ORTHOGONAL POLYNOMIALS WITH EXPONENTIALLY DECAYING RECURSION COEFFICIENTS

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Dedicated to S. Molchanov on his 65th birthday

ABSTRACT. We review recent results on necessary and sufficient conditions for measures on $\mathbb{R}$ and $\partial\mathbb{D}$ to yield exponential decay of the recursion coefficients of the corresponding orthogonal polynomials. We include results on the relation of detailed asymptotics of the recursion coefficients to detailed analyticity of the measures. We present an analog of Carmona's formula for OPRL. A major role is played by the Szegő and Jost functions.

1. INTRODUCTION: SZEGŐ AND JOST FUNCTIONS

In broad strokes, spectral theory concerns the connection between the coefficients in differential or difference equations and the spectral measures associated to those equations. The process of going from coefficients to the measures is the direct problem, and the other direction is the inverse spectral problem. The gems of spectral theory are ones that set up one-one correspondences between classes of measures and coefficients with some properties. Examples are Verblunsky's form of Szegő's theorem [25] and the Killip-Simon theorem [12]. In this paper, our goal is to describe (mainly) recent results involving such gems for orthogonal polynomials whose recursion coefficients decay exponentially. These are technically simpler systems than the L^2 results just quoted but have more involved details.

The two classes we discuss are orthogonal polynomials on the real line (OPRL) and on the unit circle (OPUC). For the OPRL case, we have a probability measure, $d\rho$, on $\mathbb{R}$ of bounded but infinite support whose orthonormal polynomials, $p_n(x)$, obey

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_np_{n-1}(x) \tag{1.1}$$

Date: March 2, 2006.

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Submitted to the Proceedings of S. Molchanov's 65th Birthday Conference.

with $b_n \in \mathbb{R}$ and $a_n \in (0,\infty)$ and called Jacobi parameters. $\{a_n, b_n\}_{n=1}^\infty$ is a description of $d\rho$ in that there is a one-one correspondence between bounded sets of such Jacobi parameters and such $d\rho$'s. For background discussion of OPRL, see [4, 9, 18, 24].

For the OPUC case, $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, and $d\mu$ is a probability measure on $\partial\mathbb{D}$ whose support is not a finite set. The orthonormal polynomials, $\varphi_n(z)$, obey the Szegő recursion relation:

$$z\varphi_n(z) = \rho_n \varphi_{n+1}(z) + \bar{\alpha}_n \varphi_n^*(z) \tag{1.2}$$

$$\varphi_n^*(z) = z^n\, \overline{\varphi_n(1/\bar{z})} \tag{1.3}$$

$$\rho_n = (1-|\alpha_n|^2)^{1/2} \tag{1.4}$$

with $\alpha_n \in \mathbb{D}$ and called Verblunsky coefficients. $\{\alpha_n\}_{n=0}^\infty$ is a description of $d\mu$ in that there is a one-one correspondence between sequences of α_n obeying $|\alpha_n| < 1$ and such $d\mu$'s. For background discussion of OPUC, see [10, 17, 18, 19, 24].

The measure theoretic side of the equivalences will be in terms of a derived object, rather than the measures themselves. For OPUC, the object is $D(z)$, the Szegő function [18, Section 2.4]. One says the Szegő condition holds if and only if

$$d\mu(\theta) = w(\theta)\,\frac{d\theta}{2\pi} + d\mu_{\mathrm{s}} \tag{1.5}$$

where $d\mu_{\mathrm{s}}$ is singular and

$$\int \log(w(\theta))\,\frac{d\theta}{2\pi} > -\infty \tag{1.6}$$

(which is known to be equivalent to $\sum_{n=0}^\infty |\alpha_n|^2 < \infty$). In that case, $D(z)$ is defined by

$$D(z) = \exp\biggl(\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\log(w(\theta))\,\frac{d\theta}{4\pi}\biggr) \tag{1.7}$$

which obeys

$$\varphi_n^*(z) \to D(z)^{-1} \tag{1.8}$$

if $|z| < 1$.

