A CUSPIDALITY CRITERION FOR THE FUNCTORIAL PRODUCT ON $\mathbf{GL(2)\times GL(3)}$, WITH A COHOMOLOGICAL APPLICATION

DINAKAR RAMAKRISHNAN AND SONG WANG

1. Introduction

A strong impetus for this paper came, at least for the first author, from a question of Avner Ash, asking whether one can construct non-selfdual, non-monomial cuspidal cohomology classes for suitable congruence subgroups Γ of $\mathrm{SL}(n,\mathbb{Z})$, say for $n=6$. Such a construction, in special examples, has been known for some time for $n=3$ ([AGG1984], [vGT1994], [vGKTV1997], [vGT2000]); it is of course not possible for $n=2$. One can without trouble construct non–selfdual, *monomial* classes for any $n=2m$ with $m\geq 2$, not just for constant coefficients (see the Appendix, Theorem E). In the Appendix we also construct non-monomial, non-selfdual classes for $n=4$ using the automorphic induction to $\mathbb{Q}$ of suitable Hecke character twists of non-CM cusp forms of "weight 2" over imaginary quadratic fields, but they admit quadratic self-twists and are hence imprimitive. The tack pursued in the main body of this paper, and which is the natural thing to do, is to take a non-selfdual (non-monomial) $n=3$ example π, and take its functorial product $\boxtimes$ with a cuspidal π' on $\mathrm{GL}(2)/\mathbb{Q}$ associated to a holomorphic newform of weight 4 for a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. The resulting (cohomological) $n=6$ example can be shown to be *non-selfdual* for suitable π'. (This should be the case for all π', but we cannot prove this with current technology – see Remark 4.1.) Given that, the main problem is that it is not easy to show that such an automorphic tensor product $\Pi:=\pi\boxtimes\pi'$, whose modularity was established in the recent deep work of H. Kim and F. Shahidi ([KSh2002-1]), is *cuspidal*. This has led us to prove a precise cuspidality criterion (Theorem A) for this product, not just for those of cohomological type, which hopefully justifies the existence of this paper. The second author earlier proved such a criterion when π is a twist of the symmetric square of a cusp form on $\mathrm{GL}(2)$ ([Wa2003]; such forms are essentially selfdual, however, and so do not help towards the problem of constructing non-selfdual classes. One of the reasons we are able to prove the criterion *in general* is the fact that the associated, degree 20 exterior cube L-function is nicely behaved and analyzable. This helps us rule out, when the forms on $\mathrm{GL}(2)$ and $\mathrm{GL}(3)$ are non-monomial, the possible decomposition of Π into an isobaric sum of two cusp forms on GL(3) (see section 7). This is the heart of the matter.

We will also give a criterion (Theorem B) as to when the base change of Π to a solvable Galois extension remains cuspidal. We will derive a stronger result for the cohomological examples (Theorem C), namely that each of them is *primitive*, i.e., not associated to a cusp form on $\mathrm{GL}(m)/K$ for *any* (possibly non-normal) extension $K/\mathbb{Q}$ of degree $d > 1$ with $dm = 6$. Furthermore, each of the three main non-selfdual GL(3) examples π of [vGT1994], [vGKTV1997] and [vGT2000] comes equipped, confirming a basic conjecture of Clozel ([C1988]), with a certain 3-dimensional ℓ-adic representation ρ_ℓ whose Frobenius traces $a_p(\rho_\ell)$ agree with the Hecke eigenvalues $a_p(\pi)$ for small p. For π' on $\mathrm{GL}(2)/\mathbb{Q}$ defined by a suitable holomorphic newform of weight 4, with associated Galois representation ρ'_ℓ, we will show (Theorem D) that the six-dimensional $R_\ell := \rho_\ell \otimes \rho'_\ell$, which should correspond to Π, remains irreducible under restriction to *any* open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

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2. The Cuspidality Criterion

Throughout this paper, by a cusp form on $GL(n)$ (over a global field F) we will mean an irreducible, cuspidal automorphic representation $\pi = \pi_\infty \otimes \pi_f$ of $GL_n(\mathbb{A}_F)$. We will denote its central character by ω_π. One says that π is *essentially self-dual* iff its contragredient $\pi^\vee$ is isomorphic to $\pi \otimes \nu$ for some character ν of (the idele classes of) F; when $n = 2$, one always has $\pi^\vee \simeq \pi \otimes \omega_\pi^{-1}$. Note that π is *unitary* iff $\pi^\vee$ is the *complex conjugate representation* $\overline{\pi}$. Given any cusp form π, we can find a real number t such that $\pi_u := \pi \otimes |.|^t$ is unitary.

For any cusp form π' on $GL(2)$, put $Ad(\pi') = \mathrm{sym}^2(\pi') \otimes \omega_{\pi'}^{-1}$ and $A^4(\pi') = \mathrm{sym}^4(\pi') \otimes \omega_{\pi'}^{-2}$. Recall that π' is *dihedral* iff it admits a self-twist by a quadratic character; it is *tetrahedral*, resp. *octahedral*, iff $sym^2(\pi')$, resp. $sym^3(\pi')$, is cuspidal and admits a self-twist by a cubic, resp. quadratic, character. (The automorphy of $sym^3(\pi')$ was shown by Kim and Shahidi in [KSh2002-1].) We will say that π' is of *solvable polyhedral type* iff it is dihedral, tetrahedral or octahedral.

Theorem A. *Let π', π be cusp forms on $GL(2)$, $GL(3)$ respectively over a number field F. Then the Kim–Shahidi transfer $\Pi = \pi \boxtimes \pi'$ on $GL(6)/F$ is cuspidal unless one of the following happens:*

(a) π' *is not dihedral, and* π *is a twist of* $Ad(\pi')$*;*
(b) π' *is dihedral,* $L(s,\pi) = L(s,\chi)$ *for an idele class character* χ *of a cubic, non-normal extension* K *of* F*, and the base change* π_K *is Eisensteinian.*

Furthermore, when (a) *(resp.* (b)*) happens,* Π *possesses an isobaric decomposition of type* $(2,4)$ *or* $(2,2,2)$ *(resp. of type* $(3,3)$*). More precisely, when we are in case* (a)*,* Π *is of type* $(2,2,2)$ *if* π' *is tetrahedral, and* $(2,4)$ *otherwise.*

Remark: By [KSh2002-1], $\Pi = \pi \boxtimes \pi'$ is automorphic on $\mathrm{GL}(6)/F$, and its L–function agrees with the Rankin–Selberg L–function $L(s, \pi \times \pi')$. Theorem A implies in particular that Π is cuspidal if (i) π' is not dihedral *and* (ii) π is not a twist of $Ad(\pi')$.

A partial cuspidality criterion was proved earlier by the second author in [Wa2003]; but he only treated the case when π is twist equivalent to the Gelbart–Jacquet symmetric square transfer of some cusp form on $GL(2)$.

Theorem B. *Let* F *be a number field and* π', π *be cusp forms on* $GL(2)/F$*,* $GL(3)/F$ *respectively. Put* $\Pi = \pi \boxtimes \pi'$*. Assume that* π' *is not of solvable polyhedral type, and* π *not essentially selfdual. Then we have the following:*
(a) *If* π *does not admit any self twist,* Π *is cuspidal without any self twist. Furthermore, if* π *is not monomial, then* Π *is not induced from any non-normal cubic extension.*
(b) *If* π *is not of solvable type, i.e., its base change to any solvable Galois extension is cuspidal,* Π *is cuspidal and not of solvable type; in particular, there is no solvable extension* K/F *of degree* $d > 1$ *dividing* 6*, and a cuspidal automorphic representation* η *of* $GL_{6/d}(\mathbb{A}_F)$*, such that* $L(s,\Pi) = L(s,\eta)$*.*

Remark: If π is regular algebraic at infinity, and F is not totally imaginary, then π is not monomial (See Lemma 9.3).

We will prove Theorem A in sections 6 through 8, and Theorem B in section 9.
Before proceeding with the proofs of these theorems, we will digress and discuss the cohomological application.

3. Preliminaries on cuspidal cohomology

The experts can skip this section and go straight to the statement (in section 4) and the proof (in section 5) of Theorems C, D. Let

$$\Gamma \subset \mathrm{SL}(n, \mathbb{Z}),$$

be a congruence subgroup of $G_n^0 := \mathrm{SL}(n,\mathbb{R})$, which has finite covolume. Γ acts by left translation on the symmetric space $X_n^0 := \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. The cohomology of Γ is the same as that of the locally symmetric orbifold $\Gamma\backslash X_n^0$.

If $H^*_{\rm cont}$ denotes the *continuous group cohomology*, a version of Shapiro's lemma gives an isomorphism

$$H^*(\Gamma, \mathbb{C}) \simeq H^*_{\rm cont}(G_n^0, \mathcal{C}^\infty(\Gamma\backslash G_n^0)).$$

The constant functions are in this space, and the contribution of $H^*_{\rm cont}(G_n^0, \mathbb{C})$ to $H^*(\Gamma, \mathbb{C})$ is well understood and plays an important role in Borel's interpretation of the values of the Riemann zeta function $\zeta(s)$ at negative integers.

We will be interested here in another, more mysterious, piece of $H^*(\Gamma, \mathbb{C})$, namely its *cuspidal part*, denoted $H^*_{\rm cusp}(\Gamma, \mathbb{C})$, which injects into $H^*(\Gamma, \mathbb{C})$ by a theorem of Borel. Furthermore, one knows by L. Clozel ([C1988]) that the cuspidal summand is defined over $\mathbb{Q}$, preserved by the Hecke operators. The cuspidal cohomology is represented by cocycles defined by smooth cusp forms in $L^2(\Gamma\backslash G_n^0)$, i.e., one has

$$H^*_{\rm cusp}(\Gamma, \mathbb{C}) = H^*_{\rm cont}(G_n^0, L^2_{\rm cusp}(\Gamma\backslash G_n^0)^\infty),$$

where $L^2_{\rm cusp}(\Gamma\backslash G_n^0)$ denotes the space of cusp forms, and the superscript ∞ signifies taking the subspace of smooth vectors. If $\mathfrak{G}_n$ denotes the *complexified* Lie algebra of G_n, the passage from continuous cohomology to the *relative Lie algebra cohomology* ([BoW1980]) furnishes an isomorphism

$$H^*_{\rm cusp}(\Gamma, \mathbb{C}) \simeq H^*(\mathfrak{G}_n^0, K : L^2_{\rm cusp}(\Gamma\backslash G_n^0)^\infty).$$

It is a standard fact (see [BoJ1979], for example) that the right action of G_n on $L^2_{\rm cusp}(\Gamma\backslash G_n^0)$ is completely reducible, and so we may write

$$L^2_{\rm cusp}(\Gamma\backslash G_n^0) \simeq \hat{\oplus}_\pi \, m_\pi \mathcal{H}_\pi,$$

where π runs over the irreducible unitary representations of G_n^0 (up to equivalence), $\mathcal{H}_\pi$ denotes the space of π, $\hat{\oplus}$ signifies taking the Hilbert direct sum, and m_π is the multiplicity. Consequently,

$$H^*_{\rm cusp}(\Gamma, \mathbb{C}) \simeq \oplus_\pi \, H^*(\mathfrak{G}^0_{n,\mathbb{C}}, K; \mathcal{H}_\pi^\infty)^{m_\pi}.$$

