathcal{K}_{P,1-Q}$, $\mathcal{K}_{1-P,Q}$, $\mathcal{K}_{1-P,1-Q}$ are invariant for P and Q and their mutual orthogonal complement has a kind of 2×2 matrix structure. Böttcher-Spitkovsky [4] have a comprehensive review of this approach. Following them, we'll call this the Halmos approach since his paper had the clearest version of it.

A second approach, introduced by Avron–Seiler–Simon [2],uses the operators

$$A = P - Q, \qquad B = 1 - P - Q \tag{7}$$

which, by simple calculations, obey

$$A^2 + B^2 = 1, \quad AB + BA = 0, \quad [P, A] = [Q, A] = [P, B] = [Q, B] = 0 \tag{8}$$

The last equations (at least for A) go back to the 1940's and were realized by Dixmier, Kadison and Mackey. The definition of B and first equation in (8) were noted by Kato [9] who found the middle equation in 1971 but never published it. Because (8) involves a vanishing anticommutator, we call the use of the operators in (7) the supersymmetric approach. One consequence of (8) is that it implies that if $P - Q$ is trace class, then its trace is an integer–indeed, as we'll discuss below, it is the index of a certain Fredholm operator.

The two approaches are related as shown by Amerein–Sinha [1] (see also Takesaki [14, pp 306-308] and Halpern [8]). In [16], Wang, Du and Dou proved the following lovely theorem

Theorem 1. *Let P and Q be two orthogonal projections on a separable Hilbert space, $\mathcal{H}$. Then there exists a unitary obeying* (3) *if and only if*

$$\dim(\mathcal{K}_{P,Q}) = \dim(\mathcal{K}_{1-P,1-Q}) \tag{9}$$

The literature on pairs of projections is so large that it is possible this was also proven elsewhere. Their proof uses the Halmos representation. Our goal here is to provide a supersymmetric proof which seems to us simpler and more algebraic (although we understand that simplicity is in the eye of the beholder). Our proof will also have a simple explicit form for U. Before turning to the proof, we want to note two corollaries of Theorem 1.

One notes first that since $\mathrm{ran} R = \ker(1-R)$ for any projection R and $P, Q \geq 0$, we have that

$$\mathcal{K}_{P,Q} = \{\varphi \mid A\varphi = \varphi\}, \quad \mathcal{K}_{1-P,1-Q} = \{\varphi \mid A\varphi = -\varphi\}$$

Thus (1) $\Rightarrow \dim \mathcal{K}_{P,Q} = \mathcal{K}_{1-P,1-Q} = 0$, so Theorem 1 implies

Corollary 2. (1) $\Rightarrow$ *the existence of U obeying* (3).

The second corollary concerns the case where $P-Q$ is compact. In that case $K = QP \restriction \mathrm{ran}\, P$ as a map of $\mathrm{ran}\, P$ to $\mathrm{ran}\, Q$ is Fredholm and $\mathcal{K}_{P,Q} = \ker K$ while $\mathcal{K}_{1-P,1-Q} = \mathrm{ran}\, K^\perp$ so (9) is equivalent to saying that the index of K is 0 so we get

Corollary 3. *If $P-Q$ is compact, then there exists a U obeying* (3) *if and only if* $\mathrm{Index} = 0$.

Avron el al [2] essentially had these two corollaries many years before [16] and this note points out that while [2] didn't consider the general case of Theorem 1, there is a small addition to their argument that proves the general result.

Lemma 4. *To prove Theorem 1, it suffices to prove it in the case where $\mathcal{K}_{P,Q} = \mathcal{K}_{1-P,1-Q} = \{0\}$.*

Proof. Let $\mathcal{H}_1 = \mathcal{K}_{P,Q} \oplus \mathcal{K}_{1-P,1-Q}$ and $\mathcal{H}_2 = \mathcal{H}_1^\perp$. Note that $\mathcal{K}_{P,Q}$ is orthogonal to $\mathcal{K}_{1-P,1-Q}$ since $\mathrm{ran}\, P$ is orthogonal to $\ker P$. P and Q leave $\mathcal{H}_1$ invariant and so $\mathcal{H}_2$.

If there is U obeying (3), then U is a unitary map of $\mathcal{K}_{P,Q}$ to $\mathcal{K}_{1-P,1-Q}$ so their dimensions are equal and (9) holds. On the other hand, if (9) holds, there is a unitary map V on $\mathcal{H}_1$ that maps $\mathcal{K}_{P,Q}$ to $\mathcal{K}_{1-P,1-Q}$ and vice versa. Clearly $VP \restriction \mathcal{H}_1 V^{-1} = Q \restriction \mathcal{H}_1$ and $VQ \restriction \mathcal{H}_1 V^{-1} = P \restriction \mathcal{H}_1$ since $P \restriction \mathcal{K}_{P,Q} = \mathbf{1}, P \restriction \mathcal{K}_{1-P,1-Q}, Q \restriction \mathcal{K}_{P,Q} = 0, Q \restriction \mathcal{K}_{1-P,1-Q} = \mathbf{1}$.

$P_2 = P \restriction \mathcal{H}_2, Q_2 = Q \restriction \mathcal{H}_2$ obey $\mathcal{K}_{P_2,Q_2} = \mathcal{K}_{1-P_2,1-Q_2} = \{0\}$. Thus the special case of the theorem implies there is a unitary $W : \mathcal{H}_2 \to \mathcal{H}_2$ with $WP_2W^{-1} = Q_2, WQ_2W^{-1} = P_2$. $U = V \oplus W$ solves (3) □

Proof of Theorem 1. By the lemma we can suppose that A doesn't have eigenvalues ± 1, so $B^2 = 1 - A^2$ has $\ker B^2 = 0$. Thus $\ker B = 0$. It follows that

$$s - \lim_{\epsilon \downarrow 0} B(|B| + \epsilon)^{-1} = \operatorname{sgn}(B) \equiv U \tag{10}$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases} \tag{11}$$

so that $\operatorname{sgn}(B)$ is unitary since $\ker B = 0$.

Since

$$BA = -AB \tag{12}$$

we see that

$$B^2 A = AB^2 \tag{13}$$

so by properties of the square root ([12, Thm. 2.4.4])

$$(|B| + \epsilon)A = A(|B| + \epsilon) \tag{14}$$

Thus (12) implies that

$$(|B| + \epsilon)^{-1} BA = -AB(|B| + \epsilon)^{-1} \tag{15}$$

By (10), we see that

$$UAU^{-1} = -A \tag{16}$$

Since U is a function of B

$$UB = BU \Rightarrow UBU^{-1} = B \tag{17}$$

We have that

$$P = \tfrac{1}{2}(A - B + \mathbf{1}), \qquad Q = \tfrac{1}{2}(-A - B + \mathbf{1}) \tag{18}$$

so, by (16) and (17), we have (3). □

To understand the difference between (4) and (5), we note that in case $\mathcal{H} = \mathbb{C}^2$ and P, Q are two one-dimensional projections with $\operatorname{Tr}(PQ) = \cos^2 \theta$ (so θ is the angle between $\operatorname{ran} P$ and $\operatorname{ran} Q$), the U of (5) is rotation by angle θ while the U of (4) is reflection in the perpendicular bisector.

One interesting open question is whether there are extension of Theorem 1 (with U unitary replaced by U invertible) to non-self-adjoint

Hilbert space projections and to general pairs of projections on a Banach space.

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