$D(z)$ does not uniquely determine $d\mu$, but it does if $d\mu_{\mathrm{s}} = 0$, as it will be in our cases of interest, since

$$w(\theta) = \lim_{r\uparrow 1} |D(re^{i\theta})|^2 \tag{1.9}$$

for a.e. θ.

For OPRL, the object is the Jost function. The situation is not as clean as the OPUC case in that there are not simple necessary and sufficient conditions for existence in terms of the measure. There are necessary and sufficient conditions in terms of the Jacobi parameters (see [5] and [19, Section 13.9]) but not for the measure. However, there are sufficient conditions for existence that suffice for us here. Suppose

$$d\rho(x) = f(x)\,dx + d\rho_{\rm s}(x) \tag{1.10}$$

where f is supported on $[-2,2]$, and outside this set, the singular part, $d\rho_{\rm s}$, has only pure points $\{E_j^\pm\}_{j=0}^{N_\pm}$ with

$$E_1^- < E_2^- < \cdots < -2 < 2 < \cdots < E_2^+ < E_1^+ \tag{1.11}$$

and suppose

$$\sum_{j,\pm} (|E_j^\pm| - 2)^{1/2} < \infty \tag{1.12}$$

and that

$$\int_{-2}^{2} \log(f(x))(4-x^2)^{-1/2}\,dx > -\infty \tag{1.13}$$

Then (originally in Peherstorfer–Yuditskii [15]; see also Simon–Zlatoš [23] and [19, Theorem 13.8.9]) there is an analytic function u on $\mathbb{D}$ so that its zeros are precisely those points $z_j^\pm$ in $\mathbb{D}$ given by

$$z_j^\pm + (z_j^\pm)^{-1} = E_j^\pm \tag{1.14}$$

and if B is the Blaschke product (convergent by (1.12))

$$B(z) = \prod_{j=\pm 1} \frac{(z - z_j^\pm)}{1 - \bar z_j^\pm z} \tag{1.15}$$

then $Bu^{-1} \in H^2$ and the boundary values of u obey

$$|u(e^{i\theta})|^2 \operatorname{Im} M(e^{i\theta}) = \sin\theta \tag{1.16}$$

where

$$M(z) = \int \frac{d\rho(x)}{z + z^{-1} - x} \tag{1.17}$$

(so $\operatorname{Im} M(e^{i\theta})$ is related to $f(2\cos\theta)$).

These properties determine u uniquely. Unlike the OPUC case, u does not determine $d\rho$ even if $d\rho_{\rm s} \restriction [-2,2] = 0$ for u only determines f and the localization of the pure points of $d\rho$ on $\mathbb{R}\backslash[-2,2]$. To recover $d\rho$, we also need to know the weights of the pure points; equivalently, the residues of the poles of M at the $z_j^\pm$.

The theme of this review is that detailed results on exponential decay of recursion coefficients are equivalent to analyticity results on D^{-1} in the OPUC case and u in the OPRL case. That exponential decay implies analyticity has been in the physics literature for Schrödinger operators for over fifty years. The subtle aspect is the strict equivalence — an idea that appeared first in Nevai–Totik [14].

In Section 2, we discuss some aspects of finite range potentials, and in Section 3, following Nevai–Totik [14] and Damanik–Simon [6], the initial equivalence. In Section 4, following Simon [20, 21], we discuss detailed exponential asymptotics and meromorphic S and u.

I would like to thank J. Christiansen, L. Golinskii, P. Nevai, and V. Totik for useful comments.

Stas Molchanov is a leading figure in spectral theory. It is a pleasure to present this birthday bouquet to him.

2. Finite Range

In this section, we present new results on approximation by finite range "potentials." We begin with an OPRL analog of Carmona's result [3] on boundary condition averaging for Schrödinger operators. We will also see that Bernstein–Szegő measures for OPUC can be viewed through the Carmona lens. Carmona's proof relies on computing derivatives of Prüfer variables — our proof here is spectral averaging making the relation to [22] transparent.

