FINE STRUCTURE OF HARMONIC MEASURE

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Introduction

These notes are intended to review some results on the singularities of harmonic measure in the complex plane.

Recall that the harmonic measure of a domain Ω is a family $\{\omega_a\}_{a\in\Omega}$ of probability Borel measures on $\partial\Omega$ such that for a fixed set e, the function $a \mapsto \omega_a(e)$ is harmonic.

If the domain is simply connected, the measures ω_a can be described in terms of the Riemann map

$$\phi : \mathbb{D} \equiv \{|z| < 1\} \quad \to \quad \Omega, \quad \phi(0) = a.$$

Namely, we extend ϕ to $\partial\mathbb{D}$ in the sense of the angular limits, which exist a.e. with respect to the Lebesgue measure m, and define

$$\omega_a \;=\; \phi_* m.$$

This definition extends to arbitrary plane domains if one considers the universal cover map instead of ϕ. However, the universal cover is quite difficult to deal with, and it is more common to define harmonic measure in terms of potential theory – as an equilibrium distribution or as a solution to the Dirichlet problem:

$$\omega_a(e) \;=\; u(a),$$

where u denotes the harmonic function in Ω with "boundary values" 1 on e and 0 on the complement of e.

There are some other approaches to harmonic measure, in particular it is sometimes helpful to think of it probabilistically. Let $z(t)$ be a Brownian particle starting at a. Then

$$\omega_a \;=\; \text{Prob}\{z(\tau) \in e\},$$

where τ is the first time the partical leaves Ω: $\tau = \inf\{t : z(t) \notin \Omega\}$.

We would like to know how harmonic measure is distributed over the the boundary. Typically, the concentration of harmonic measure varies widely — it is higher on the more "exposed" parts of the boundary and smaller on the "screened" parts. It has become usual (see, e.g., [H]) to consider the dimension spectrum of a measure — a continuum of parameters characterising the size of the sets where the mass

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concentration has a given power law singularity, say $\omega_a B(z,\delta) \approx \delta^\alpha$ for small δ. Actually we will distinguish between the box counting and the Hausdorff dimension spectra, see Section 1 for precise definitions. Since $\omega_{a_1} \asymp \omega_{a_2}$ for any two points $a_1, a_2 \in \Omega$, the dimension spectrum does not depend on the choice of the pole, and when appropriate we suppress the notation to ω.

The study of the dimension spectrum involves a general form of the large deviations theory. In particular, one considers the entropy functions defined in terms of some covers and packings of the boundary satisfying certain conditions. For simply connected domains, these functions are closely related to the behavior of the derivative of the Riemann map, in particular to the *integral means spectrum*

$$\beta(t) \;=\; \overline{\lim_{r\to 1}}\, \frac{\log \int_{\partial\mathbb{D}} |\phi'(r\zeta)|^t |d\zeta|}{\log \frac{1}{1-r}}.$$

The study of the dimension spectrum becomes more interesting if there is some additional structure such as a certain type of selfsimilarity of the boundary. Then the harmonic measure also exhibits some selfsimilarity features. For example, the harmonic measure on a Cantor set J is almost multiplicative in the sense that if X and Y are two cylinder sets of J, then

$$\omega(XY) \;\asymp\; \omega(X)\omega(Y),$$

where XY denotes the image of Y if we rescale J to X. (This fact is intuitively obvious if we think of harmonic measure in terms of the Brownian motion.) One can expect strong ergodic properties of harmonic measure on such fractal boundaries.

Conformal dynamics is a rich source of interesting fractals. In some cases, the values of the entropy functions can be expressed in terms of the topological pressure of certain functions related to the dynamical system, and one can apply the powerful machinery of the Perron–Frobenius operator. As usual, the better studied case is that of the hyperbolic dynamics. If, for instance, $\partial\Omega$ is a mixing repeller (in the sense of Ruelle [R2]), then the dimension spectrum and the entropy functions have various nice properties such as real analyticity, the existence of "thermodynamical" limits, the coincidence of the Hausdorff and the box-counting versions.

The non-hyperbolic case is more difficult. There are very few general results but it is possible to analyze some special cases. It turns out that some of the properties of the hyperbolic case are no longer true. According to the physical meaning of the pressure, we will use the term "phase transition" to describe the points at which the entropy function is not smooth or real analytic. Sometimes we will use this concept in the non-dynamical situation, in particular when considering the universal bounds of the dimension spectrum.

One of the possible reasons for the phase transition is the following. Let us consider a conformal map ϕ. The growth of the integral means is usually determined by the pole type singularities of $|\phi'|^t$ at some isolated points, and by the complexity of the boundary as a whole. The behavior of the $\beta(t)$-spectrum depends on which of these two factors is predominant.

The paper is organized as follows. The first two sections concern generalities: we discuss the multifractal formalism of the dimension spectrum and study the properties of harmonic measure on regular fractals ("hyperbolic case"). In Section 3 we consider simply connected domains and following [CJ] establish a relation between the box dimension spectrum and the integral means spectrum. In Section 4 some general estimates of harmonic measure are provided. We introduce the notion of the universal dimension spectrum and show in Section 5 that the universal bounds can be approximated by the bounds for regular fractals. The method of "fractal approximation" is then applied to some problems concerning the coefficients and integral means of univalent functions. The material of Sections 3-5 is partially based on a joint work with Peter Jones [JM2]. In the last section of the paper we consider two classes of non-regular fractals.

We conclude this introduction with a remark concerning estimates of harmonic measure, cf. [C2].

In many cases, the estimates are based on some general properties of harmonic functions such as the maximum principle or Harnack's inequality. Recall that the latter states that if u is a positive harmonic function in Ω, then for every compact set $K \subset \Omega$, we have

$$z_1, z_2 \in K \implies u(z_1) \asymp u(z_2)$$

with constants depending only on the pair (Ω, K). Succesive application of Harnack's inequality provides the following statement which we will use in the study of harmonic measure on Cantor sets.

0.1. Lemma. *Suppose ω contains the annuli $R_j = B_j \setminus \overline{A}_j$, $1 \le j \le n$, where A_j and B_j are Jordan domains such that*

$$A_1 \subset B_1 \subset \cdots \subset A_n \subset B_n,$$

and suppose that the moduli of R_j's are bounded from below by $\rho > 0$. If u and v are two positive harmonic functions in Ω vanishing on $\partial\Omega \setminus A_1$, then for any $z_1, z_2 \in \Omega \setminus B_n$ we have

$$\left| \log \left[\frac{u(z_1)}{v(z_1)} : \frac{u(z_2)}{v(z_2)} \right] \right| \le C q^n,$$

where the constants $C < \infty$ and $q \in (0,1)$ depend only on ρ.

Another type of estimates comes from the relation with logarithmic capacity. Let, for example, E be a compact set inside the disc $\lambda\mathbb{D} \equiv \{|z| < \lambda\}$, $\lambda < 1$, and let ω denote the harmonic measure of the domain $\mathbb{D} \setminus E$. Then

$$\omega_z(E) \ge \text{const}\, |\log \operatorname{cap} E|^{-1}, \qquad \forall z \in \lambda\mathbb{D}.$$

The inequality is reversable if $\operatorname{dist}(z, E) \ge \delta > 0$. (The constants depend on λ and δ.)

More delicate considerations show that we can replace a sufficiently separated part of the boundary by a disc of comparable capacity without greatly affecting harmonic measure. The following is the "Main Lemma" in [JW].