One knows completely which representations π of G_n^0 have non-zero $(\mathfrak{G}_n^0, K)$-cohomology ([VZ1984]; see also [Ku1980]). An immediate consequence (see [C1988], page 114) is the following (with $[x]$ denoting, for any $x \in \mathbb{R}$, the integral part of x):

Theorem 3.1.

$$H^i_{\rm cusp}(\Gamma, \mathbb{C}) = 0 \quad \text{unless} \quad d(n) \leq i \leq d(n) + [(n-1)/2],$$

where

$$d(n) = m^2 \quad \text{if} \quad n = 2m \quad \text{and} \quad d(n) = m(m+1) \quad \text{if} \quad n = 2m+1.$$

It will be necessary for us to work with the $\mathbb{Q}$-group $G_n := \mathrm{GL}(n)$ with center Z_n, and also work adelically. Let $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ be the adele ring of $\mathbb{Q}$,

$K_\infty = O(n)$, and $X_n = G_n(\mathbb{R})/K_n$, whose connected component is X_n^0. For any compact open subgroup K of $G_n(\mathbb{A}_f)$, we have

$$S_K := G_n(\mathbb{Q})Z_n(\mathbb{R})^0\backslash G_n(\mathbb{A})/K_\infty K \simeq \cup_{j=1}^r \Gamma_j\backslash X_n^0, \tag{3.6}$$

where the Γ_j are congruence subgroups of $\mathrm{SL}(n,\mathbb{Z})$ and $Z_n(\mathbb{R})^0$ is the Euclidean connected component of $Z_n(\mathbb{R})$. We need the following, which follows easily from the discussion in section 3.5 of [C1988]:

Theorem 3.2. (i)

$$H^*_{\mathrm{cusp}}(S_K,\mathbb{C}) \simeq \oplus_{\pi\in\mathrm{Coh}_K} H^*(\tilde{\mathfrak{G}}_{n,\infty},K_\infty;\pi_\infty)\otimes\pi_f^K,$$

where $\tilde{\mathfrak{G}}_{n,\infty}$ *consists of matrices in* $M_n(\mathbb{C})$ *with purely imaginary trace, and* Coh_K *is the set of (equivalence classes) of cuspidal automorphic representations* $\pi = \pi_\infty\otimes\pi_f$ *of* $G_n(\mathbb{A})$ *such that* $\pi_f^K \neq 0$, π_∞ *contributes to the relative Lie algebra cohomology, and* $(\omega_\pi)_\infty$ *is trivial on* $Z(\mathbb{R})^0$.

(ii) *Suppose* $\pi = \pi_\infty\otimes\pi_f$ *is a cuspidal automorphic representation of* $G_n(\mathbb{A})$ *with* $\pi_f^K \neq 0$ *such that the restriction* r_∞ *of the Langlands parameter of* π_∞ *to* $\mathbb{C}^*$ *is given by the* n*-tuple*

$$\{(z/|z|)^{n-1},(\overline{z}/|z|)^{n-1},(z/|z|)^{n-3},(\overline{z}/|z|)^{n-3},\\ \ldots,(z/|z|),(\overline{z}/|z|)\}\otimes(z\overline{z})^{n-1}$$

if n *is even, and*

$$\{(z/|z|)^{n-1},(\overline{z}/|z|)^{n-1},(z/|z|)^{n-3},(\overline{z}/|z|)^{n-3},\\ \ldots,(z/|z|)^2,(\overline{z}/|z|)^2,1\}\otimes(z\overline{z})^{n-1}$$

if n *is odd. Then* π *contributes to* Coh_K *in degree* $d(n)$.

□

Given any cohomological π as above, the fact that the cuspidal cohomology at any level K has a $\mathbb{Q}$-structure ([C1988]) preserved by the action of the Hecke algebra $\mathcal{H}_\mathbb{Q}(G_f,K)$ (consisting of $\mathbb{Q}$-linear combinations of K-double cosets), implies that the G_f-module π_f is *rational over a number field* $\mathbb{Q}(\pi_f)$. When $n=2$, such a π is defined by a holomorphic newform h of weight 2, and then $\mathbb{Q}(\pi_f)$ is none other than the field generated by the Fourier coefficients of h.

4. Non-selfdual, cuspidal classes for $\Gamma\subset\mathrm{SL}(6,\mathbb{Z})$

The principle of functoriality predicts that given cuspidal automorphic representations π,π' of $G_n(\mathbb{A}),G_m(\mathbb{A})$ respectively, there exists an isobaric automorphic representation $\pi\boxtimes\pi'$ of $G_{nm}(\mathbb{A})$ such that for every place v of $\mathbb{Q}$, one has

$$\sigma((\pi\boxtimes\pi')_v) \simeq \sigma(\pi_v)\otimes\sigma_v(\pi'_v),$$

where σ is the map (up to isomorphism) given by the local Langlands correspondence ([HaT2000], [He2000]) from admissible irreducible representations of $G_r(\mathbb{Q}_v)$ to r-dimensional representations of W'_v, which is the real Weil group $W_{\mathbb{R}}$ if $v = \infty$ and the extended Weil group $W_{\mathbb{Q}_p} \times \mathrm{SL}(2, \mathbb{C})$ if v is defined by a prime number p.

This prediction is known to be true for $n = m = 2$ ([Ra2000]), More importantly for the matter at hand, it is also known for $(n, m) = (3, 2)$ by a difficult theorem of H. Kim and F. Shahidi ([KSh2002-1]).

Put
$$T = T_1 \cup T_2,$$
with
$$T_1 = \{53, 61, 79, 89\} \quad \text{and} \quad T_2 = \{128, 160, 205\}.$$
By the article [AGG1984] of Ash, Grayson and Green (for $p \in T_1$), and the works [vGT1994], [vGT2000], [vGKTV1997] of B. van Geemen, J. Top, et al (for $p \in T_2$), one knows that for every $q \in T$, there is a non-selfdual cusp form $\pi(q)$ on $\mathrm{GL}(3)/\mathbb{Q}$ of level q, contributing to the (cuspidal) cohomology (with constant coefficients) .

Theorem C. *Let π' be a cusp form on $GL(2)/\mathbb{Q}$ defined by a non-CM holomorphic newform g of weight 4, level N, trivial character, and field of coefficients K. Let π denote an arbitrary cusp form on $GL(3)/\mathbb{Q}$ contributing to the cuspidal cohomology in degree 2, and let $\pi(q)$, $q \in T$, be one of the particular forms discussed above. Put $\Pi = \pi \boxtimes \pi'$ and $\Pi(q) = \pi(q) \boxtimes \pi'$. Then*

(a) Π *contributes to the cuspidal cohomology of $GL(6)$.*

(b) $\Pi(q)$ *is not essentially selfdual when $N \leq 23$ and $K = \mathbb{Q}$.*

(c) *If N is relatively prime to q, then the level of $\Pi(q)$ is N^3q^2. Now let $N \leq 23$ and $K = \mathbb{Q}$. Then $\Pi(q)$ does not admit any self-twist. Moreover, there is no cubic non-normal extension $K/\mathbb{Q}$ with a cusp form η on $GL(2)/K$ such that $L(s, \Pi(q)) = L(s, \eta)$, nor is there a sextic extension (normal or not) $E/\mathbb{Q}$ with a character λ of E such that $L(s, \Pi(q)) = L(s, \lambda)$.*

We note from the *Modular forms database* of William Stein ([WSt2003]) that there exist newforms g of weight 4 with $\mathbb{Q}$-coefficients, for instance for the levels $N = 5, 7, 13, 17, 19, 23$.

Remark 4.1 Part (b) should be true for any π'. Suppose that for a given any cusp form π on $\mathrm{GL}(3)$, cohomological or not, the functorial product $\Pi = \pi \boxtimes \pi'$ satisfies $\Pi^\vee \simeq \Pi \otimes \nu$ for a character ν. Then at any prime p where $a_p(\pi') \neq 0$, which happens for a set of density 1, we can of course conclude that $a_p(\pi^\vee) = a_p(\pi)\nu(p)$. But this does not suffice, given the state of knowledge right now concerning the refinement of the strong multiplicity one theorem, to conclude that $\pi^\vee$ is isomorphic to a twist of π. In the case of the $\pi(q)$, we have information at a small set of primes and we have to make sure that $a_p(\pi') \neq 0$ *and* $\overline{a_p(\pi)} \neq a_p(\pi)$ for one of those p. The hypothesis that $K = \mathbb{Q}$ is made for convenience, however, and the proof will extend to any totally real field.

In [vGT1994], [vGT2000], [vGKTV1997] one finds in fact an algebraic surface $S(q)$ over $\mathbb{Q}$ for each $q \in T_2$, and a 3-dimensional ℓ-adic representation $\rho(q)$ (for any prime ℓ), occurring in $H^2_{\rm et}(S(q)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$, such that

$$(*) \qquad L_p(s, \rho(q)) \,=\, L_p(s, \pi(q)),$$

for all odd primes $p \leq 173$ not dividing q. Here is a conditional result.

Theorem D. *Let π' be a cusp form on $GL(2)/\mathbb{Q}$ defined by a non-CM holomorphic newform g of weight 4, level N, trivial character, and field of coefficients $\mathbb{Q}$, with corresponding $\overline{\mathbb{Q}}_\ell$-representation ρ'. Let $\pi(q), \rho(q), \Pi(q)$ be as above for $q \in T_2$. Put $R(q) = \rho(q) \otimes \rho'$. Suppose $(*)$ holds at all the odd primes p not dividing $q\ell$. Then $R(q)$ remains irreducible when restricted to any open subgroup of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.*

5. Proof of Theorems C, D modulo Theorems A, B

In this section we will assume the truth of Theorems A and B.

Proof of Theorem C: As π' is associated to a holomorphic newform of weight 4, we have

$$\sigma(\pi'_\infty)|_{\mathbb{C}^*} \,\simeq\, \big((z/|z|)^3 \oplus (\overline{z}/|z|)^3\big) \otimes (z\overline{z})^3.$$

And since π contributes to cohomology, we have (cf. part (ii) of Theorem 3.1)

$$\sigma(\pi_\infty)|_{\mathbb{C}^*} \,\simeq\, \big((z/|z|)^2 \oplus 1 \oplus (\overline{z}/|z|)^2\big) \otimes (z\overline{z})^2.$$

Since Π_∞ corresponds to the tensor product $\sigma(\pi_\infty) \otimes \sigma(\pi'_\infty)$, we get part (a) of Theorem C in view of Theorem A and part (ii) of Theorem 3.2.