Let J be the semi-infinite Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{2.1}$$

associated to an OPRL with measure $d\rho$; $J_{n;F}$, the matrix obtained from the top n rows and n left columns; and J_n^b, the matrix $J_{n;F}$ with b_n replaced by $b_n + b$. Here $b \in \mathbb{C}$. Notice that if $\operatorname{Im} b \le 0$, then $\operatorname{spec}(J_n^b) \subset \mathbb{C} \backslash \mathbb{C}_+$, so

$$m_n^{(b)}(z) = \langle \delta_0, (J_n^{(b)} - z)^{-1} \delta_0 \rangle$$

is analytic for $b \in \overline{\mathbb{C}}_-$ and z fixed in $\mathbb{C}_+$.

If $d\rho$ is a determinate moment problem, then J is essentially selfadjoint on finite sequences [16], so

$$m_n^{(b)}(z) \to m(z) \equiv \int \frac{d\rho(x)}{x - z} \tag{2.2}$$

for any b. Thus, if $d\nu_n$ is defined by

$$\widetilde{m}_n(z) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} m_n^{(b)}(z) \, \frac{db}{1 + b^2} \tag{2.3}$$

$$= \int \frac{d\nu^{(n)}(x)}{x - z} \tag{2.4}$$

then

$$d\nu^{(n)} \to d\rho \tag{2.5}$$

weakly. $d\nu^{(n)}$ is thus the average over b of the pure point spectral measures of $J_n^{(b)}$.

Theorem 2.1. *If $p_n(x)$ are the orthonormal OPRL, then*

$$d\nu^{(n)}(x) = \frac{dx}{\pi(a_n^2 p_n^2(x) + p_{n-1}^2(x))} \tag{2.6}$$

In particular, the right-hand side of (2.6) converges weakly to $d\rho$. More is true, for Gaussian quadrature implies that if $m_n^{(b)}(z) = \int d\rho_n^{(b)}(x)(x-z)^{-1}$, then $\int x^\ell\, d\rho_n^{(b)}(x) = \int x^\ell\, d\rho(x)$ for $\ell \leq 2n-2$, and thus,

$$\int x^\ell\, d\nu^{(n)}(x) = \int x^\ell\, d\rho \qquad \ell = 0, \dots, 2n-2 \tag{2.7}$$

Of course, $d\nu^{(n)}$ does not have *all* moments finite; indeed, $\int |x|^\ell\, d\nu^{(n)} = \infty$ for $\ell \geq 2n-1$.

Proof of Theorem 2.1. It is well known (see [18, Section 1.2]) that

$$\det(z - J_{n;F}) = P_n(z) \tag{2.8}$$

the monic OPRL, and if $J_{n;F}^{(1)}$ is the matrix obtained by removing the top row and leftmost column (i.e., 11 minor), then

$$\det(z - J_{n;F}^{(1)}) = Q_n(z) \tag{2.9}$$

the monic second kind polynomial of degree $n-1$.

By expanding $\det(z - J_n^{(b)})$ in minors, we see

$$\det(z - J_n^{(b)}) = P_n(z) - bP_{n-1}(z) \tag{2.10}$$

$$= (a_1 \dots a_{n-1})(a_n p_n(z) - bp_{n-1}(z)) \tag{2.11}$$

and thus,

$$m_n^{(b)}(z) = -\frac{(a_n q_n(z) - bq_{n-1}(z))}{(a_n p_n(z) - bp_{n-1}(z))} \tag{2.12}$$

As noted above, if $\operatorname{Im} b \leq 0$, $m_n^{(b)}(z)$ has its poles in $\operatorname{Im} z \leq 0$ and thus, if $\operatorname{Im} z > 0$, $m_n^{(b)}(z)$ is analytic in $\operatorname{Im} b \leq 0$. Thus, we can close the contour in the lower half-plane and find for $\operatorname{Im} z > 0$,

$$\begin{aligned}\widetilde{m}_n(z) &= m_n^{(b=-i)}(z) \\ &= -\frac{(a_n q_n(z) + iq_{n-1}(b))}{(a_n p_n(z) + ip_{n-1}(z))}\end{aligned} \tag{2.13}$$