0.2. Lemma. *For any $\varepsilon > 0$ there is a number $M = M(\varepsilon) < \infty$ such that if $\Omega \ni \infty$, and*

$$\partial\Omega \cap (M\mathbb{D} \setminus \mathbb{D}) = \emptyset,$$

then the harmonic measure $\tilde{\omega}$ of the new domain

$$\tilde{\Omega} = (\Omega \cup \mathbb{D}) \setminus r\overline{\mathbb{D}}, \qquad r \stackrel{\text{def}}{=} [\operatorname{cap}(\partial\Omega \cap \mathbb{D})]^{1+\varepsilon},$$

satisfies the estimates

$$\tilde{\omega}_\infty(e) \geq \omega_\infty(e), \qquad \forall e \subset \partial\Omega \setminus \overline{\mathbb{D}},$$
$$\tilde{\omega}_\infty(\overline{\mathbb{D}}) \geq \frac{1}{2}\omega_\infty(\overline{\mathbb{D}}).$$

Sometimes it is possible to estimate Green's function. Recall that if $\infty \in \Omega$, and $g(z) = g(z, \infty)$ is Green's function of Ω with pole at ∞, then

$$g(z) = \gamma - \int \log \frac{1}{|z - \zeta|}\, d\omega_\infty(\zeta),$$
$$\exp\{-\gamma\} = \operatorname{cap} \partial\Omega,$$

and

$$d\omega_\infty(\zeta) = \frac{1}{2\pi}\|\nabla g(\zeta)\|\,|d\zeta|, \qquad \zeta \in \partial\Omega,$$

if the boundary is piecewise smooth.

Let now Ω be a simply connected domain, and $\phi : \mathbb{D} \to \Omega$, $\phi(0) = \infty$, be the Riemann map. In this case,

$$g(z) = \log \frac{1}{|\phi^{-1}(z)|}, \qquad z \in \Omega.$$

We refer to the monographs [P1], [P2] for properties and estimates of conformal maps. It is a traditional problem of geometric function theory to understand how analytic properties of a conformal map are related to the geometric properties of the boundary.

An example of such relation is provided by the Koebe lemma:

$$(1 - |z|)|\phi'(z)| \asymp \delta(\phi(z)), \qquad z \in \mathbb{D},$$

where $\delta(\cdot) = \operatorname{dist}(\cdot, \partial\Omega)$. For an arc $I \subset \partial\mathbb{D}$ with center $\zeta_I \in \partial\mathbb{D}$, we denote $z_I = (1 - |I|)\zeta_I$. Then

$$\delta(\phi(z_I)) \leq \operatorname{const} \operatorname{diam} \phi(\sigma)$$

for every crosscut σ of $\mathbb{D}$ joining the endpoints of I. In the opposite direction, we have the inequality

$$\omega_\infty B(\phi(z_I), C\delta(\phi(z_I))) \geq \operatorname{const} |I|$$

with an absolute constant C.

One of the most efficient tools in estimating harmonic measure of a simply connected domain is the method of extremal lengths, see [AB], [O]. We will write $\lambda_\Omega(A, B)$ for the extremal distance between the sets A and B. The following statement is due to Beurling.

0.3. Lemma. *Let Ω be a simply connected domain and $K \subset \Omega$ a fixed continuum. Then*

$$\operatorname{cap} \phi^{-1}(e) \asymp \exp\{-\pi \ \lambda_\Omega(e, K)\}, \qquad e \subset \partial\Omega,$$

where the constants depend on (Ω, K).

Since $\omega_\infty(e) \le \operatorname{cap} \phi^{-1}(e)$, this gives a upper bound for harmonic measure, and we have a two-sided estimate if $\phi^{-1}(e)$ is an arc. Suppose $e = \partial\Omega \cap \Delta$ for some disc Δ. Though the set $\phi^{-1}(e)$ can be quite complicated, the following lemma (cf. [C1], [M2]) is sometimes useful. We denote by Δ' the concentric disc of radius twice the radius of Δ.

0.4. Lemma. *Let Ω and K be as in the previous lemma, and let $\alpha' > \alpha > 0$. Then there is $\delta_0 > 0$ such that if Δ is a disc of radius $\delta \le \delta_0$, and*

$$\omega_\infty \Delta \ge \delta^\alpha,$$

then there is a crosscut $l \subset \partial\Delta'$ of Ω such that

$$\lambda_\Omega(l, K) \le \frac{\alpha'}{\pi} \log \frac{1}{\delta}.$$

1. Multifractal formalism

We recall here some general concepts of the multifractal analysis that will be used in the paper (cf.,e.g.,[F]). Throughout this section, μ is a fixed *atom-free* probability measure with compact support $J = \operatorname{supp}\ \mu$.

Box dimension spectrum.

For $\alpha \in \mathbb{R}$ and $\delta > 0$, let $N^+(\delta; \alpha)$ and $N^-(\delta; \alpha)$ denote the maximal number of disjoint discs $B = B(z, \delta),\ z \in J$, satisfying

$$\mu B \ge \delta^\alpha \quad \text{and} \quad \mu B \le \delta^\alpha \quad \text{respectively}.$$

We define

$$f^\pm(\alpha) = \lim_{\eta \to 0+} \overline{\lim_{\delta \to 0}} \ \frac{\log N^\pm(\delta; \alpha \pm \eta)}{|\log \delta|}.$$

These functions are monotone and upper semicontinuous: $f^\pm(\alpha) = f^\pm(\alpha\pm)$. Clearly, $f^+(0-) = -\infty$ and $f^-(0) = M(J)$, where $M(J)$ is the *upper Minkowski dimension* of J:

$$M(J) = \overline{\lim_{\delta \to 0}} \frac{\log N(\delta)}{|\log \delta|},$$

$$N(\delta) \stackrel{\text{def}}{=} \max\{N : \ \exists N \ \text{ disjoint discs } \ B(z, \delta) \ \text{ with } \ z \in J\}.$$

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For every α we have

$$f^+(\alpha) \le \alpha, \tag{1.1}$$

$$(1.2) \qquad \max\{f^+(\alpha),\ f^-(\alpha)\} = M(J).$$

Denote

$$\alpha_{\min} = \inf\{\alpha : f^+(\alpha) \geq 0\},$$

$$\alpha_{\max} = \sup\{\alpha : f^-(\alpha) \geq 0\}.$$

The inequality $\alpha_{\min} > \alpha$, ($\alpha_{\max} < \alpha$, resp.), implies that

$$\mu B(z,\delta) \leq \delta^{\alpha}, \qquad (\geq,\ \text{resp.}),$$

for any $z \in J$ and $\delta \leq \delta_0$. In fact, we can define $\alpha_{\min}$, ($\alpha_{\max}$, resp.), as the supremum (infimum, resp.) of such α's.