Pick any q in T and denote by $\mathbb{Q}(\pi(q))$ the field of rationality of the finite part $\pi(q)_f$ of $\pi(q)$. Then it is known by [AGG1984] that for $q \in T_1$,

$$\mathbb{Q}(\pi(53)) = \mathbb{Q}(\sqrt{-11}),\ \mathbb{Q}(\pi(61)) = \mathbb{Q}(\sqrt{-3}),\ \mathbb{Q}(\pi(79)) = \mathbb{Q}(\sqrt{-15}),\ \mathbb{Q}(\pi(89)) = \mathbb{Q}(i),$$

while by [vGT1994], [vGKTV1997] and [vGT2000],

$$\mathbb{Q}(\pi(q)) \,=\, \mathbb{Q}(i), \ \ \forall\, q \in T_2.$$

By hypothesis, π' is non-CM, and by part (a), $\Pi(q)$ is cuspidal. Suppose there exists a character ν such that for some $q \in T$,

$$\Pi(q)^\vee \,\simeq\, \Pi(q) \otimes \nu.$$

Comparing central characters, we get $\nu^6 = 1$. We claim that $\nu^2 = 1$. Suppose not. Then there exists an element σ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing $\mathbb{Q}(\pi(q))$ such that $\nu \neq \nu^\sigma$. Since π' has $\mathbb{Q}$-coefficients and $\pi(q)$ has coefficients in $\mathbb{Q}(\pi(q))$, we see that $\Pi(q)_f$ must be isomorphic to the Galois conjugate $\Pi(q)_f^\sigma$, which exists because the cuspidal cohomology group has, by Clozel (see section 3), a $\mathbb{Q}$-structure preserved by the Hecke operators. If we put $\mu = \nu/\nu^\sigma \neq 1$, we then see that $\Pi(q) \simeq \Pi(q) \otimes \mu$. But we will see below that $\Pi(q)$ admits no non-trivial self-twist. This gives the desired contradiction, proving the claim. If ν is non-trivial, the quadratic extension $F/\mathbb{Q}$ it cuts out will need to have discriminant dividing $q^a N^b$ for suitable integers a, b. For any prime p which is unramified in F, we will have

$$\overline{a}_p(\pi) a_p(\pi') \,=\, \pm a_p(\pi) a_p(\pi').$$

For each $j \leq 3$ and for each π' with $N \leq 23$ and $K = \mathbb{Q}$, we can find, using the tables in [AGG1984], [vGT1994],[vGT2000], [vGKTV1997] and [WSt2003], a prime p such that $a_p(\pi') \neq 0$, $\nu(p) \neq 0$ and $\overline{a}_p(\pi) \neq a_p(\pi)$. This proves part (b) of Theorem C.

When N is relatively prime to q, the conductor of $\Pi(q)$ must be N^3q^2 as can be seen from the way epsilon factors change under twisting (see section 4 of [BaR1994] for example).

From now on, let $N \leq 23$ and $K = \mathbb{Q}$. One knows that as π' is holomorphic and not dihedral, the associated Galois representation ρ' remains irreducible when restricted to any open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It follows that the base change of π' to any solvable Galois extension remains cuspidal. In particular, it is not of solvable polyhedral type. We claim that $\pi(q)$ is not monomial. Indeed, the infinite type of $\pi(q)$ is regular algebraic [C1988], and to be monomial there needs to be a cubic, possibly non-normal, extension $K/\mathbb{Q}$ which can support an algebraic Hecke character which is *not* a finite order character times a power of the norm. By [We1955], for such a character to exist, K must contain a CM field, i.e., a totally imaginary quadratic extension of a totally real field, which forces K to be imaginary. But any cubic extension of $\mathbb{Q}$ has a real embedding, and this proves our claim. Note also that as $\pi(q)$ is not essentially self-dual, it is not a twist of the symmetric square of any cusp form, in particular π', on $\mathrm{GL}(2)/\mathbb{Q}$. Now it follows from Theorem B that $\Pi(q)$ does not admit any self-twist.

Suppose K is a non-normal cubic field together with a cusp form η on $\mathrm{GL}(2)/K$ such that $L(s, \Pi(q)) = L(s, \eta)$. Let L be the Galois closure of K (with Galois group S_3), and let E be the quadratic extension of $\mathbb{Q}$ contained in L. Then $\Pi(q)_E$ will be cuspidal and automorphically induced by the cusp form η_L of $\mathrm{GL}(3, \mathbb{A}_L)$. In other words, $\Pi(q)_E$ admits a non-trivial self-twist. To contradict this, it suffices, in view of Theorem B, to show that $\pi(q)_E$ admits no self-twist relative to L/E, i.e., that $\pi(q)_E$ is not automorphically induced by a character μ of L. But as noted above, this forces L to be a totally imaginary number field containing a CM field L_0. Then either

$L = L_0$ or $L_0 = E$. In the latter case, by [We1955], μ will be a finite order character times the pullback by norm of a character μ_0 of E, forcing $I_L^E(\mu)$ to be *not* regular at infinity, and so this case cannot happen. So L itself must be a CM field, with its totally real subfield F. Then $\mathrm{Gal}(F/\mathbb{Q})$ would be cyclic of order 3 and a quotient of S_3, which is impossible. So this case does not arise either. So $\pi(q)_E$ doe snot admit any self-twist, and $\Pi(q)$ is not associated to any η as above.

Now suppose $L(s, \Pi(q)) = L(s, \lambda)$ for a character λ of a sextic field L. If L contains a proper subfield $M \neq \mathbb{Q}$, then since $m := [L : M] \leq 3$, one can induce λ to M and get an automorphic representation β of $\mathrm{GL}_m(\mathbb{A}_M)$ such that $L(s, \lambda) = L(s, \beta) = L(s, \Pi(q))$, which is impossible by what we have seen above. So L must not contain any such M. But on the other hand, since $\Pi(q)_\infty$ is algebraic and regular, we need L to contain, by [We1955], a CM subfield L_0, and hence also its totally real subfield F. Either $F = \mathbb{Q}$, in which case L_0 is imaginary quadratic, or $F \neq \mathbb{Q}$. Either way there will be a proper subfield M of degree ≤ 3, and so the purported equality $L(s, \Pi(q)) = L(s, \lambda)$ cannot happen. We are now done with the proof of Theorem C.

Proof of Theorem D: By assumption, the ℓ-adic representation ρ is functorially associated to the cuspidal cohomological form $\pi(q)$ on $\mathrm{GL}(3)/\mathbb{Q}$ with $q \in T_2$.

Lemma 5.1. *ρ is irreducible under restriction to any open subgroup.*

Proof. Suffices to show that the restriction ρ_E to $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ is irreducible for any finite *Galois* extension $E/\mathbb{Q}$. Pick any such extension and write $G = \mathrm{Gal}(E/\mathbb{Q})$. Suppose ρ_E is reducible. Then we have *either*

(i) $\rho_E \simeq \tau \oplus \chi$ with τ irreducible of dimension 2 and χ of dimension 1; *or*

(ii) $\rho_E \simeq \chi_1 \oplus \chi_2 \oplus \chi_3$, with each χ_j one-dimensional.

Let V be the 3-dimensional $\overline{\mathbb{Q}}_\ell$-vector space on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via ρ. Suppose we are in case(i), so that there is a line L in V preserved by $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ and acted upon by χ. Note that G acts on $\{\tau, \chi\}$ and, by the dimension consideration, it must preserve $\{\chi\}$. Hence the line L is preserved by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which contradicts the fact that ρ is irreducible.

So we may assume that we are in case (ii). We claim that $\chi_i \neq \chi_j$ if $i \neq j$. Indeed, since ρ arises as (the base change to $\overline{\mathbb{Q}}_\ell$ of) a summand of the ℓ-adic cohomology of a smooth projective variety, it is Hodge-Tate, and so is each χ_j. So each χ_j is locally algebraic and corresponds to an algebraic Hecke character χ'_j of E. By the identity of the L-functions, we will have $L^S(s, \pi) = \prod_j L^S(s, \chi'_j)$ for a suitable finite set S of places S. By the regularity of π, each χ'_j must appear with multiplicity one, which proves the claim. Now let L_j denote, for each $j \leq 3$, the (unique) line in V stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$

and acted upon by χ_i. And G acts by permutations on the set $\{\chi_1, \chi_2, \chi_3\}$. In other words, there is a representation $r : G \to S_3$ such that the G-action is via r. Put $H = \mathrm{Ker}(r)$, with corresponding intermediate field M. Then each L_j is stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$, so that $\rho_M \simeq \nu_1 \oplus \nu_2 \oplus \nu_3$, where each ν_j is a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/M)$. Also, $M/\mathbb{Q}$ is Galois with $\mathrm{Gal}(M/\mathbb{Q}) \subset S_3$. But from the proof of Theorem C that the base change π_M of π to any such M is cuspidal. However, if ν_j^1 denotes the algebraic Hecke character of M defined by ν_j, the twisted L-function $L^S(s, \pi_M \otimes {\nu'_j}^{-1})$ will have a pole at $s = 1$, leading to a contradiction. We have now proved Lemma 5.1.

□

Note that Lemma 1 implies in particular that for any finite extension $F/\mathbb{Q}$, ρ_F does not admit any self-twist.

Lemma 5.2. *For any finite extension $E/\mathbb{Q}$, the restriction ρ_E is not essentially self-dual.*

Proof. Again we may assume that $E/\mathbb{Q}$ is Galois with group G. As before let V denote the space of ρ, and suppose that we have an isomorphism $\rho \simeq \rho^\vee \otimes \nu$, for a character ν. Then there is a line L in $V \otimes V$ on which $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ acts via ν. By Schur's lemma (and this is why we have to work over $\overline{\mathbb{Q}}_\ell$), the trivial representation appears with multiplicity one in $V \otimes V^\vee$. It implies that ν must appear with multiplicity one in $V \otimes V$. We claim that $V \otimes V$ contains no other character. Indeed, if we have another character ν', we would have $\rho \simeq \rho \otimes \mu$, where $\mu = \nu/\nu'$. But as noted above, ρ_E admits no self-twist, and so $\mu = 1$, and the claim is proved. Consequently, the action of G on $V \otimes V$ must preserve ν. In other words, the line L is stable under all of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, contradicting the fact that ρ is not essentially self-dual. Done

□

Now consider $R = \rho \otimes \rho'$. We know that both ρ and ρ' remain irreducible upon restriction to any open subgroup and moreover, such a restriction of ρ is *not* essentially self-dual. It then follows easily that the restriction of R is irreducible.

This finishes the proof of Theorem D.

□

6. Proof of Theorem A, Part #1

By twisting we may assume that π, π' are unitary, so that $\pi^\vee \simeq \overline{\pi}$ and ${\pi'}^\vee \simeq \overline{\pi'}$, with respective central characters ω, ω'.