Thus, $\widetilde{m}_n$ is analytic on $\overline{\mathbb{C}}_+$, so

$$d\nu_n(x) = \pi^{-1} \operatorname{Im} \widetilde{m}_n(x)\, dx$$

Since p_n, p_{n-1}, q_n, q_{n-1} are real on $\mathbb{R}$,

$$\operatorname{Im} \widetilde{m}_n(x) = \frac{a_n(p_{n-1}(x)q_n(x) - p_n(x)q_{n-1}(x))}{(a_n^2 p_n(x)^2 + p_{n-1}(x)^2)} \tag{2.14}$$

which is (2.8) by a standard Wronskian calculation (see (1.2.51) of [18]). □

By this same calculation, one can recover Carmona's formula for the Schrödinger operator case.

One can ask about the analog of this for OPUC. Given a nontrivial measure, $d\mu$, on $\partial\mathbb{D}$ and $\omega = e^{i\theta} \in \partial\mathbb{D}$, we define $d\mu_n^{(\omega)}$ to be the trivial measure with Verblunsky coefficients

$$\begin{aligned} &\alpha_j = \alpha_j(d\mu) \qquad j = 0, \dots, n-1 \\ &\alpha_n = \omega \end{aligned}$$

Then $d\mu_n^{(\omega)}$ is the measure with $n+1$ pure points at the zeros of the paraorthogonal polynomial (POPUC),

$$\Phi_{n+1}^{(\omega)}(z) = z\Phi_n(z) - \bar{\omega}\Phi_n^*(z) \tag{2.15}$$

Theorem 2.2. *$d\mu_n \equiv \int \frac{d\theta}{2\pi}\, d\mu_n(e^{i\theta})$ is the Bernstein–Szegő measure*

$$d\mu_n = \frac{d\theta}{2\pi|\varphi_n(e^{i\theta})|^2} \tag{2.16}$$

Proof. If ψ_n are the second kind polynomials, Geronimus' formula for $F(z)$ (see [18, Theorem 3.2.4]) implies ($F(z; d\mu) = \int \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)$)

$$F(z; d\mu_n^{(\omega)}) = \frac{\psi_n^*(z) - \omega z \psi_n(z)}{\varphi_n^*(z) - \omega z \varphi_n(z)} \tag{2.17}$$

Averaging ω over $\frac{d\theta}{2\pi}$ gives the value at $\omega = 0$ since this function is analytic in ω for z fixed in $\mathbb{D}$. It follows that

$$F(z; d\mu_n) = \frac{\psi_n^*(z)}{\varphi_n^*(z)} \tag{2.18}$$

and yields (2.16) by (3.2.35) of [18]. □

The Bernstein–Szegő approximation also has the property of being the measure associated to extending the α's up to n to be free beyond n (i.e., $\alpha_j = 0$ for $j \geq n$). One can ask about the analogous approximation for OPRL. We will get the function S_n used by Dombrowski–Nevai [8]:

Let J_ℓ be the Jacobi matrix with parameters

$$a_n(J_\ell) = \begin{cases} a_n(J) & n = 1, \dots, \ell - 1 \\ 1 & n \geq \ell \end{cases} \tag{2.19}$$

$$b_n(J_\ell) = \begin{cases} b_n(J) & n = 1, \dots, \ell \\ 0 & n \geq \ell \end{cases} \tag{2.20}$$

According to Theorem 13.6.1 (with a_ℓ replaced by 1), its Jost function is ($x = z + 1/z$)

$$g_\ell(z) = z^\ell \biggl(p_\ell \biggl(z + \frac{1}{z} \biggr) - z p_{\ell-1} \biggl(z + \frac{1}{z} \biggr) \biggr) \tag{2.21}$$

Define $S_\ell(x)$ by

$$S_\ell \biggl(z + \frac{1}{z} \biggr) \equiv g_\ell(z) g_\ell \biggl(\frac{1}{z} \biggr) \tag{2.22}$$

Then, by (2.21),

$$S_\ell(x) = p_\ell(x)^2 + p_{\ell-1}(x)^2 - x p_\ell(x) p_{\ell-1}(x) \tag{2.23}$$

Taking into account the different normalization (for us, "free" is $a_k = 1$; for them, $a_k = \frac{1}{2}$), this is the function $S_\ell(x)$ of Dombrowski–Nevai [8]. The approximating measure has a.c. part related to $dx/|g_\ell(z)|^2$ on $[-2, 2]$ which is $dx/S_\ell(x)$. The eigenvalues of J_ℓ are zeros of $S_\ell(x)$ but not all zeros since S_ℓ also vanishes if $g_\ell(1/z) = 0$, that is, at antibound state and resonance energies.