We also need the following parameter related to the function f^+:

$$t_* \stackrel{\text{def}}{=} f^+(\infty).$$

Observe that $0 \leq t_* \leq M(J)$, and by (1.2) we have

$$(1.3) \qquad (\alpha_{\max} < \infty) \Rightarrow t_* = M(J).$$

FIGURE 1

A typical gragh of $f^{\pm}$ is shown in Figure 1. The *box dimension spectrum* $f(\alpha)$ of μ is, roughly speaking, the minimum of these two functions. More precisely, we define

$$f(\alpha) = \lim_{\eta\to 0+} \overline{\lim_{\delta\to 0}} \frac{\log N(\delta;\alpha,\eta)}{|\log\delta|},$$

where $N(\delta;\alpha,\eta)$ is

$$\max\{N: \exists N \text{ disjoint discs } B = B(z,\delta) \text{ satisfying } \delta^{\alpha+\eta} \le \mu B \le \delta^{\alpha-\eta} \text{ and } z \in J\}.$$

It is clear that

$$f(\alpha) \le \min\{f^+(\alpha), f^-(\alpha)\},$$

and that $f(\alpha) = f^+(\alpha)$ if α is a *growth* point of f^+ (i.e. $f^+(\alpha-\eta) \ne f^+(\alpha+\eta)$, $\forall\eta > 0$). Similarly, we have $f(\alpha) = f^-(\alpha)$ at the growth points of f^-. In particular,

$$t_* = \sup_\alpha f(\alpha),$$

and since μ has no atoms, $\alpha_{\min}$, and $\alpha_{\max}$ can be defined as the *inf* and *sup* of the set $\{f \ge 0\}$.

Hausdorff dimension spectrum.

We will write "dim" for the Hausdorff dimension. By definition, $\dim\emptyset = -\infty$. For $p > 0$, Λ_p denotes the corresponding Hausdorff measure.

The Hausdorff dimension spectrum is defined in terms of the Hausdorff dimension of the sets where the local dimension of μ has a given bound. Since we only need an analogue of $f^+(\alpha)$, we consider the *lower pointwise dimension:*

$$\underline{\alpha}(z) = \underline{\lim}_{\delta\to 0} \frac{\log\mu B(z,\delta)}{\log\delta}. \tag{1.4}$$

We define

$$\tilde{f}^+(\alpha) = \lim_{\eta\to 0+} \dim\{\underline{\alpha}(z) \le \alpha+\eta\}.$$

It is easy to see that

$$\tilde{f}^+(\alpha) \le f^+(\alpha). \tag{1.5}$$

Together with (1.1) and (1.4), this gives the following equation:

$$\sup\{\mu E: \ \dim E \le \alpha\} = \mu\{\underline{\alpha}(z) \le \alpha\}. \tag{1.6}$$

Let $\dim\mu$ denote the (Hausdorff) *dimension of the measure:*

$$\begin{aligned}\dim\ \mu &= \inf\ \{\dim E: \ \mu E = 1\}\\ &= \inf\ \{p: \ \mu \perp \Lambda_p\},\end{aligned}$$

(" $\perp$ " means "singular"). We also define

$$\begin{aligned} \underline{\dim}\, \mu &= \inf\{\dim E : \ \mu E > 0\} \\ &= \inf\{p : \ \mu \ll \Lambda_p\} \end{aligned}$$

(" $\ll$ " means "absolutely continuous": $\Lambda_p(e) = 0 \implies \mu(e) = 0$). From (1.6) it follows that

$$\underline{\dim}\, \mu = \mathrm{ess}_\mu \inf \underline{\alpha}, \qquad \dim\ \mu = \mathrm{ess}_\mu \sup \underline{\alpha},$$

and we have

$$\tilde{f}^+(\alpha) = \alpha \qquad \text{for} \quad \alpha = \dim \mu \quad \text{and} \quad \alpha = \underline{\dim}\, \mu.$$

By (1.1) and (1.5), we also have $f(\alpha) = \alpha$ at these points. Hence

$$\alpha_- \le \underline{\dim}\, \mu \le \dim \mu \le \alpha_+, \tag{1.7}$$

where

$$\alpha_\pm \stackrel{\text{def}}{=} \max \ \textit{or} \ \min \ \{f(\alpha) = \alpha\},$$

see Figure 1.

Packing spectrum.

It is standard to use some kind of an entropy function to study the box dimension spectrum. The function we will use is based on partitions of J into sets of equal measure. Our definition is a natural extension of the notion of the *conformal dimension* introduced in [CJ]. It is different from a more usual definition that is based on partitions into sets of equal diameter, see Remark 2 below.

We define the *packing spectrum,* our entropy function, $\pi(t)$, $t \in \mathbb{R}$, as follows:

$$\pi(t) = \sup\{q : \ \forall \delta > 0 \ \exists \text{ a } \delta\text{-packing } \{B\} \text{ such that } \sum \delta(B)^t \mu(B)^q \ge 1\}.$$

(A collection of discs $B_j = B(z_j, \delta_j)$ is a *δ-packing* if these discs are pairwise disjoint, $z_j \in J$, and $\delta_j \le \delta$.) We always use $\delta(B)$ to denote the diameter of B.

Remarks.

1) There are several other ways to define $\pi(t)$. For instance, it is not difficult to show that

$$\pi(t) = \overline{\lim_{\varepsilon \to 0}} \frac{\log L(t;\varepsilon)}{|log\varepsilon|}, \tag{1.8}$$

where

$$L(t;\varepsilon) = \sup\{\sum \delta(B)^t : \ \{B\} \text{ is a packing satisfying } \ \mu B = \varepsilon\},$$

or (for $t \neq 0$) we can replace the last condition $\mu B = \varepsilon$ with the condition

$$\mu B \le \varepsilon, \ \mu B^* \ge \varepsilon, \qquad (B^*(z,\delta) \equiv B(z, 2\delta)). \tag{1.9}$$

For $t \geq 0$, we can also define $\pi(t)$ in terms of the covers of J with discs of equal measure:

$$\pi(t) = \overline{\lim_{\varepsilon \to 0}} \frac{\log L_1(t;\varepsilon)}{|\log \varepsilon|}, \tag{1.10}$$

where

$$L_1(t;\varepsilon) = \inf\{\sum \delta_j^t : J \subset \bigcup B_j,\ B_j = B(z_j, \delta_j),\ z_j \in J,\ \mu B_j = \varepsilon\}. \tag{1.11}$$

The same is true for the version of (1.10) with (1.9) instead of the condition $\mu B_j = \varepsilon$ in (1.11). In the special case $t = 1$, the latter version is exactly the definition of the confornal dimension in [CJ].

An example with μ equal to the sum of the area measure of the unit disc and the arclength measure of the boundary shows that (1.10) can be false for $t < 0$.

2) The usual choice of the entropy function is the following. For each $\delta > 0$, let $\mathcal{G}(\delta)$ denote the grid of squares of δ-coordinate mesh. Consider the sums

$$S(q;\delta) = \sum_{Q \in \mathcal{G}(\delta), Q \cap J \neq \emptyset} (\mu Q^*)^q,$$

and denote the function $\tau(q)$, $q \in \mathbb{R}$, by the formula

$$\tau(q) = \overline{\lim_{\delta \to 0}} \frac{\log S(q;\delta)}{|\log \delta|}.$$

If we interpret $f(\alpha)$ as the scaling exponent for the number N of squares Q satisfying $\mu Q \approx \delta^\alpha$ and assume that N obeys a power law as $\delta \to 0$:

$$N \approx \delta^{-f(\alpha)},$$

then a crude estimate

$$\begin{aligned} S(\delta;q) &\approx \int \delta^{\alpha q - f(\alpha)} d\alpha \\ &\approx \delta^{-\sup_\alpha [f(\alpha) - \alpha q]} \end{aligned}$$

suggests that the function $-\tau(q)$ is the *Legendre transform* of $f(\alpha)$:

$$-\tau(q) = \mathcal{L}f(q) \stackrel{\text{def}}{=} \inf_\alpha\, [\alpha q - f(\alpha)], \tag{1.12}$$

so that in the case when f is concave we have $f = \mathcal{L}(-\tau)$. In fact, it can be shown that (1.12) is valid for $q \neq 0$. (For $q = 0$, we have $\tau(0) = M(J)$ which can be not equal to $t_* = f(\infty)$. By (1.3), the relation (1.12) holds for all $q \in \mathbb{R}$ if $\alpha_{\max} < \infty$.)