Now we proceed in several steps. Applying Langlands's classification, ([La1979-1], [La1979-2], [JS1981]), we see that the Kim-Shahidi product $\Pi = \pi \boxtimes \pi'$ must be an isobaric sum of cusp forms whose degrees add up to 6. Thanks to the Clebsch-Gordon decomposition

$$\mathrm{sym}^2(\pi') \boxtimes \pi' \simeq \mathrm{sym}^3(\pi') \boxplus (\pi' \otimes \omega'),$$

Π is not cuspidal if π is a twist of $\mathrm{sym}^2(\pi')$.

The list of all the cases when Π is not cuspidal is the following:

Case I: Π *has a constituent of degree* 1*, i.e.,* $\Pi = \lambda \boxplus \Pi'$ *for some idele class character* λ *and some automorphic representation* Π' *of* $GL(5)$*.*

Case II: Π *has a constituent of degree* 2*, i.e.,* $\Pi = \tau \boxplus \Pi'$ *for some cusp form* τ *on* $GL(2)$ *and some automorphic representation* Π' *of* $GL(4)$*.*

Case III: Π *is an isobaric sum of two cusp forms* σ_1 *and* σ_2 *on* $GL(3)$*.*

We first deal with Cases I and II. We need some preliminaries. First comes the following basic result due to H. Jacquet and J.A. Shalika ([JS1981], [JS1990], and R. Langlands ([La1979-1], [La1979-2]).

Lemma 6.1. (i) *Let* Π*,* τ *be isobaric automorphic representations of* $GL_n(\mathbb{A}_F)$*,* $GL_m(\mathbb{A}_F)$ *respectively. Assume that* τ *is cuspidal. Then the order of the pole of* $L(s, \Pi \otimes \overline{\tau})$ *at* $s = 1$ *is the same as the multiplicity of* τ *occurring in the isobaric sum decomposition of* Π*.*

(ii) $L(s, \Pi \times \overline{\Pi})$ *has a pole at* $s = 1$ *of order* $m = \sum_i m_i^2$ *if* $\Pi = \boxplus_i m_i \pi_i$ *is the isobaric decomposition of* Π*, with the* π_i *being inequivalent cuspidal representations of smaller degree. In particular,* $m = 1$ *if and only if* Π *is cuspidal.*

An L–function $L(s)$ is said to be nice if it converges on some right half plane, admits an Euler product of some degree m, say, and extends to a meromorphic function of finite order with no pole outside $s = 1$, together with a functional equation related to another L–function $L^\vee(s)$ given by

$$L(s) = W(d_F^m N)^{1/2-s} L^\vee(1-s),$$

where W is a non-zero scalar.

If π_1, π_2 are automorphic forms on $GL(m), GL(n)$ respectively, then the Rankin-Selberg L-function $L(s, \pi_1 \times \pi_2)$ is known to be nice ([JPSS1983], [MW1995], [Sh1988]). Of course, the product of two nice L–functions is nice. Furthermore, we recall the following Tchebotarev-like result for nice L–functions ([JS1990]):

Lemma 6.2. *Let* $L_1(s) = \prod_v L_{1,v}(s)$ *and* $L_2(s) = \prod_v L_{2,v}(s)$ *be two* L*–functions with Euler products, and suppose that they are both of exactly one of the following types:*

(a) $L_i(s)$ *is an Artin* L*–function of some Galois extension;*

(b) $L_i(s)$ *is attached to an isobaric automorphic representation;*

(c) $L_i(s)$ *is a Rankin–Selberg* L*–function of two isobaric automorphic representations.*

If $L_{1,v}(s) = L_{2,v}(s)$ *for all but finite places* v *of* F*, then* $L_1(s) = L_2(s)$*.*

Proof of Theorem A for Cases I and II.

Firstly, Case I can never happen. The reason is the following: If λ is a constituent of $\Pi = \pi \boxtimes \pi'$, then $L(s, \Pi \times \bar{\lambda})$ has a pole at $s = 1$ (Lemma 6.1), hence so does

$$L(s, \pi' \times \pi \otimes \bar{\lambda}) = L(s, \pi \boxtimes \pi' \otimes \bar{\lambda}).$$

However, π' and $\pi \otimes \bar{\lambda}$ are cuspidal of different degrees, hence $L(s, \pi' \times \pi \otimes \bar{\lambda})$ is entire, and we get the desired contradiction.

Now we treat Case II, where Π has a constituent τ of degree 2. We will show that this can happen IF AND ONLY IF π is twist equivalent to $\mathrm{sym}^2(\pi')$ in which case τ is twist equivalent to π'.

In fact, for each finite v where π and π' are unramified,

$$L_v(s, \Pi \otimes \bar{\tau}) = L_v(s, \pi \times (\pi' \boxtimes \bar{\tau})),$$

where $\pi' \boxtimes \bar{\tau}$ is the functorial product of π' and $\bar{\tau}$ whose modularity (in GL(4)) was established in [Ra2000]. One may check the following: If $\pi'_v = \alpha_{v,1} \boxplus \alpha_{v,2}$, $\pi_v = \beta_{v,1} \boxplus \beta_{v,2} \boxplus \beta_{v,3}$, and $\tau'_v = \gamma_{v,1} \boxplus \gamma_{v,2}$, then both sides of the equality is the same as $\prod_{i,j,k} L(s, \alpha_{v,i} \beta_{v,j} \bar{\gamma}_{v,k})$ where the product is over all i, j, k such that $1 \leq i, k \leq 2$ and $1 \leq j \leq 3$.

Hence by Lemma 6.2,

$$L(s, \Pi \otimes \bar{\tau}) = L(s, \pi \times (\pi' \boxtimes \bar{\tau}))$$

As τ is a constituent of Π, the L–functions on both sides above have a pole at $s = 1$. As π is cuspidal, this means by Lemma 6.1, $\bar{\pi}$ is a constituent of $\pi' \boxtimes \bar{\tau}$. Hence $\pi' \boxtimes \bar{\tau}$ should possess a constituent of degree 1, namely a character μ.

Thus $L(s, \pi' \times \bar{\tau} \otimes \bar{\mu}) = L(s, \pi' \boxtimes \bar{\tau} \otimes \bar{\mu})$ has a pole at $s = 1$, implying that π' is equivalent to $\tau \otimes \mu$. Hence

$$\pi' \boxtimes \bar{\tau} \cong \mu \boxplus Ad(\tau) \otimes \mu,$$

which means that $\pi \cong Ad(\tau) \otimes \mu \cong Ad(\pi') \otimes \mu$.

Finally, it is clear that if Case II happens, then π' cannot be dihedral. Furthermore, Π is Eisensteinian of type $(2, 2, 2)$ if π' is tetrahedral, and $(2, 4)$ otherwise. we can see this by observing that

$$\pi' \boxtimes Ad(\pi') \cong \mathrm{sym}^3(\pi') \otimes \omega_{\pi'} \boxplus \pi' \otimes \omega_{\pi'}^2,$$

□

7. Proof of Theorem A, Part #2

It remains to treat Case III. Here again, π' denotes a cusp form on $GL(2)$ and π a cusp form on $GL(3)$. Assume that $\Pi = \sigma_1 \boxplus \sigma_2$ where σ_1 and σ_2 are cusp forms on $GL(3)$.

We will divide Case III into two subcases: In this section, we will assume that π' is not dihedral. The (sub)case when π' is dihedral will be treated in the next section.

The following equality is crucial, and it holds for all cusp forms π' on $GL(2)$ and π on $GL(3)$:

Proposition 7.1.

$$\begin{aligned} &L(s,\pi \times \pi'; \Lambda^3 \otimes \omega_\pi^{-1}\chi) L(s, \pi' \otimes \omega_{\pi'}\chi) \\ &= L(s, \mathrm{sym}^3(\pi') \otimes \chi) L(s, (\pi \boxtimes \pi') \times \tilde{\pi} \otimes \omega_{\pi'}\chi) \end{aligned} \tag{1}$$

where $\omega_{\pi'}$ and ω_π are the respective central characters of π' and π.

Proof of Proposition 7.1.
We claim that both sides of (1) are nice. Indeed, we see that formally, the admissible representation $\Lambda^3(\pi \boxtimes \pi') \otimes \omega_\pi^{-1}$ is equivalent to $\mathrm{sym}^3(\pi') \boxplus (Ad(\pi) \boxtimes \pi' \otimes \omega_{\pi'})$. So the left hand side is nice. And the right hand side is nice by [KSh2002-1], whence the claim. So by Lemma 6.2 it suffices to prove this equality given by the Proposition locally at v for almost all v. It then suffices to prove the following identity (as admissible representations) for almost all v:

$$\begin{aligned} &\Lambda^3(\pi'_v \boxtimes \pi_v) \otimes \omega_{\pi_v}^{-1} \boxplus \pi'_v \otimes \omega_{\pi'_v} \\ &= \mathrm{sym}^3(\pi'_v) \boxplus \pi'_v \boxtimes \pi_v \boxtimes \tilde{\pi}_v \otimes \omega_{\pi'_v} \end{aligned} \tag{2}$$

Let v be any place where π' and π are unramified. Say $\pi'_v = \alpha_{v,1} \boxplus \alpha_{v,2}$, $\pi_v = \beta_{v,1} \boxplus \beta_{v,2} \boxplus \beta_{v,3}$. Note that $\omega_{\pi'_v} = \alpha_{v,1}\alpha_{v,2}$ and $\omega_{\pi_v} = \beta_{v,1}\beta_{v,2}\beta_{v,3}$. Then it is routine to check that the left and the right hand sides of (2) are equal to the sum of the following terms:

Terms A (2 terms): $\alpha_{v,1}^3 \boxplus \alpha_{v,2}^3$;
Terms B ($2 \times 4 = 8$ terms): 4 copies of $(\alpha_{v,1} \boxplus \alpha_{v,2}) \otimes \omega_{\pi'_v}$;
Terms C (12 terms): $\boxplus_{1\le i\le 2, 1\le j\ne k\le 3}\, \alpha_{v,i}\omega_{\pi'_v}\beta_{v,j}\beta_{v,k}^{-1}$.

In fact, Terms A, B and C are obtained by expanding the right hand side of (2). Since

$$Ad(\pi_v) = 3 \cdot \underline{1} \boxplus (\boxplus_{1\le j\ne k\le 3}, \beta_{v,j}\beta_{v,k}^{-1}),$$

the Terms C and (three of) the Terms B are obtained from $\pi'_v \boxtimes \pi_v \boxtimes \tilde{\pi}_v \otimes \omega_{\pi'}$, and the Terms A and (one of) the Terms B arise from $\mathrm{sym}^3(\pi')$.

The left hand side is easy to handle since we have the following:

$$\begin{aligned}\Lambda^3(\pi'_v \otimes \pi_v) = &\boxplus_{1\le i\le 2, 1\le j,k\le 3, j\ne k}\, \alpha_{v,i}\omega_{\pi'_v}\beta_{v,j}^2\beta_{v,k}\\ &\boxplus\, \omega_{\pi_v}\alpha_{v,1}^3 \boxplus \omega_{\pi_v}\alpha_{v,2}^3\\ &\boxplus\, 3\omega_{\pi_v}\alpha_{v,1}\omega_{\pi'_v} \boxplus 3\omega_{\pi_v}\alpha_{v,2}\omega_{\pi'_v}\end{aligned}$$

In fact, the $\omega_{\pi_v}^{-1}$ twist of the thing above contributes the Terms C, Terms A and (three of) Terms B.