For most purposes, (2.8) is a more useful representation than the one associated to S_ℓ.

3. Necessary and Sufficient Conditions on Exponential Decay

The starting point of the recent results on exponential decay is the following result of Nevai–Totik for OPUC:

Theorem 3.1 ([14]). *Let $d\mu$ be a nontrivial probability measure on $\partial\mathbb{D}$ and $R > 1$. Then the following are equivalent:*

(a)
$$\limsup_{n\to\infty} |\alpha_n(d\mu)|^{1/n} \leq R^{-1} \tag{3.1}$$

(b) $d\mu_\mathrm{s} = 0$*, the Szegő condition* (1.6) *holds, and* $D(z)^{-1}$ *has an analytic continuation to* $\{z \mid |z| < R\}$.

Remark. Since $R^{-1} < 1$, (3.1) is an expression of exponential decay.

The proof is easy. If (3.1) holds, Szegő recursion first implies inductively that for $|z|=1$,

$$|\Phi_{n+1}(e^{i\theta})| \leq (1+|\alpha_n|)|\Phi_n(e^{i\theta})| \tag{3.2}$$

so

$$\begin{aligned}\sup_{n,|z|\leq 1} |\Phi_n^*(z)| &= \sup_{n,\theta} |\Phi_n^*(e^{i\theta})| \quad \text{(by the maximum principle)} \\ &\leq \prod_{j=0}^{\infty}(1+|\alpha_j|) \equiv C < \infty \end{aligned} \tag{3.3}$$

and thus, for $|z|>1$,

$$|\Phi_n(z)| \leq C|z|^n \tag{3.4}$$

Iterating

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z \Phi_n(z) \tag{3.5}$$

we get

$$\Phi_n^*(z) = 1 - \sum_{j=0}^{n-1} \alpha_j z \Phi_j(z) \tag{3.6}$$

(3.1), (3.4), and (3.6) imply that for any $\varepsilon>0$,

$$\sup_{n,|z|<R-\varepsilon} |\Phi_n^*(z)| < \infty$$

which implies that $\varphi_n^*(z)$ has a limit for $|z|<R$. This limit defines the analytic continuation of $D(z)^{-1}$.

For the other direction, one can use either of two similar-looking but distinct formulae relating D to α_n. One can use a formula of Geronimus [10] and Freud [9] as Nevai–Totik [14] do (it requires $d\mu_{\text{s}}=0$)

$$\alpha_n = -\kappa_\infty \int \overline{\Phi_{n+1}(e^{i\theta})}\, D(e^{i\theta})^{-1} d\mu(\theta) \tag{3.7}$$

or the following formula of Simon [20] derived from iterated Szegő recursion:

$$\alpha_n = -\kappa_\infty^{-1}\kappa_n^2 \int \overline{\Phi_n(e^{i\theta})}\, [D(e^{i\theta})^{-1} - D(0)^{-1}] e^{-i\theta}\, d\mu(\theta) \tag{3.8}$$

In these formulae,

$$\kappa_n = \prod_{j=0}^{n-1}(1-|\alpha_j|^2)^{-1/2} \qquad \kappa_\infty = \lim_{n\to\infty} \kappa_n$$

To get exponential decay of α_n from (3.7) or (3.8), one uses $\int \overline{\Phi_n(e^{i\theta})} e^{-ij\theta}\, d\mu(\theta) = 0$ for $j<n$ and the Taylor series for D^{-1} to

see that α_n is bounded by the tail of the Taylor series of $D(z)^{-1}$ which, of course, decays exponentially if $D(z)^{-1}$ is analytic in $|z| < R$.