The packing spectrum $\pi(t)$ is essentially the inverse function of τ and it has a similar Legendre type relation with the box dimension spectrum.

1.1. Proposition. *1) If $0 < t \le t_*$, then*

$$\pi(t) = \sup_{\alpha>0} \frac{f^+(\alpha) - t}{\alpha}.$$

If $\alpha_{\min} > 0$, this is true for all $t \le t_*$.   2) If $t \ge M(J)$, then

$$\pi(t) = \sup_{\alpha>0} \frac{f^-(\alpha) - t}{\alpha}.$$

Since $f(\alpha)$ coincides with $f^+(\alpha)$ or $f^-(\alpha)$ at the growth points of these functions, we have

Corollary. *If $t \neq 0$, then*

$$\pi(t) = \sup_{\alpha>0} \frac{f(\alpha) - t}{\alpha}. \tag{1.13}$$

If $\alpha_{\min} > 0$, (1.13) holds for all $t \in \mathbb{R}$. If $\alpha_{\min} = 0$, the both sides of (1.13) are equal to $+\infty$ for $t < 0$. However, it can happen that

$$\pi(0) = 1 > \sup_{\alpha>0} \frac{f(\alpha)}{\alpha},$$

but only if $\dim \mu = 0$. Observe also that $\alpha_{\max} = \infty$ implies $\pi(t) = 0$ for $t \ge t_*$.

The packing spectrum is a decreasing convex function satisfying $\pi(0) = 1$. If $\alpha_{\max} > 0$, the only case we will be considering, $\pi(t)$ is finite, continuous on $(0, +\infty)$. If $\alpha_{\min} > 0$, this is so for the whole real line.

The inverse (Legendre type) transform of (1.13) gives the concave envelope $\hat{f}(\alpha)$ of $f(\alpha)$:

$$\hat{f}(\alpha) = \inf_t (t + \alpha\pi(t)), \qquad \alpha > 0.$$

As usual, one can determine the values of $\hat{f}(\alpha)$ from the graph Γ of $\pi(t)$: $\hat{f}(\alpha)$ is the coordinate of the point where the tangent T_α with slope $-1/\alpha$ intersects the t-axis. Similarly, we can determine the values of $\pi(t)$ from the graph of $\hat{f}(\alpha)$, see Figure 2.

The formula

$$\begin{cases} \alpha \mapsto \{t : (t, \pi(t)) \in \Gamma \cap T_\alpha\}, \\ t \mapsto \{\alpha : (\alpha, \hat{f}(\alpha)) \in \Gamma \cap T^t\}, \end{cases} \tag{1.14}$$

sets a one-to-one correspondence between the sets where $\pi(t)$ and $\hat{f}(\alpha)$ are differentiable and not locally linear. On such sets we have

$$\begin{cases} \alpha \mapsto t(\alpha) = f(\alpha) - \alpha f'(\alpha), \\ t \mapsto \alpha(t) = -1/\pi'(t). \end{cases}$$

Figure 2

The points where $\pi(t)$ is not smooth correspond to the intervals where $\hat{f}(\alpha)$ is linear and visa versa. In particular, $\pi(t)$ is smooth if and only if $\hat{f}(\alpha)$ is strictly convex, and then $f = \hat{f}$ so we can recover the box dimension spectrum from $\pi(t)$.

We can use the correspondence (1.14) to describe the parameters of μ in terms of $\pi(t)$:

$$\begin{gathered} t_* = \text{the first zero of } \pi(t), \\ \alpha_{\min} = 1/|\pi'(-\infty)|, \qquad \alpha_{\max} = 1/|\pi'(+\infty)|, \\ \alpha_{\pm} = 1/|\pi'(0\pm)|. \end{gathered}$$

If π is differentiable at 0, then the latter implies (see (1.7)):

$$\dim \mu = \underline{\dim}\ \mu = 1/|\pi'(0)|. \tag{1.15}$$

Of course, we can not determine the value of $M(J)$ from the $\pi(t)$ spectrum.

Covering spectrum.

Similarly to the definition of the packing spectrum, we can define the function

$$c(t) = \inf\{q :\ \forall \delta > 0\ \exists \text{ a } \delta\text{-cover } \{B\} \text{ such that} \sum \delta(B)^t \mu(B)^q \leq 1\}.$$

(A δ-*cover* is a collection of balls $B_j = B(z_j, \delta_j)$ such that $J \subset \bigcup B_j,\ z_j \in J,\ \delta_j \leq \delta$.)

The function $c(t)$ is somehow related to the Hausdorff dimension spectrum though this relation is not as nice as between $\pi(t)$ and $f(\alpha)$. Instead of (1.13), we only have the inequalities

$$c(t) \leq \max\left\{0,\ \sup_\alpha \frac{\tilde{f}^+(\alpha) - t}{\alpha}\right\}, \tag{1.16}$$

and

$$\max\{c(t), 0\} \geq \sup_\alpha \frac{\underline{\tilde{f}}^+(\alpha) - t}{\alpha}, \tag{1.17}$$

where

$$\underline{\tilde{f}}^+(\alpha) \stackrel{\text{def}}{=} \dim \{\overline{\alpha}(z) \leq \alpha\},$$

and $\overline{\alpha}(z)$ is the *upper* pointwise dimension (replace $\underline{\lim}$ with $\overline{\lim}$ in (1.4)).

We will see in Section 4 that the function $c(t)$, in contract to the packing spectrum, has nontrivial bounds for harmonic measure. In fact, this will be established for a larger function $\tilde{\pi}(t)$ which is defined by the equation

$$\tilde{\pi}(t) = \overline{\lim_{\varepsilon \to 0}} \frac{\log \tilde{L}(\varepsilon; t)}{|\log \varepsilon|}, \tag{1.18}$$

where

$$\tilde{L}(\varepsilon; t) = \inf\{\sum \delta(B)^t : J \subset \bigcup B,\ \mu B \leq \varepsilon\}.$$

It is clear that

$$c(t) \leq \max\{0, \tilde{\pi}(t)\},$$

and

$$\tilde{\pi}(t) \leq \pi(t).$$

It is perhaps interesting to mention that if J is connected (e.g., μ is the harmonic measure of a simply connected domain), then the parameter $t = 1$ (the case of "conformal dimension") plays a special role:

$$J \text{ is connected} \quad \Longrightarrow \quad \pi(1) = \tilde{\pi}(1), \tag{1.19}$$

Since

$$\tilde{\pi}(d-) \geq 0, \qquad \tilde{\pi}(d+) \leq 0, \qquad (d \stackrel{\text{def}}{=} \dim\ J),$$

it is possible that $\pi(t) \neq \tilde{\pi}(t)$ for $t > 1$. It is also easy to give an example such that $\pi(t) \neq \tilde{\pi}(t)$ for $t \in (0, 1)$.

2. Harmonic measure on regular fractals

Conformal Cantor sets.