So we have proved (2), and hence (1).
□

Let σ_1 and σ_2 be cusp forms on $GL(3)$.

Lemma 7.2. *Let η_1 and η_2 be the central characters of σ_1 and σ_2 respectively. Then*

$$\begin{aligned}&L(s, \sigma_1 \boxplus \sigma_2; \Lambda^3 \otimes \chi') = L(s, \eta_1\chi')L(s, \eta_2\chi')\\ &\qquad L(s, \sigma_1 \times \tilde{\sigma}_2 \otimes \eta_2\chi')L(s, \sigma_2 \times \tilde{\sigma}_1 \otimes \eta_1\chi') \end{aligned} \tag{3}$$

Proof. of Lemma 7.2.
This is easy since at each place v where the σ_i are unramified,

$$\Lambda^3(\sigma_{1,v} \boxplus \sigma_{2,v}) = \boxplus_{0\le i\le 3} \left(\Lambda^i(\sigma_{1,v}) \boxtimes \Lambda^{3-i}(\sigma_{2,v})\right),$$

$$\Lambda^2(\sigma_{i,v}) \cong \tilde{\sigma}_{i,v} \otimes \eta_i$$

and

$$\Lambda^3(\sigma_i) \cong \eta_i.$$

Done by applying Lemma 6.2.
□

Before we apply Proposition 7.1 and Lemma 7.2, let us first investigate a special instance of Case III when σ_1 and σ_2 are both twists of π:

Lemma 7.3. *If $\pi \boxtimes \pi' = (\pi \otimes \chi_1) \boxplus (\pi \boxplus \chi_2)$ then*

$$\mathrm{sym}^3(\pi') \cong (\pi' \otimes \omega_{\pi'}) \boxplus \chi_1^3 \boxplus \chi_2^3$$

Hence if π' is not dihedral or tetrahedral, this cannot happen.

Proof of Lemma 7.3.
Let v be any place where π'_v and π_v are unramified. Write

$$L_v(s, \pi') = (1 - U_v(Nv)^{-s})^{-1}(1 - V_v(Nv)^{-s})^{-1}$$

and

$$L_v(s, \pi) = (1 - A_v(Nv)^{-s})^{-1}(1 - B_v(Nv)^{-s})^{-1}(1 - C_v(Nv)^{-s})^{-1}.$$

Then

$$L_v(s, \pi' \times \pi) =$$

$$(1 - U_v A_v (Nv)^{-s})^{-1} (1 - U_v B_v (Nv)^{-s})^{-1} (1 - U_v C_v (Nv)^{-s})^{-1}$$
$$(1 - V_v A_v (Nv)^{-s})^{-1} (1 - V_v B_v (Nv)^{-s})^{-1} (1 - V_v C_v (Nv)^{-s})^{-1}$$

Let $X_v = \chi_1(\mathrm{Frob}_v)$ and $Y_v = \chi_2(\mathrm{Frob}_v)$. Then

$$L_v(s, \pi \otimes \chi_1) L_v(s, \pi \otimes \chi_2) =$$

$$(1 - X_v A_v (Nv)^{-s})^{-1} (1 - X_v B_v (Nv)^{-s})^{-1} (1 - X_v C_v (Nv)^{-s})^{-1}$$
$$(1 - Y_v A_v (Nv)^{-s})^{-1} (1 - Y_v B_v (Nv)^{-s})^{-1} (1 - Y_v C_v (Nv)^{-s})^{-1}$$

Consequently we have

$$\begin{aligned} &\{ U_v A_v, U_v B_v, U_v C_v, V_v A_v, V_v B_v, V_v C_v \} = \\ (4) \qquad &\{ X_v A_v, X_v B_v, X_v C_v, Y_v A_v, Y_v B_v, Y_v C_v \}. \end{aligned}$$

In particular,

$$(U_v^n + V_v^n)(A_v^n + B_v^n + C_v^n) = (X_v^n + Y_v^n)(A_v^n + B_v^n + C_v^n)$$

for each positive integer n. And besides, taking the products of the elements on each side of (4) and equating, we get

$$U_v^3 V_v^3 = X_v^3 Y_v^3$$

Now we apply the following lemma.

Lemma 7.4. *If X, Y, U, V, A, B, C are nonzero complex numbers such that (for all $n > 0$)*

$$(U^n + V^n)(A^n + B^n + C^n) = (X^n + Y^n)(A^n + B^n + C^n),$$

and $U^3 V^3 = X^3 Y^3$, then $\{ U^3, V^3 \} = \{ X^3, Y^3 \}$.

Proof of Lemma 7.4.
If $A^3 + B^3 + C^3 \neq 0$, then $U^3 + V^3 = X^3 + Y^3$. Hence $\{ U^3, V^3 \} = \{ X^3, Y^3 \}$ as $U^3 V^3 = X^3 Y^3$.

If $A^3 + B^3 + C^3 = 0$, we claim that $A + B + C \neq 0$. Otherwise,

$$-3ABC = 3AB(A+B) = (A+B)^3 - (A^3 + B^3) = -C^3 + C^3 = 0$$

Thus a, b or c is zero. This leads to a contradiction.

In fact we will prove the following statement:
Claim: *If $a, b, c \neq 0$, then $a + b + c$ or $a^3 + b^3 + c^3$ is not zero.*

So we claim also that $A^9 + B^9 + C^9 \neq 0$, and, $A^2 + B^2 + C^2$ or $A^6 + B^6 + C^6$ is not zero.

Hence

$$U^n + V^n = X^n + Y^n$$

holds for $n = 1$ and 9, and for one of 2 or 6.

If this equality holds for $n = 1$, and 2, then

$$UV = \frac{(U+V)^2 - (U+V^2)}{2} = \frac{(X+Y)^2 - (X+Y^2)}{2} = XY,$$

implying that $\{ U, V \} = \{ X, Y \}$, and the lemma will follow.

Now assume that $U^n + V^n = X^n + Y^n$ holds for $n = 1$, 6 or 9. As we have already assumed that $U^3V^3 = X^3Y^3$, $U^{3n}V^{3n} = X^{3n}Y^{3n}$. So we have

$$\{ U^n, V^n \} = \{ X^n, Y^n \}$$

for $n = 6$ and 9.

Without loss of generality, assume that $U^9 = X^9$ and $V^9 = Y^9$. If $U^6 = X^6$ and $V^6 = Y^6$, then of course we have $U^3 = X^3$ and $V^3 = Y^3$ and the lemma follows. If $U^6 = Y^6$ and $V^6 = X^6$, then U, V, X and Y have the same norm. However, since $U + V = X + Y$, the pairs $\{U, V\}$ and $\{X, Y\}$ are the same. hence implying the lemma. The reason for this comes from the following statement which is elementary: (Note that even when $U + V = X + Y = 0$, although we cannot directly apply this statement, we still have $U^3 + V^3 = X^3 + Y^3 = 0$, so that $\left\{ U^3, V^3 \right\} = \left\{ X^3, Y^3 \right\}$.)

Statement *The pair (z_1, z_2) such that $|z_1| = |z_2| = R$ and $z_1 + z_2 = Z$ is uniquely determined by $R > 0$ and Z with $0 < |Z| < 2R$.*

So in all cases, $\left\{ U^3, V^3 \right\} = \left\{ X^3, Y^3 \right\}$.
□

Proof of Lemma 7.3 (contd.)
By the previous lemma,

$$\left\{ U_v^3, V_v^3 \right\} = \left\{ X_v^3, Y_v^3 \right\}$$

at any unramified finite place v.
Hence

$$\begin{aligned}
&L(s, \mathrm{sym}^3(\pi'_v)) \\
&\quad = (1 - U_v^3(Nv)^{-s})^{-1}(1 - V_v^3(Nv)^{-s})^{-1} \\
&\quad (1 - V_v^2U_v(Nv)^{-s})^{-1}(1 - U_v^2V_v(Nv)^{-s})^{-1} \\
&\quad = (1 - X_v^3(Nv)^{-s})^{-1}(1 - Y_v^3(Nv)^{-s})^{-1} \\
&\quad (1 - V_v\omega_{\pi'_v}(\mathrm{Frob}_v)(Nv)^{-s})^{-1}(1 - U_v\omega_{\pi'_v}(\mathrm{Frob}_v)(Nv)^{-s})^{-1} \\
&\quad = L(s, \chi_{1,v}^3)L(s, \chi_{2,v}^3)L(s, \pi' \otimes \omega_{\pi'_v})
\end{aligned}$$

Here we have used

$$U_vV_v = \omega_{\pi'_v}(\mathrm{Frob}_v)$$

Hence
$$\mathrm{sym}^3(\pi'_v) \simeq \chi^3_{1,v} \boxplus \chi^3_{2,v} \boxplus (\pi'_v \otimes \omega_{\pi'_v}),$$
and by Lemma 6.2 we get what we desire, namely,
$$\mathrm{sym}^3(\pi') \simeq \chi^3_1 \boxplus \chi^3_2 \boxplus (\pi' \otimes \omega_{\pi'}).$$
□

Lemma 7.5. *If π' is tetrahedral with* $\mathrm{sym}^2(\pi')$ *invariant under twisting by a cubic character χ, then*
$$\mathrm{sym}^3(\pi') \cong (\pi' \otimes \omega_{\pi'}\chi) \boxplus (\pi' \otimes \omega_{\pi'}\chi^{-1}).$$

Hence the situation of Lemma 7.3 will not happen if π' is tetrahedral.

Proof of Lemma 7.5.
Consider $\pi' \boxtimes \mathrm{sym}^2(\pi') = \mathrm{sym}^3(\pi') \boxplus (\pi' \otimes \omega_{\pi'})$, which obviously contains $\pi' \otimes \omega_{\pi'}$ as an isobaric constituent .
Since $\mathrm{sym}^2(\pi')$ allows self twists by χ and χ^{-1}, the isobaric sum above should also contain $\pi' \otimes \omega_{\pi'}\chi$ and $\pi' \otimes \omega_{\pi'}\chi^{-1}$. Together with $\pi' \otimes \omega_{\pi'}$, they are pairwise inequivalent, the reason being that if a cusp form on $GL(2)$ admits a self twist by a character, then such character has to be trivial or quadratic.
Thus, by the uniqueness of the isobaric decomposition, $\mathrm{sym}^3(\pi')$ should have $\pi' \otimes \omega_{\pi'}\chi$ and $\pi' \otimes \omega_{\pi'}\chi^{-1}$ as its constituents, and there is no room for any other constituent.
□

Proof of Case III when π' is not dihedral.

Assume that $\Pi = \pi \boxtimes \pi' = \sigma_1 \boxplus \sigma_2$ where σ_1 and σ_2 are cusp forms on $GL(3)$ with central characters η_1 and η_2 respectively. Also, assume that π' is not dihedral.