For OPRL, the analogs of Theorem 3.1 are due to Damanik–Simon [6]. The result is simpler if there are no bound states or resonances where

Definition. We say a measure $d\rho$ on $\mathbb{R}$ has no bound states or resonances if

$$d\rho(x) = f(x)\,dx + d\rho_{\rm s} \tag{3.9}$$

where

$$\operatorname{supp}(d\rho) \subset [-2,2] \tag{3.10}$$

and

$$\int (4-x^2)^{-1} f(x)\,dx < \infty \tag{3.11}$$

Theorem 3.2 ([6])**.** *Let $R > 1$. Suppose $d\rho$ has no bound states or resonances. Then $u(z)$ has an analytic continuation to $\{z \mid |z| < R\}$ if and only if*

$$\limsup[|a_n(d\rho) - 1| + |b_n(d\rho)|]^{1/2n} \leq R^{-1} \tag{3.12}$$

[6] has several proofs, but the simplest one is in [21]. When (3.11) holds, there is a measure $d\mu$ on $\partial\mathbb{D}$ given by

$$d\mu(\theta) = w(\theta)\,\frac{d\theta}{2\pi} + d\mu_{\rm s} \tag{3.13}$$

where

$$w\biggl(\arccos\biggl(\frac{x}{2}\biggr)\biggr) = c(4-x^2)^{-1/2} f(x) \tag{3.14}$$

for suitable c and $d\mu_{\rm s}$. The Verblunsky coefficients α_n for $d\mu$ and Jacobi parameters for $d\rho$ are related by ([2, 11]; [19, Section 13.2])

$$b_{n+1} = \alpha_{2n} - \alpha_{2n+2} - \alpha_{2n+1}(\alpha_{2n} + \alpha_{2n+2}) \tag{3.15}$$

$$a_{n+1}^2 - 1 = \alpha_{2n+1} - \alpha_{2n+3} - \alpha_{2n+2}^2(1-\alpha_{2n+3})(1+\alpha_{2n+1}) - \alpha_{2n+3}\alpha_{2n+1} \tag{3.16}$$

and the Jost function for $d\rho$ and Szegő function for $d\mu$ by

$$u(z) = (1-|\alpha_0|^2)(1-\alpha_1)D(z)^{-1} \tag{3.17}$$

From this, it is easy to derive Theorem 3.2 from Theorem 3.1.

To understand the situation when J has bound states, we note the analytic continuation of (1.16) says

$$u(z)u\biggl(\frac{1}{z}\biggr)\biggl[M(z) - M\biggl(\frac{1}{z}\biggr)\biggr] = z - z^{-1} \tag{3.18}$$

(this uses also u, M real on $\mathbb{R}$). Recall that if $z_0 \in \mathbb{D}$ is such that $z_0 + z_0^{-1}$ is an eigenvalue of J, then $u(z_0) = 0$. An argument shows that if $|z_0| > R^{-1}$ and $|a_n - 1| + |b_n| \leq CR^{-2n}$, then $u(z_0^{-1}) \neq 0$ and $M(1/z)$ is regular at z_0. Thus, (3.18) implies a relation between $u'(z_0)$, $u(1/z_0)$, and the residue of the pole of $M(z)$ at z_0. This leads to

Definition. Suppose u is analytic in $\{z \mid |z| < R\}$ for some $R > 1$ and $z_0 \in \mathbb{D}$ with $u(z_0) = 0$ and $|z_0| > R^{-1}$. We say the weight of the point mass at $z_0 + z_0^{-1}$ is canonical if

$$\lim_{z \to z_0} (z - z_0) M(z_0) = (z_0 - z_0^{-1}) \left[u'(z_0) u\left(\frac{1}{z_0}\right)\right] \tag{3.19}$$

Theorem 3.3 ([6])**.** *Fix $R > 1$. Then* (3.13) *holds if and only if*

(i) *$u(z)$ has an analytic continuation to $\{z \mid |z| < R\}$.*
(ii) *The point mass at each $z_0 \in \mathbb{D}$ with $|z_0| > R^{-1}$ and $u(z_0) = 0$ is a canonical weight.*

If u is entire and has m zeros in $\mathbb{D}$, the set of measures with that u has dimension $m - 1$. A single point on this space has decay at rate faster than any exponential. Similarly, if u is a polynomial, $\{a_n - 1, b_n\}$ has finite support if and only if all weights are canonical.