We first consider the following class of fractals, see Figure 3. Suppose we have m topological discs D_j inside a simply connected domain D. We assume that the closures $\overline{D}_j$ of these discs are pairwise disjoint and $\overline{D}_j \subset D$. We fix m conformal maps $F_j \to D$ and define

$$F : \bigcup D_j \to \mathbb{C}$$

be setting $F = F_j$ on D_j. Then F determines the *Cantor set*

$$J = \bigcap F^{-n} D.$$

In the special case when the map F is piecewise linear, we get the standard self-similar Cantor sets.

Figure 3

The pair (J, F) is an example of an *analytic dynamical system.* In general it is required that $J \subset \mathbb{C}$ is a compact set and F is an analytic map defined in a neighborhood of J such that it leaves J invariant: $FJ \subset J$. The following theorem states that the harmonic measure on a Cantor set is a nice multifractal measure.

Later in this section we will indicate some other classes of (piecewise) analytic dynamical systems with similar properties.

2.1. Theorem. *Let ω denote the harmonic measure on a conformal Cantor set. Then ω has the following properties.*

(1) *The Hausdorff and the box dimension spectra of ω coincide. In fact,*

$$f(\alpha) = \dim\{z \in J : \lim_{\delta \to 0} \frac{\omega B(z,\delta)}{\log \delta} = \alpha\}.$$

We also have

$$\alpha_{\min} > 0, \quad \alpha_{\max} < \infty, \quad \text{and} \quad t_* = M(J) = \dim J.$$

(2) *The packing and the covering spectra of ω coincide and exist as the following limits (cf. (1.8), (1.10), (1.18)):*

$$\pi(t) = \tilde{\pi}(t) = \lim_{\varepsilon \to 0} \frac{\log L(t;\varepsilon)}{|\log \varepsilon|} = \lim_{\varepsilon \to 0} \frac{\log L_1(t;\varepsilon)}{|\log \varepsilon|} = \lim_{\varepsilon \to 0} \frac{\log \tilde{L}(t;\varepsilon)}{|\log \varepsilon|}.$$

(3) *The spectrum $\pi(t)$ is a real analytic function on $\mathbb{R}$, and $f(\alpha)$ is the Legendre type transform of $\pi(t)$:*

$$f(\alpha) = \inf_t (t + \alpha \pi(t)).$$

(4) *Either $\pi''(t) > 0$ for all $t \in \mathbb{R}$, or $\pi(t)$ is a linear function. In the first case, we have*

$$\dim \omega < \dim J,$$

and in the second case, we have $\omega \asymp \Lambda_{t_*}$.

(It is conjectured that the second case never happens for conformal Cantor sets, see below.)

The proof depends on the following fact, see [C3],[MV].

2.2. Lemma. *The Jacobian of F with respect to ω is a nonvanishing Hölder continuous function.*

We will use the standard *symbolic dynamics* associated with the Cantor set, see Figure 3. Let $\Sigma = \Sigma_m$ denote the space of infitite unilateral sequences

$$x = (x_1, x_2, \dots)$$

of the symbols $1, 2, \dots, m$, and let T be the shift map on Σ:

$$T : x \mapsto (x_2, x_3, \dots).$$

Every finite sequence determines a *cylinder set* in Σ:

$$(x_1, \dots, x_n) = \{y \in \Sigma : y_1 = x_1, \dots, y_n = x_n\},$$

(we call n the *rank* of the cylinder), and a domain

$$D_{(x_1,\dots,x_n)} = F_{x_1}^{-1} \dots F_{x_n}^{-n} D \subset \mathbb{C}.$$

The map

$$x \mapsto \bigcap_n D_{(x_1,\dots,x_n)}$$

establishes a one-to-one correspondence between the sets Σ and J and congugates T and $F|J$. This map is a bi-Hölder continuous homeomorphism if we consider Σ with the metric

$$\rho(x, y) = 2^{-\nu}, \qquad \nu = \min\{i : \ x_i \neq y_i\}.$$

Thus we can identify the dynamical systems (J, F) and (Σ, T). In particular, we can think of harmonic measure $\omega = \omega_\infty$ as a measure on the symbolic space Σ.

If μ is a measure on Σ satisfying

$$\mu(A) = 0 \Rightarrow \mu(TA) = 0,$$

then the *Jacobian* J_μ is an $L^1(\mu)$-function such that

$$\mu(TA) = \int_A J_\mu d\mu \tag{2.1}$$

for every set A on which T is injective. In the symbolic terms, we have

$$J(x) = \lim_{n\to\infty} \frac{\mu(x_2, \dots, x_n)}{\mu(x_1, \dots, x_n)}$$

for μ-a.e. $x \in \Sigma$.

To prove that the latter limit is a nonvanishing Hölder continuous function for $\mu = \omega$, we use the following fact which states that the harmonic measure on a Cantor set is exponentially multiplicative:

There is a constant $q \in (0, 1)$ such that for any cylinder sets X, Y, Z, we have

$$\left| \log \left[\frac{\omega(XYZ)}{\omega(XZ)} \ : \ \frac{\omega(YZ)}{\omega(Y)} \right] \right| \leq \text{const} \ \ q^{\operatorname{rank} Y}. \tag{2.2}$$

(We write $XY = (x_1, \dots, x_n, y_1, \dots, y_k)$ for $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_k)$, etc.) The estimate (2.2) is a consequence of Lemma 0.1.

Lemma 2.2 gives everything we need to know about harmonic measure to derive the theorem. It turns out that every measure μ on a Cantor set such that the function

$$\Theta_\mu = -\log J_\mu$$

is Hölder continuous has the properties stated in Theorem 2.1. This follows from some standard facts of ergodic theory which we want to recall now. For details concerning the next subsection we refer to the monographs [Bo1],[R1].

We will use the notation $S_n\Theta$ for the ergodic sums:

$$S_n\Theta = \sum_{j=0}^{n-1} \Theta \circ T^j.$$

By (2.1),

$$\mu(T^n A) = \int_A e^{-S_n\Theta_\mu} d\mu, \qquad (T^n|A \text{ is injective}),$$

and by the Hölder condition we have

$$\mu(X) \asymp \exp\{S_n\Theta_\mu(x)\}, \qquad \forall x \in X, \quad n = \operatorname{rank} X. \tag{2.3}$$

Elementary ergodic theory for symbolic dynamics.

Let Θ be a continuous function on the symbolic space $\Sigma = \Sigma_m$. The *topological pressure* of Θ is the number

$$P(\Theta) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\operatorname{rank} X = n} e^{S_n\Theta(x)},$$

where x is an arbitrary point in X. The limit exists and is independent of the choice of the points x.

The *Perron–Frobenius operator* $L = L_\Theta$ acts on $C(\Sigma)$, the space of continuous functions, by the formula

$$Lf(x) = \sum_{y\in T^{-1}x} f(y)e^{\Theta(y)}.$$

We have

$$L^n f(x) = \sum_{y\in T^{-1}x} f(y)e^{S_n\Theta(y)},$$

and since

$$\|L^n\| = \|L^n 1\|_\infty,$$

the number

$$\lambda(\Theta) \overset{\text{def}}{=} \log P(\Theta)$$

is the *spectral radius* of L_Θ.