Subcase A: π does not allow a self twist by a nontrivial character.

From (1) (Proposition 7.1) and (3) (Lemma 7.2), we have

$$\begin{aligned} &L(s, \mathrm{sym}^3(\pi') \otimes \chi) L(s, (\pi \boxtimes \pi') \times \tilde{\pi} \otimes \omega_{\pi'}\chi) \\ &\qquad = L(s, \pi' \times \pi; \Lambda^3 \otimes \omega_\pi^{-1}\chi) L(s, \pi' \otimes \omega_{\pi'}\chi) \\ &\qquad = L(s, \eta_1\omega_\pi^{-1}\chi) L(s, \eta_2\omega_\pi^{-1}\chi) L(s, \pi' \otimes \omega_{\pi'}\chi) \\ &\qquad L(s, \sigma_1 \times \tilde{\sigma}_2 \otimes \eta_2\omega_\pi^{-1}\chi) L(s, \sigma_2 \times \tilde{\sigma}_1 \otimes \eta_1\omega_\pi^{-1}\chi) \end{aligned} \tag{5}$$

Hence, take $\chi = \omega_\pi \eta_i^{-1}$, then the right hand side has a pole at $s = 1$ (as the remaining factors do not vanish st $s = 1$). Then the left hand side also has a pole at $s = 1$.

However, since π' is not dihedral, then $\mathrm{sym}^3(\pi')$ is either cuspidal or an isobaric sum of two cusp forms on $GL(2)$ (Lemma 7.5), so any twisted L–function of $\mathrm{sym}^3(\pi')$ has to be entire. So the only pole at $s = 1$ should come from $L(s, (\pi \boxtimes \pi') \times \tilde{\pi} \otimes \omega_{\pi'}\omega_\pi \eta_i^{-1})$.

As π is cuspidal, $\Pi = \pi \boxtimes \pi'$ should have both $\sigma_i' = \pi \otimes \omega_{\pi'}^{-1}\omega_\pi^{-1}\eta_i$ as constituents.

Since π is not monomial, it does not allow a self twist. Hence, if $\eta_1 \neq \eta_2$ then σ_i' are different. Hence σ_1' and σ_2' are the only constituents of Π which are also twists of π. Furthermore, if $\eta_1 = \eta_2$, then the order of the pole of both sides of (5), and hence also of $L(s, (\pi \boxtimes \pi') \times \tilde{\pi} \otimes \omega_{\pi'}\omega_\pi \eta_i^{-1})$, is 2. Hence $\sigma_1' = \sigma_2$ should be a constituent of Π with multiplicity 2.

Thus we get an isobaric decomposition of Π as a sum of two twists of π. Thus, from lemma 7.3 and Lemma 7.5, this cannot happen if π' is not dihedral. This completes the treatment of Subcase A.

Subcase B: π admits a self twist by a nontrivial cubic character χ.

In this subcase, recall that we are assuming $\Pi = \pi \boxtimes \pi' \cong \sigma_1 \boxplus \sigma_2$, where σ_1 and σ_2 are cusp forms with respective central characters η_1 and η_2.

We claim that σ_1 and σ_2 are also invariant when twisted by χ. Otherwise $\sigma_i \otimes \chi$ and $\sigma_i \otimes \chi^{-1}$ will be distinct from σ_i, while they should both be constituents of $\Pi \simeq \Pi \otimes \chi \simeq \Pi$. Hence Π has at least degree $3 \times 3 = 9$, which is impossible as it is automorphic on $GL(6)$.

Moreover, let v be any place where π and π' are unramified. Write (for $i = 1, 2$) $\pi_v' = \alpha_v \boxplus \alpha_v'$, $\pi_v = \beta_v \boxplus \beta_v\chi_v \boxplus \beta_v\chi_v^{-1}$ (this form being implied by $\pi \cong \pi \otimes \chi$), and $\sigma_{i,v} = \theta_{i,v} \boxplus \theta_{i,v}\chi_v \boxplus \theta_{i,v}\chi_v^{-1}$.

Then we have

$$\Pi_v = (\alpha_v \boxplus \alpha_v') \otimes \beta_v \otimes (1 \boxplus \chi_v \boxplus \chi_v^{-1})$$

and

$$\sigma_{1,v} \boxplus \sigma_{2,v} = (\theta_{1,v} \boxplus \theta_{2,v}) \otimes (1 \boxplus \chi_v \boxplus \chi_v^{-1})$$

Since the sets of all cubes of characters occurring in the previous two isobaric decompositions should be the same, and since $\beta_v^3 = \omega_{\pi_v}$ and $\theta_v^3 = \eta_{i,v}$, we must have

$$\begin{aligned} \left\{ \alpha_v^3\omega_{\pi_v}, {\alpha_v'}^3\omega_{\pi_v} \right\} &= \left\{ \alpha_v^3\beta_v^3, {\alpha_v'}^3\beta_v^3 \right\} \\ &= \left\{ \theta_{1,v}^3, \theta_{2,v}^3 \right\} \\ &= \left\{ \eta_{1,v}, \eta_{2,v} \right\} \end{aligned}$$

So

$$\begin{aligned} \mathrm{sym}^3(\pi_v') \otimes \omega_{\pi_v} &\cong \alpha_v^3\omega_{\pi_v} \boxplus {\alpha_v'}^3\omega_{\pi_v} \boxplus (\alpha_v \boxplus \alpha_v') \boxtimes \omega_{\pi_v'}\omega_{\pi_v} \\ &\simeq \eta_{1,v} \boxplus \eta_{2,v} \boxplus \pi_v' \otimes \omega_{\pi_v'}\omega_{\pi_v}, \end{aligned}$$

that is

$$\mathrm{sym}^3(\pi'_v) \cong \eta_{1,v}\omega_{\pi_v}^{-1} \boxplus \eta_{2,v}\omega_{\pi_v}^{-1} \boxplus \pi'_v \otimes \omega_{\pi'_v}$$

Thus by Lemma 6.2 or the strong multiplicity one theorem, we have

$$\mathrm{sym}^3(\pi') \cong \eta_1\omega_{\pi}^{-1} \boxplus \eta_2\omega_{\pi}^{-1} \boxplus \pi' \otimes \omega_{\pi'}$$

However, since π' is not dihedral, $\mathrm{sym}^3(\pi')$ is by Lemma 7.5 either cuspidal or an isobaric sum of two cusp forms on $GL(2)$. This gives a contradiction.

This completes the treatment of Subcase B.

□

8. Proof of Theorem A, Part #3

In this part, we will treat the case when π' is dihedral. After that, we will analyze precisely the cuspidality criterion when π is an adjoint of a form on GL(2). Again, F denotes a number field.

In fact, we will prove the following:

Theorem 8.1. *Let π', π be cusp forms on $GL(2)/F, GL(3)/F$ respectively, with π' dihedral.*

Then $\Pi = \pi \boxtimes \pi'$ is cuspidal unless both the following two conditions hold:

(a) *There is a non-normal cubic extension K' of F such that $\pi'_{K'}$ is Eisensteinian; equivalently, π' is dihedral of type D_6.*

(b) *π is monomial and $\pi = I_{K'}^F(\chi')$ for some idele class character χ' of K.*

If (a) *and* (b) *both hold, then Π is an isobaric sum of two cuspidal representations of degree 3, which are both twist equivalent to π.*

Before we prove this theorem, let us recall that a dihedral Galois representation ρ' of $\mathcal{G}_F$ is said to be of type D_{2n} if its projective image is D_{2n}. It is clear that ρ' is not irreducible if and only if $n = 1$ (Note that the projective image of D_{4n} must be a quotient of D_{2n} since D_{4n} has a nontrivial center). If $6|n$, then D_{2n} has a unique cyclic subgroup with quotient isomorphic to $D_6 \cong S_3$. Suppose K' is a non-normal cubic extension of F, and ρ' restricted to $\mathcal{G}_{K'}$ is reducible. Then the projective image of $\mathcal{G}_{K'}$ should be a subgroup of that of $\mathcal{G}_F$ of index 3, hence is isomorphic to $D_{2n/3}$. Thus $n = 3$, and ρ must be dihedral of type D_6. Similarly, we conclude that if π' is dihedral and $\pi_{K'}$ is not cuspidal, then π' is of type D_6.

Proof of Theorem 8.1.

First assume (a) and (b). Note that $\pi'_{K'} = v_1 \boxplus v_2$, and $\pi = I_{K'}^F(\chi')$. Then

$$\begin{aligned}\Pi = \pi \boxtimes \pi' &\cong \pi' \boxtimes I_{K'}^F(\chi') \\ &\cong I_{K'}^F(\pi'_{K'} \otimes \chi') \cong I_{K'}^F(v_1\chi') \boxplus I_{K'}^F(v_2\chi')\end{aligned}$$

Hence Π is not cuspidal.

Note that $\pi_i = I_{K'}^F(v_i\chi')$ MUST be cuspidal as from Section 5, Π cannot have a character as its constituent.

Next, we prove that if Π is not cuspidal, then (a), (b) and the remaining statements of the theorem hold.

Step 1: π_K is cyclic cubic monomial.

Assume that $\pi' = I_K^F(\tau)$ where τ is some idele class character of C_K with K a quadratic extension of F. and also assume that K/F is cut out by δ.

From Section 5, Case I and II cannot happen, so we are in Case III. Say $\Pi = \sigma_1 \boxplus \sigma_2$ where σ_i are some cusp forms on $GL(3)/F$. As π' allows a self twist by δ, so does $\Pi = \pi \boxtimes \pi'$. Thus $\sigma_1 \cong \sigma_2 \otimes \delta$ as the only possible characters that either σ_i allows (for self-twisting) should be trivial or cubic.

Let θ be the nontrivial Galois conjugation of K/F. Then $\pi'_K \cong \tau \boxplus \tau^\theta$. Hence the base change $\Pi_K = \pi'_K \boxtimes \pi_K$ is equivalent to $\pi_K \otimes \tau$ plus $\pi_K \otimes (\tau\circ\theta)$. As $\Pi = \sigma_1 \boxplus \sigma_1 \otimes \delta$ hence Π_K is equivalent to the isobaric sum of two copies of σ_{1K}.
Thus $\pi_K \otimes \tau \cong \pi_K \otimes (\tau \circ \theta) \cong \sigma_1$. As $\pi' = I_K^F(\tau)$ is cuspidal, $\tau \neq \tau \circ \theta$. Hence π_K is forced to be cyclic monomial.

Step 2: π is non-normal cubic monomial.

By Step 1, $\mu = \tau^{-1}(\tau \circ \theta)$ is a cubic character of C_K. Let M be the cubic field extension of K cut out by μ. As $\mu \circ \theta = \mu^{-1}$, $M^\theta = M$, thus M/F is normal, and $\mathrm{Gal}(M/F) \cong S_3$.

Furthermore, $\pi_K = I_M^K(\lambda)$ for some character λ of C_M. And also, π_M is of the form $\lambda \boxplus \lambda' \boxplus \lambda''$.