4. Detailed Asymptotics

Let S be defined by

$$S(z) = -\sum_{j=0}^{\infty} \alpha_{j-1} z^j \tag{4.1}$$

where $\alpha_{-1} = -1$. Of course, when D exists, both $D(z)^{-1}$ and $S(z)$ are analytic near $z = 0$. Theorem 3.1 can be rephrased.

Theorem 4.1. *The Taylor series of $D(z)^{-1}$ and $S(z)$ have the same radius of convergence.*

Barrios, López, and Saff [1] extend this to show $S(z)$ is meromorphic in $\{z \mid |z| < R + \varepsilon\}$ with a single simple pole at $z = R$ if and only if $D(z)^{-1}$ is meromorphic in a similar region. This condition on S is, of course, equivalent to

$$\alpha_n = CR^{-n} + O(R^{-n(1+\delta)}) \tag{4.2}$$

which is how they phrased their result. To go further, it is useful to define

$$r(z) = \overline{D(1/\bar{z})}\, D(z)^{-1} \tag{4.3}$$

which is analytic in $\{z \mid 1 - \varepsilon < |z| < R\}$ if (3.1) holds. Simon [18] proved that $r(z) - S(z)$ is analytic in $\{z \mid 1 - \varepsilon < |z| < R^2\}$ when (3.1) holds, thereby generalizing [1]. The ultimate result of this genre was found independently by Deift–Ostensson [7] and Martínez-Finkelshtein et al. [13]; an alternate proof was then found by Simon [20].

Theorem 4.2. *If* (3.1) *holds for some* $R > 1$*, then* $r(z) - S(z)$ *is analytic in* $\{z \mid 1 - \varepsilon < |z| < R^3\}$.

This is optimal in that there are examples [13, 20] where S (and r) have a simple pole at $z = R$ but $S - r$ has a pole at $z = R^3$.

Motivated by this, Simon [20] proved:

Theorem 4.3. $S(z)$ *is an entire meromorphic function if and only if* $D(z)^{-1}$ *is.*

One can even relate the poles. Given a set S in $\{z \mid |z| > 1\}$ which is discrete, one defines $\mathbb{G}(S)$ to be the set of all products $z_1 \dots z_{n+1} \bar{z}_{n+2} \dots \bar{z}_{2n+1}$ where $z_j \in S$. Then

Theorem 4.4 ([20])**.** *Let* $S(z)$ *be entire meromorphic and let* P *be the poles of* $D(z)^{-1}$ *and* T *the set of poles of* $S(z)$*. Then* $P \subset \mathbb{G}(T)$ *and* $T \subset \mathbb{G}(P)$.

Simon [21] studies analogs of the results for OPRL. In the Jacobi case, define

$$B(z) = 1 - \sum_{n=0}^{\infty} [b_{n+1} z^{2n+1} + (a_{n+2} - 1) z^{2n+2}] \tag{4.4}$$

The analog of Theorem 4.2 is

Theorem 4.5 ([21])**.** *Suppose* $R > 1$ *and*

$$\limsup_{n\to\infty} \left(|a_n^2 - 1| + |b_n|\right)^{1/2n} = R^{-1}$$

Then $(1 - z^2)u(z) + z^2 u(1/z) B(z)$ *is analytic in* $\{z \mid R^{-1} < |z| < R^2\}$.

As explained there, R^2 is optimal. The analog of Theorem 4.3 is

Theorem 4.6. $B(z)$ *is an entire meromorphic function if and only if* $u(z)$ *is.*

The connection between poles, that is, the analog of Theorem 4.4 is complicated but appears in [21].

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