Now we assume that Θ is a Hölder continuous function, namely that $\Theta \in \mathcal{H}_\alpha$ for some $\alpha > 0$, where

$$\mathcal{H}_\alpha = \{f \in C(\Sigma) : \|f\|_\alpha \overset{\text{def}}{=} \|f\|_\infty + \sup_{x,y} \frac{|f(x)-f(y)|}{\rho(x,y)^\alpha} < \infty\}.$$

Then L_Θ acts on the Banach space $\mathcal{H}_\alpha$, and has the following important property:

The number $\lambda(\Theta)$ is the spectral radius and a simple, isolated eigenvalue of the operator $L_\Theta : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$.

The spectral radius $\lambda(\Theta)$ can also be identified as a unique positive eigenvalue of the congugate operator L^* acting on $M(J)$, the space of Borel complex measures. The existence of such an eigenvalue follows from the Schauder theorem and the uniqueness from the formula

$$J_\nu = \lambda e^{-\Theta} \tag{2.4}$$

for the Jacobian of any measure ν satisfying $L^*\nu = \lambda\nu$. (From (2.3) and (2.4), we have $\nu(X) \asymp \lambda^{-n} e^{S_n\Theta(x)}$, $x \in X$, and hence $\lambda = \lambda(\Theta)$.)

One can choose the eigenvectors ν, $L^*\nu = \lambda\nu$, and h, $Lh = \lambda(\Theta)h$, such that

$$\nu \geq 0, \quad h > 0, \quad \int h \; d\nu = 1.$$

Then the probability measure μ,

$$d\mu \stackrel{\text{def}}{=} h \; d\nu, \tag{2.5}$$

is T-invariant. On can also prove that μ is ergodic and therefore μ can be characterized as a unique measure satisfying

$$\mu(X) \asymp e^{-P(\Theta)n} e^{S_N\Theta(x)}, \quad x \in X, \quad n = \operatorname{rank} X. \tag{2.6}$$

We denote $\mu = \mu_\Theta$ and call μ_Θ the *Gibbs measure* with *potential* Θ.

The *entropy* of an invariant probability measure μ is

$$h_\mu = \int \log J_\mu \; d\mu.$$

If μ is ergodic, then

$$\frac{1}{n} \log \frac{1}{\mu(x_1, \dots, x_n)} \to h_\mu, \qquad \text{for } \mu\text{-a.e. } x \in \Sigma, \tag{2.7}$$

(This is a version of the ergodic theorem.) If $\mu = \mu_\Theta$ for some Hölder continuous function Θ, then by (2.4) and (2.5), we have

$$\log J_\mu = P(\Theta) - \Theta + \gamma \circ T - \gamma, \qquad \gamma = \log h, \tag{2.8}$$

$$P(\Theta) = h_\mu + \int \Theta \; d\mu. \tag{2.9}$$

In fact, the *variational principle* states that for $\Theta \in C(\Sigma)$, the pressure $P(\Theta)$ is the supremum of the functional $\mu \mapsto h_\mu + \int \Theta \, d\mu$ on the class of invariant probability measures, and in the case of a Hölder continuous Θ, $\mu = \mu_\Theta$ is the only measure satisfying (2.9).

Returning to the proof of theorem 2.1, we consider two Hölder continuous functions

$$\Theta = -\log J_\omega,$$
$$\Psi = -\log |F'|.$$

Observe that for a cylinder set X of rank n, we have

$$e^{S_n\Theta(x)} \asymp \omega(X), \qquad x \in X, \tag{2.10}$$

and

$$e^{S_n\Theta(x)} = |(F^n)'(x)|^{-1} \asymp \delta(X) \equiv \operatorname{diam} D_X \tag{2.11}$$

(apply the distortion theorem to the conformal map $F^n : D_X \to D$).

Define the *pressure function*

$$P(s,t) = P(s\Theta + t\Psi), \qquad (s,t) \in \mathbb{R}^2.$$

By the perturbation theory, this function is real analytic on $\mathbb{R}^2$. It is also clear that $P(s,t)$ is convex and strictly decreasing (from $+\infty$ to $-\infty$) in t and in s. Therefore, the equation

$$P(s,t) = 0 \tag{2.12}$$

uniquely determines a convex, real analytic function

$$s = s(t), \qquad t \in \mathbb{R}.$$

By (2.10), (2.11), the identity $P(s(t),t) = 0$ means that for any n, we have

$$\sum_{\operatorname{rank} X = n} \omega(X)^{s(t)} \delta(X)^t \asymp 1,$$

which implies

$$\pi(t) \le s(t),$$

and

$$f(\alpha) \le l(\alpha) \stackrel{\text{def}}{=} \inf_t [t + \alpha s(t)].$$

We will now show that

$$\dim \left\{ z : \lim_{\delta \to 0} \frac{\log \omega B(z,\delta)}{\log \delta} = \alpha \right\} \ge l(\alpha). \tag{2.13}$$

Since $s(t)$ is a convex function, taking the inverse Legendre transform in (2.13), we then obtain

$$\tilde{\pi}(t) \geq s(t),$$

cf. (1.16), (1.17). This completes the proof of the first three statements of Theorem 2.1.

To check (2.13), let $\mu_{s,t}$ denote the Gibbs measure with potential $s\Theta+t\Psi$. Applying the analytic perturbation theory to the isolated eigenvalue (= spectral radius) of the Perron–Frobenius operator, one can justify the following (intuitively "obvious") formulae for the partial derivatives of the pressure function:

$$\frac{\partial P}{\partial s} = \int \Theta \, d\mu_{s,t}, \qquad \frac{\partial P}{\partial t} = \int \Psi \, d\mu_{s,t}. \tag{2.14}$$

Next we fix $t \in \mathbb{R}$ and define $\mu_{(t)} = \mu_{s(t),t}$, $\alpha = -1/s'(t)$. Since

$$h_{\mu_{(t)}} \overset{(2.9)}{=} -\int (s(t)\Theta + t\Psi) \, d\mu_{(t)}, \tag{2.15}$$

we have

$$\begin{aligned} l(\alpha) = t + \alpha s(t) = t - \frac{s(t)}{s'(t)} &\overset{(2.14)}{=} t + \frac{s(t) \int \Theta \, d\mu_{(t)}}{\int \Psi \, d\mu_{(t)}} \\ &\overset{(2.15)}{=} \frac{h_{\mu_{(t)}}}{\int \log |F'| d\mu_{(t)}} = \dim \mu_{(t)}. \end{aligned}$$

The latter equality follows from (2.7) and the ergodic theorem applied to the sequence

$$S_n(\log |F'(x)|) = -\log \delta(x_1, \ldots, x_n) + O(1).$$

This formula for the dimension of a measure is in fact quite general. For instance, in the case of an analytic dynamics, it holds for every ergodic measure with positive entropy, see [Man].

On the other hand, for $\mu_{(t)}$-a.e. z, we have

$$\lim_{\delta \to 0} \frac{\log \omega B(z, \delta)}{\log \delta} = \frac{\int \Theta \, d\mu_{(t)}}{\int \Psi \, d\mu_{(t)}} \overset{(2.14)}{=} -1/s'(t) = \alpha,$$

and (2.13) is proved.

To explain the last statement of the theorem, let us assume that $s''(t_0) = 0$ for some $t_0 \in \mathbb{R}$. Define the function

$$P(\tau) = P(A + \tau B), \qquad \tau \in \mathbb{R},$$

where A and B denote the Hölder continuous functions $s(t_0)\Theta + t_0\Psi$ and $s'(t_0)\Theta + \Psi$ respectively. By the definition of $s(t)$, we have $P''(0) = 0$.