Let K' be a non-normal cubic extension of F contained in M. Then $[M : K'] = 2$ and π_M is a quadratic base change of $\pi_{K'}$.
We claim that $\pi_{K'}$ is Eisensteinian, i.e., not cuspidal. The reason is that, if $\pi_{K'}$ were cuspidal, then its quadratic base change π_M would be either cuspidal or the isobaric sum of two cusp forms of the same degree. Since $\pi_{K'}$ is a cusp form on $GL(3)$, we see from [AC1989] that this is impossible.

So $\pi_{K'}$ must admit a character as an isobaric constituent, which means that π is induced from some character of $C_{K'}$.

Step 3: $\pi'_{K'}$ is not cuspidal, hence π' is dihedral of type D_6.

Recall that $\pi'_K = \tau \boxplus (\tau \circ \theta) = \tau \boxplus \tau\mu$, so that $\pi'_M = \tau_M \boxplus \tau_M$ as M/K is cut out by μ.

Thus the projective image of ρ'_M is trivial, where ρ' is the representation $\mathrm{Ind}_K^F(\tau)$ of the Weil group W_F, and ρ'_M is the restriction of ρ to $\mathcal{G}_M$. Hence the projective image of ρ must be D_6.

Remark: Even if π' is selfdual, τ may be a character of order 3 or 6.

Step 4: σ_1 and σ_2 are all twist equivalent to π.

Observe that

$$\begin{aligned}(\tau\mu^{-1})\circ\theta &= (\tau\circ\theta)(\mu\circ\theta)^{-1}\\ &= \tau\mu\mu = \tau\mu^{-1}.\end{aligned}$$

So $\tau\mu^{-1}$ is a base change of some character, say ν, of C_F to K/F.

So $\pi' = I_K^F(\mu)\otimes\nu$ and $\pi'_{K'} \cong \nu_{K'} \boxplus \nu_{K'}\delta_{K'}$. By Step 2, we may assume that $\pi = I_{K'}^F(\lambda)$ for a character λ of a non-normal cubic extension K' of F. We get

$$\begin{aligned}\Pi &= \pi\boxtimes\pi' = \pi'\boxtimes I_{K'}^F(\lambda)\\ &= I_{K'}^F(\pi'_{K'}\otimes\lambda) = I_{K'}^F(\nu_{K'}\lambda\boxplus\nu_{K'}\lambda\delta_{K'})\\ &= I_{K'}^F(\lambda)\otimes\nu\boxplus I_{K'}^F(\lambda)\otimes\nu\delta\\ &= (\pi\otimes\nu)\boxplus(\pi\otimes\nu\delta)\end{aligned}\tag{6}$$

Now the proof of Theorem 8.1 is completed.
□

Remark. When π is twist equivalent to $Ad(\pi_0)$, and π' is dihedral, we claim that the only way $\Pi = \pi\boxtimes\pi'$ can be *not* cuspidal is for $Ad(\pi)$ to be non-normal cubic monomial, implying that π is of octahedral type. We get the following theorem which is more precise than the result in [Wa2003]:

Theorem 8.2. *Let π', and π'' be two cusp form on $GL(2)/F$, and suppose that π'' is not dihedral. then $\Pi = \pi'\boxtimes Ad(\pi'')$ is cuspidal unless one of the following happens:*
(d) *π' and π'' are twist equivalent.*
(e) *π' and π'' are octahedral attached with the same $\tilde{S}_4$-extension, and $Ad(\pi')$ and $Ad(\pi'')$ are twist equivalent.*
(f) *π'' is octahedral, $\pi' = I_K^F(\mu)\otimes\nu$ is dihedral, where μ is the cubic character which is allowed by $Ad(\pi''_K)$.*

The proof of the theorem above uses the following proposition.

Proposition 8.3. *Let π_1 and π_2 are two non-dihedral cusp forms on $GL(2)/F$, and $Ad(\pi_1)$ and $Ad(\pi_2)$ are twist equivalent. Then one of the following holds.*
(g) *π_1 and π_2 are twist equivalent (so that their adjoints are equivalent).*
(h) *π_1 and π_2 are octahedral attached with the same $\tilde{S}_4$-extension, and $Ad(\pi_1)$ and $Ad(\pi_2)$ are twist equivalent by a quadratic characters.*

Proof of Proposition 8.3. (cf. [Ra2000])
It is clear that (g) and (h) imply that π_1 and π_2 are twist equivalent. So it suffices to show the other side.

First assume that $Ad(\pi_1)$ and $Ad(\pi_2)$ are equivalent.

Consider $\Pi = \pi_1 \boxtimes \pi_2$. Note that

$$\Pi \boxtimes \overline{\Pi} \cong 1 \boxplus Ad(\pi_1) \boxplus Ad(\pi_2) \boxplus Ad(\pi_1) \boxtimes Ad(\pi_2)$$

admits two copies 1 as its constituents. Hence Π is not cuspidal.

If Π contains a character ν, then $\overline{\pi}_2 \cong \pi_1 \otimes \nu$. If Π is an isobaric sum of two cusp forms σ_1 and σ_2 on $GL(2)$, then check that $\Lambda^2(\pi_1 \boxtimes \pi_2)$ is equivalent to

$$(Ad(\pi_1) \boxplus Ad(\pi_2)) \otimes \omega_{\pi_1}\omega_{\pi_2}$$

which does not contain any character constituent; However, $\Lambda^2(\sigma_1 \boxplus \sigma_2)$ is equivalent

$$\omega_{\sigma_1} \boxplus \omega_{\sigma_1} \boxplus (\sigma_1 \boxtimes \sigma_2)$$

which contains two $GL(1)$-constituents. Thus we get a contradiction, and (g) holds.

Furthermore, assume that

$$Ad(\pi_2) \cong Ad(\pi_1) \otimes \epsilon$$

where ϵ is a character. Then

$$Ad(\pi_2) \cong Ad(\pi_1) \otimes \epsilon^{-1}$$

and hence

$$Ad(\pi_1) \boxtimes Ad(\pi_1) \cong Ad(\pi_2) \boxtimes Ad(\pi_2).$$

However,

$$Ad(\pi_i) \boxtimes Ad(\pi_i) \cong 1 \boxplus Ad(\pi_i) \boxplus A^4(\pi_i).$$

Hence

$$Ad(\pi_1) \boxplus A^4(\pi_1) \cong Ad(\pi_2) \boxplus A^4(\pi_2).$$

If $Ad(\pi_1)$ and $Ad(\pi_2)$ are equivalent, we get (g). Otherwise, $Ad(\pi_1)$, which is a nontrivial twist of $Ad(\pi_2)$, must be contained in $A^4(\pi_2)$.

So π_1 and π_2 are of solvable polyhedral type.

If π_2 is tetrahedral, then

$$A^4(\pi_2) \cong Ad(\pi_2) \boxplus \omega \boxplus \omega^2$$

where ω is some cubic character that $Ad(\pi_2)$ admits as as self-twist. So this cannot happen.

Thus π_2 and π_1 are octahedral, and

$$A^4(\pi_2) \cong I_K^F(\mu) \boxplus Ad(\pi_2) \otimes \epsilon$$

where K is a quadratic extension of F such that $Ad(\pi_{2K})$ allows a self twist by μ, and ϵ is the quadratic character cuts out K.

So they must come from the same $\tilde{S}_4$-extension, and hence (h) holds.
□

Proof of Theorem 8.2.

Set $\pi = Ad(\pi'')$ From what we have seen (including the proof of Theorem 8.1), $\Pi = \pi' \boxtimes Ad(\pi')$ is cuspidal unless (i) $Ad(\pi')$ and $Ad(\pi'')$ are twist equivalent; or (ii) π' is dihedral of type D_6, $\pi = Ad(\pi'')$ is non-normal cubic monomial, and (a) and (b) of Theorem 8.1 hold.

If (ii) holds, then π'' must be octahedral, and (f) must hold. If (1) holds, then $Ad(\pi')$ and $Ad(\pi'')$ are twist equivalent. Then part (g) of Proposition 8.3 implies (d), and part (h) of this proposition implies (e).
□

The proof of Theorem A is now complete.

9. Proof of Theorem B

In this part we deduce Theorem B from Theorem A. First we need some preliminaries.

Recall that a cusp form π on $GL(n)$ over F is essentially selfdual if $\overline{\pi}$ is twist equivalent to π. Throughout this section, π' and π denote cusp forms on $GL(2)$ and $GL(3)$ over F. We assume that π' is not dihedral, and π is not twist equivalent to $Ad(\pi'')$ for any cusp form π''.

First, from Theorem A, we have the following:

Corollary 9.1. *If π' is not of solvable polyhedral type and π is not essentially selfdual, then $\Pi = \pi \boxtimes \pi'$ is cuspidal.*

Lemma 9.2. *Let K be any solvable extension of F. If π is not essentially selfdual, and if π_K does not admit any self twist, then π_K is not essentially selfdual.*

Proof of Lemma 9.2.
First, assume that $[K : F] = l$ a prime so that K/F is cyclic. Let $\theta \neq 1$ be a Galois conjugation of K over F, and τ be a character cutting out K/F.

Assume that π_K is essentially selfdual. Say $\overline{\pi}_K \cong \pi_K \otimes \mu$, for a character μ. Applying θ, and being aware of the fact that π_K is fixed by θ, we get

$$\overline{\pi}_K \cong \pi_K \otimes (\mu \circ \theta)$$

Since π_K does not allow a self twist, then $\mu \circ \theta = \mu$, hence μ must be a base change of some character α of C_F to K.

Hence, $\overline{\pi}$ and $\pi \otimes \alpha$ have the same base change over K/F, and thus must be twist equivalent. This shows that π is also essentially selfdual.

In general case, let $K_0 = F \subset K_1 \subset \ldots \subset K_n = K$ be a tower of cyclic extensions of prime degree. Assume that $\pi_{K_n} = \pi_K$ is essentially selfdual, then as π_{K_n} does not allow a self twist, neither does π_{K_i} for any smaller i, thus applying the arguments above inductively, we claim that π_{K_i} is essentially selfdual. In particular, π must be essentially selfdual.

□

Proof of Theorem B.

First prove (a). Π is cuspidal from Corollary 9.1. First, assume only that π is not essentially selfdual and does not allow a self twist. Assume that Π allows a self twist by some character ν. Without loss of generality, may assume that ν is of order 2 or 3. Let K/F be cut out by ν. Thus $\Pi_K = I_K^F(\eta)$ is Eisensteinian of type $(2, 2, 2)$ or $(3, 3)$.

However, π_K is cuspidal as π does not allow a self twist. From Theorem A (and the remark at the end of Section 5), Π_K cannot be of type $(3, 3)$ as π'_K is not dihedral, type $(2, 2, 2)$ as π'_K is not tetrahedral (as π' and hence π'_K is not of solvable polyhedral type). Thus Π_K must be cuspidal, and hence Π does not allow a self twist.

Moreover, assume that π is not monomial, in particular, π is not induced from a non-normal cubic extension. Want to prove that Π is not either.

Assume that $\Pi \cong I_{K'}^F(\eta)$ where η is some cusp form on $GL(2)$ over K' which is non-normal cubic over F. Let M be the normal closure of K' over F and E be the unique quadratic subextension of M over F. Then π'_E is still not of solvable polyhedral type. And π_E is not cyclic monomial as π is not monomial. From Lemma 9.2, π_E is not essentially selfdual. Thus the first part of (a) implies that Π_E does not allow a self twist. Hence Π_M is still cuspidal. However, $\Pi_M \cong I_{K'}^F(\eta)_M = \eta_M \boxplus (\eta_M \circ \theta') \boxplus (\eta_M \circ \theta'^2)$ where θ' is the character cutting out M/E. We get a contradiction.

Thus (a) is proved.

Next prove (b). It suffices to prove that Π_K is cuspidal for any solvable extension K of F.

Since π' is not of solvable polyhedral type, neither is π'_K. As π is not of solvable type, then π_K must be cuspidal. We claim that π_K cannot allow a self twist. Otherwise, say $\pi_K \cong \pi_K \otimes \nu$. ν must be a cubic character. Let K_1/K be cut out by ν, then π_{K_1} should be Eisensteinian. However, K_1/F is contained in some solvable normal extension. Thus π is of solvable type. Contradiction.

Hence the claim. From Lemma 9.2, π_K is not essentially selfdual. Corollary 9.1 then implies that Π_K is cuspidal. Thus Π is not of solvable type.

□

Before we finish this section, we would like to prove the following lemma.

Lemma 9.3. *Let π be a cusp form on $GL(2m+1)$ over F. Assume that π is regular algebraic at infinity, and F is not totally complex. Then π is not monomial.*

Proof of Lemma 9.3.

Assume that $\pi = I_{K'}^F(\nu)$ where K' is a field extension of F of degree $2m+1$, and ν is an algebraic character of C_K. As F is not totally complex, neither is K' as $[K' : F]$ is odd. Thus from Weil ([We1955]), ν must be of the form $\nu_0\| \cdot \|^k$ where ν_0 is a character of finite order. Thus $I_{K'}^F(\nu)$ does not contain any nontrivial algebraic character at infinity, and hence cannot be regular at infinity. □

Appendix

The object here is to justify the statement made in the Introduction that it is possible to construct, for $n > 2$ even (resp. $n = 4$), non-selfdual, *monomial* (resp. non-monomial, but imprimitive), cuspidal cohomology classes for $\Gamma \subset \mathrm{SL}(n, \mathbb{Z})$.

Theorem E. (a) *Fix any integer $m > 1$. Then there exists a cuspidal automorphic representation π of $GL(2m, \mathbb{A}_{\mathbb{Q}})$, which is not essentially self-dual, contributing to the cuspidal cohomology in degree m^2, and admitting a self-twist relative to a character of order $2m$. In fact, this can be done for any, not necessarily constant, coefficient system.*

(b) *There exists a cusp form π on $GL(4)/\mathbb{Q}$ contributing to the cuspidal cohomology in degree 4, which is not essentially self-dual. It admits a self-twist relative to a quadratic character, but not relative to any quartic character.*

In both cases it will be apparent from the proof below that there are infinitely many such examples, and by the discussion in section 3, they give rise, for arbitrary coefficient systems V, to non-selfual, cuspidal cohomology classes in $H^*(\Gamma, V)$ for suitable congruence subgroups $\Gamma \subset \mathrm{SL}(2m, \mathbb{Z})$ ($m \geq 2$). Moreover there are naturally associated Galois representations, which are monomial in case (a), and are imprimitive but non-monomial in case (b).

Proof. (a) For any $m > 1$, fix a finite-dimensional coefficient system V. Then a cuspidal automorphic representation π of $\mathrm{GL}(2m, \mathbb{A}_F)$ contributes to the cuspidal cohomology with coefficients in V iff it is algebraic with infinity type (in the *unitary* normalization): (cf. [C1988])

$$\{(z/|z|)^{k_1}, (\overline{z}/|z|)^{k_1}, (z/|z|)^{k_2}, (\overline{z}/|z|)^{k_2}, \ldots, (z/|z|)^{k_m}, (\overline{z}/|z|)^{k_m}\},$$

where $(k_1, k_2, \ldots, k_m)$ is an ordered m-tuple of integers (determined by V) satisfying

$$k_1 > k_2 > \cdots > k_m.$$

In particular, π_∞ is regular. If $V \simeq \mathbb{C}$, then as seen in Theorem 3.2, $k_j = 2(m-j)+1$. Now pick any cyclic, totally real extension F of $\mathbb{Q}$ of degree m, and a totally imaginary quadratic extension K of F which is normal over $\mathbb{Q}$. Let $v_1, \ldots, v_m$ denote the archimedean places of F, and for each j let w_j

be a complex place of K above v_j. Choose an algebraic Hecke character χ of of K such that

$$\chi_{w_j}(z) \;=\; \left(\frac{z}{|z|}\right)^{2(m-j)+1} |z\overline{z}|^{2m-1} \;\forall\; j \leq m. \tag{1}$$

That such a character exists is a consequence of the discussion on page 3 of [We1955]. To elaborate a bit for the sake of the uninitiated, the necessary and sufficient condition for the existence of χ as above is that there be a positive integer M such that the following holds for all units $u \in \mathfrak{O}_K^*$ with components u_j at w_j:

$$\prod_{j\leq m} \left(\frac{u_j}{\overline{u}_j}\right)^{Mk_j} \;=\; 1. \tag{2}$$

But the index of the real units $\mathfrak{O}_F^*$ in $\mathfrak{O}_K^*$ is finite by the Dirichlet unit theorem, and hence for a suitable M, u_j^M is real for all j. The desired identity (2) follows.

Next pick a finite order character ν of K and set

$$\Psi \;=\; \chi\nu. \tag{3}$$

Let τ be the non-trivial automorphism (*complex conjugation*) of K/F, and δ the quadratic character of F attached to K. Then we may, and we will, choose ν in such a way that

$$\Psi(\Psi\circ\tau) \;\neq\; \mu\circ N_{K/\mathbb{Q}} \tag{4}$$

for any character μ of $\mathbb{Q}$, which is possible – and this is *crucial* – because $[K:\mathbb{Q}] \geq 4$ and so $F = \{x \in K \,|\, x^\tau = x\} \neq \mathbb{Q}$. Put

$$\pi := I_K^{\mathbb{Q}}(\Psi),$$

where I denotes *automorphic induction* ([AC1989]). Note that π makes sense because $K/\mathbb{Q}$ is solvable and normal, allowing automorphic induction to be defined. By looking at the infinity type (1) we see that π is regular and algebraic.

By construction, π_∞ contributes to cohomology, and π is cuspidal because the infinity type of Ψ precludes it from being $\Psi\circ\sigma$ for any non-trivial automorphism σ of $K/\mathbb{Q}$. To elaborate, note first that $\eta := I_K^F(\Psi)$ is cuspidal and algebraic, corresponding to a Hilbert modular newform on $\mathrm{GL}(2)/F$ of the prescribed weight at infinity. Since the automorphic induction is compatible with doing it in stages, π is just $I_F^{\mathbb{Q}}(\eta)$, and since $F/\mathbb{Q}$ is cyclic, it suffices to check that for any automorphism τ of F, η and $\eta\circ\tau$ are not isomorphic, which is clear from the properties of Ψ.

It remains to check that π is not essentially self-dual, which comes down to checking the same for the $2m$-dimensional representation ρ of $W_{\mathbb{Q}}$ induced by the character Ψ of W_K. For this we need, by Mackey, to check that $\Psi^{-1} \neq (\mu_K)\Psi\circ\sigma$ for any automorphism σ of K and any character μ of $\mathbb{Q}$. By our choice of the infinity type, this is clear for any $\sigma \neq \tau$ (and any μ).

For $\sigma = \tau$, this is the content of (4). So we are done with the proof of part (a).

(b) Let $K/\mathbb{Q}$ be an imaginary quadratic field and β a non-dihedral cusp form of weight 2 over K with $\mathbb{Q}$ coefficients, such that no twist of β is a base change from $\mathbb{Q}$. Here *weight 2* signifies the fact that the Langlands parameter of β_∞ is given by

$$\sigma(\beta_\infty) \;=\; \{z/|z|, \overline{z}/|z|\} \otimes (z\overline{z})$$

Here are two explicit (known) examples with these properties: First consider the (non-CM) elliptic curve

$$E_1 : y^2 + xy \;=\; x^3 + (3 + \sqrt{-3})x^2/2 + (1 + \sqrt{-3})x/2$$

over $K_1 = \mathbb{Q}(\sqrt{-3})$. This was shown to be associated to a cusp form β_1 of weight 2 and trivial central character on $\mathrm{GL}(2)/K_1$ by R. Taylor ([Ta1994]) such that $a_P(E_1) = a_P(\beta_1)$ for a set of primes P of density 1. (In fact, recent results can be used to show that this equality holds outside a finite set of primes P.)

For the second example, set $K_2 = \mathbb{Q}(i)$ and Q the prime ideal generated by $8{+}13i$. Then there is a corresponding cusp form β_2 of weight 2, conductor Q and trivial central character, as seen on page 344 of the book [EGM1998] by Elstrodt, Grunewald and Mennicke. Its conjugate by the non-trivial automorphism θ of K_2 will have conductor Q^θ and so no twist of β_2 can be a base change from $\mathbb{Q}$. There is a corresponding elliptic curve

$$E_2 : y^2 + iy \;=\; x^3 + (1 + i)x^2 + ix$$

over K_2 of conductor Q, and one knows for many P that $a_P(E_2) = a_P(\beta_2)$.

Next choose an algebraic Hecke character χ of K such that

$$\chi_\infty(z) \;=\; (z/|z|)^2|z\overline{z}|^2.$$

For example, we can choose χ to be the square of a Hecke character associated to a CM elliptic curve. Now consider, for $j = 1, 2$, the automorphic induction

$$\pi_j := I_K^{\mathbb{Q}}(\beta_j \otimes \chi).$$

The infinity types chosen imply that either $\beta_j \otimes \chi$ is not isomorphic to its transform by the non-trivial automorphism of K_j. So π_j is a cusp form on $\mathrm{GL}(4)/\mathbb{Q}$. It is cohomological, as easily seen by its archimedean parameter. That π_j is not essentially self-dual is an immediate consequence of the infinity types of χ and β. Finally, π_j admits a quadratic self-twist, namely by the character of $\mathbb{Q}$ associated to K_j, but it admits no quartic self-twist as β_j is non-dihedral. We are now done.

□

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Dinakar Ramakrishnan
253-37 Caltech
Pasadena, CA 91125, USA.
dinakar@caltech.edu

Song Wang
Department of Mathematics
Yale University
New Haven, CT 06520, USA.
song.wang@yale.edu