Consider the stationary process $\{B \circ T^n\}_{n\geq 0}$ in $L^2(\mu_A)$. It turns out that this process has exponentially decreasing correlations and therefore the *asymptotic variance*

$$\sigma^2 \equiv \sigma^2_{\mu_A}(B) \stackrel{\text{def}}{=} \lim_{n\to\infty} \int \left[S_n B - \int B \; d\mu_A \right]^2 \; d\mu_A$$

exists and is finite. On the other hand, it can be shown that

$$\sigma^2 = P''(0).$$

The variance σ^2 is zero if and only if the sequence $\{S_n B\}$ is bounded in $L^2(\mu_A)$, and in this case we have

$$B = u \circ T - u. \tag{2.16}$$

It is a nontrivial fact of the theory of Gibbs measures that we can then find a *Hölder continuous* function u satisfying the homological relation (2.16), and therefore $s(t)$ is a linear function.

Regular fractals.

The properties of harmonic measure that we established for conformal Cantor sets are also valid for some other classes of fractals.

Let Ω be a domain such that the boundary $J = \partial\Omega$ is a *mixing repeller* (see [R2]) with respect to an analytic dynamics F. By definition, this means that

(1) we can choose U, the domain of F, so that

$$J = \{z \in U : \; F^n z \in U \text{ for all } n > 0\},$$

in particular, J is *completely invariant:* $F^{-1}J = J$;

(2) F is *expanding* on J:

$$\exists Q > 1 \quad \forall z \in J : \quad |(F^n)'(z)| \geq \text{const } \; Q^n;$$

(3) F is *topological mixing* on J, i.e. for every non-empty open set O intersecting J, there is an $n > 0$ such that $J \subset F^n O$.

A crucial property of a mixing repeller is the existence of an appropriate Markov partition such that we can apply the methods of the symbolic dynamics in essentially the same way as we did for Cantor sets. (One has to consider a slightly more general form of the symbolic space Σ, namely the Markov shift space. The coding map $\Sigma \to J$ is Hölder continuous but in general it is no longer one-to-one. However, this map establishes a one-to-one correspondence between ergodic measures on Σ and those on J nonvanishing on open sets.) The key issue is again the fact that the harmonic measure of Ω is equivalent to some Gibbs measure. This allows us to extend Theorem 2.1 to the case of general mixing repellers.

Conformal Cantor sets are examples of mixing repellers. Another important class of examples comes from the iteration theory. We refer to the books [CG] and [Mil] for general information on this subject.

Let F be a complex polynomial of degree $d \geq 2$. Consider the domain

$$\Omega = \{z : F^n \to \infty\},$$

the *basin of attraction* to ∞. The *Julia set* $J = J_F$ is defined as the boundary of Ω. The components of $\mathbb{C} \setminus J$ are called *Fatou components.* The Julia set is completely invariant and F is a topological mixing on J. Thus J is a mixing repeller if and only if the dynamics F is expanding on J. Polynomials with the latter property are called *hyperbolic.* One can show that F is hyperbolic iff the trajectory $\{F^n c\}$ of every critical point (= zero of the derivative of F) c tends to ∞ or to some periodic cycle $\{a, Fa, \ldots, F^p a = a\}$ with $|(F^p)'(a)| < 1$. If all critical point tend to ∞, the Julia set is a conformal Cantor set.

Let us consider the harmonic measure ω on J_F evaluated at ∞. It turns out that ω is already invariant with respect to F: if $u(z), z \in \Omega$, is the solution to the Dirichlet problem with boundary values the characteristic function of $e \subset J$, then $u \circ F$ is the solution with boundary values the characteristic function of $F^{-1}e$. In a similar way one can show that Green's function $g(\cdot) \equiv g(\cdot, \infty)$ satisfies the functional equation

$$g \circ F(z) = d\, g(z), \qquad z \in \Omega, \tag{2.17}$$

and therefore the Jacobian of ω is a constant function: $J_\omega = d$. We see that in the hyperbolic case, ω is a Gibbs measure. By variational principle, ω can be characterized as a unique measure of maximal entropy ($h_\omega = \log d$). In fact, this characterization remains valid for arbitrary polynomial Julia sets (see [Br]).

The Julia set J_F is connected (and Ω is simply connected) if and only if the trajectories of all critical points are bounded. The Mandelbrot set $\mathcal{M}$ is the set of the parameters $c \in \mathbb{C}$ such that the Julia set J_c of the quadratic polinomial $F_c = z^2 + c$ is connected. Thus if $c \notin \mathcal{M}$, F_c is hyperbolic and J_c is a Cantor set. The set $\mathcal{H} = \{c \in \mathcal{M} : F_c \text{ is hyperbolic}\}$ consists of infinitely many components corresponding to different periodic cycles. The "main cardioid" is the component corresponding to the fixedpoint case:

$$\mathcal{H}_\heartsuit \stackrel{\text{def}}{=} \{c : F_c \text{ has a finite attracting fixedpoint}\} = \{\lambda/2 - \lambda^2/4 : |\lambda| < 1\}.$$

For $c \in \mathcal{H}_\heartsuit$, J_c is a Jordan curve, e.g., $J_c = \partial\mathbb{D}$ for $c = 0$, but if $c \in \mathcal{H} \setminus \mathcal{H}_\heartsuit$, then there are infinitely many Fatou components. Let D be a bounded periodic Fatou component. Though ∂D is not a mixing repeller (we don't have complete invariance), one can modify the argument to show that the harmonic measure of D is equivalent to some Gibbs measure and extend Theorem 2.1 to this case.

One can also extend the theorem to the case of *piecewise* analytic repellers. We will consider the following class of fractals.

Let P be a polygon with sides $\sigma_1, \dots, \sigma_k$ built on the unit interval $[0,1]$ in the following sense. The sides σ_1 and σ_k are the intervals $[0,a]$ and $[b,1]$ respectively, where $0 < a < b < 1$. The remaining sides $\sigma_2, \dots, \sigma_{k-1}$ join σ_1 and σ_k. We now repeat the construction on each side $\sigma_j, 1 \le j \le k$, by replacing σ_j with a rescaled copy of P. We obtain a second generation curve P^2. This process continues is a obvious fashion producing polygons P^n which we assume nonintersecting. In the limit we obtain a *snowflake* curve P^∞.

If we now join the endpoints of P^∞ by some smooth arc l nonintersecting P^∞, we get two domains with harmonic measures ω_+, ω_-. Let I be an arc on P^∞ *separated* from the endpoints. One can show (cf. [PUZ], [M3]) that the statements of Theorem 2.1 are true for the restriction of ω_+ or ω_- to I. The main difficulty is the constuction of a piecewise conformal dynamics that respects the structure of the domain in some neighborhood of every (closed) arc in the corresponding Markov partition. (Then we can apply an argument with moduli similar to Lemma 0.1.) Observe also that one can choose the arc l so that the measure ω_+ (or ω_-) has the same multifractal parameters $(f(\alpha), \pi(t)$ etc.) as its restriction to I.

Dimension of harmonic measure.

We will now show how ergodic properties of harmonic measure can be used, in the case of regular fractals, to estimate the size of the support. The first result of this type was established by Carleson [C3]:

2.3. Theorem. *Suppose the boundary of Ω is a regular fractal, and Ω is not simply connected. Then*

$$\dim\ \omega < 1.$$

We will explain this theorem for polynomial Julia sets (cf. [Pr]). The general case is only slightly more difficult.

Let Ω be the basin of ∞ for some polynomial $F(z) = z^d + \dots$ As we mentioned, the harmonic measure $\omega = \omega_\infty$ is an invariant, ergodic measure with entropy

$$h_\omega = \log d.$$

Let $c_1, \dots, c_{d-1}$ be the critical points of F counting with multiplicities. By assumption, at least one of c_j's lies in Ω. We have

$$\begin{aligned} \int \log |F'|\ d\omega &= \log d + \sum_{j=0}^{d-1} \int \log |z - c_j|\ d\omega(z) \\ &= \log d + \sum_{c_j \in \Omega} g(c_j), \end{aligned}$$

where $g(\cdot)$ is Green's function with pole at ∞. In the last equality we used the fact that $\operatorname{cap} J = 1$ which is a consequence of the functional equation (2.17). It follows that

$$\dim\ \omega = \frac{h_\omega}{\int \log |F'|\ d\omega} < 1.$$

Remarks.

1) The above argument also shows that $\dim \omega = 1$ in the simply connected case, and therefore we have $\dim \omega \leq 1$ for all regular fractals. These both facts extend to arbitrary plane domains ([M1], [JW], respectively), see Section 4.

2) One can obtain further information on the size of the sets supporting harmonic measure by comparing ω with general Hausdorff measures $\Lambda_{\varphi(t)}$. Assuming that $\dim \omega \neq \dim J$, i.e. $\pi''(0) > 0$ in Theorem 2.1, we can apply the standard limit theorems to the sequence of exponentially weakly dependent variables $\Phi \circ F^n$, where

$$\Phi = \log J_\omega + \kappa \log |F'|, \qquad \kappa \stackrel{\text{def}}{=} \dim\ \omega. \tag{2.18}$$

For instance, the application of Kolmogorov's test shows that ω is absolutely continuous or singular with respect to $\Lambda_{\varphi(t)}$,

$$\varphi(t) = t^\kappa \exp\{h(\log \frac{1}{t})\},$$

according as the following integral converges or diverges:

$$\int^\infty t^{-3/2} h(t) \exp\{-ct^{-1} h^2(t)\}\ dt,$$

where

$$c = \frac{1}{2}\pi''(0) \int \log |F'|\ d\mu, \tag{2.19}$$

and μ is the Gibbs measure equivalent to ω. In the simply connected case, there is a universal bound for the transition parameter (2.19).

A different type of estimates for regular fractals is the following inequality:

$$\dim\ \omega < \dim\ J \tag{2.20}$$

This inequality is certainly true if $\dim J > 1$. If Ω is simply connected,then by Theorem 2.1, we have (2.20) unless J is rectifiable. The latter happens if and only if J is a piecewise real analytic curve ([B2]).

It is conjectured that (2.20) is valid for all disconnected regular fractals. This has been verified in the following cases:

(1) Julia sets, [Z];
(2) $J \subset \mathbb{R}$, [V1];
(3) Cantor sets with piecewise linear dynamics, [V2].

By Theorem 2.1, one has to rule out the possibility of the homological relation

$$\Phi = \gamma - \gamma \circ F, \tag{2.21}$$

for the function Φ defined by (2.18) and some Hölder continuous function γ on J. We will outline the proof in the case of a piecewise linear dynamics but first we mention that there are several special cases where the proof is quite elementary (cf.[MV]).

Example 1. Let J be the Julia set of a quadratic polynomial F. Then $\Phi = \text{const} + \text{const}\ \log|F'|$, and (2.21) implies that there is $K > 1$ such that for all $n \in \mathbb{N}$ and $z \in \operatorname{Fix} F^n$, we have

$$|(F^n)'(z)| = K^n.$$

Comparing the multipliers of the fixedpoints ($n = 1$) and the cycle of period $n = 2$, we already arrive to a contradiction.

Example 2. Let J be the linear Cantor set of constant ratio a, $0 < a < 1/2$. We will use the natural coding of J with symbols 1 and 2. Consider the cylinders $X_n = (1, \ldots, 1, 1)$ and $Y_n = (1, \ldots, 1, 2)$ of rank n. We want to show that ω is not equivalent to a Hausdorff measure, and it is enough to check that $\omega(X_n) \geq (1 + \delta)\omega(Y_n)$ with $\delta > 0$ independent of n. We have

$$\omega(X_n) - \omega(Y_n) = \int (v - u)\ d\omega,$$

where u and v denote the harmonic measures of X_n and Y_n with respect to $\hat{\mathbb{C}} \setminus X_{n-1}$. Clearly, $v \geq u$ on $J \setminus X_n$, and hence

$$\omega(X_n) - \omega(Y_n) \geq \omega(Y_n) \min_{Y_{n-1}} (v - u) \geq \text{const}\, \omega(Y_{n-1})$$

because the minimum is scale invariant.

Now we will explain the proof of the following theorem due to Volberg:

2.3. Theorem. *Let J be the conformal Cantor set determined by the dynamics*

$$F : \bigcup_{j=1}^{m} D_j \to D$$

such that at least two of the maps $F_j = F|D_j$ are linear. Then

$$\dim\ \omega < \dim\ J.$$

The idea is to extend the homological relation (2.21) from J to the whole complex plane. To this end, Volberg considers the following representation of the potential $\Theta_\omega = -\log J_\omega$ in terms of Green's function $g(\cdot)$ of $\Omega = \hat{\mathbb{C}} \setminus J$ with pole at ∞:

$$\Theta_\omega(\zeta) = \lim_{z \to \zeta, z \in \Omega} \log \frac{g(z)}{g(Fz)}, \qquad \zeta \in J.$$

This formula follows from the relation

$$\left| \log \left[\frac{g(z)}{g(Fz)} : \frac{\omega(X)}{\omega(TX)} \right] \right| \leq \text{const}\ q^{\operatorname{rank} X}, \qquad \forall z \in \partial D_X, \tag{2.22}$$

with some $q \in (0,1)$. The inequality (2.22) is a consequence of Lemma 0.1.

Suppose F_1 and F_2 are linear, and denote $\lambda_j = |F_j'|$. Let $a_j \in D_j$ be the fixedpoints of F_j: $a_j = (j, j, \dots)$ in the symbolic representation. For $z \in D$ and an integer n we will write $z_j^{-n} = F_j^{-n} z$. For $j = 1, 2$, define

$$\gamma_j(z) = \gamma(a_j) + \sum_{n \geq 0} \log \frac{g(F z_j^{-n})}{\lambda_j^\kappa g(z_j^{-n})}, \qquad z \in D. \tag{2.23}$$

¿From (2.21) and (2.22), we have

$$\log \frac{g(F z_j^{-n})}{g(z_j^{-n})} \underset{(\exp)}{\approx} -\Theta_\omega(a_j) = \kappa \log \lambda_j,$$

which implies that the limit in (2.23) exists and extends to a Hölder continuous function in D. Replacing λ_j in (2.23) by $|F'(z_j^{-n})|$ and applying (2.21), we see that $\gamma_1 = \gamma_2 = \gamma$ on J. It follows that the functions

$$\tau_j(z) = g(z) e^{-\gamma_j(z)},$$

are harmonic in $D \setminus J$, subharmonic and Hölder continuous in D. They vanish exactly on J, and satisfy the functional equation

$$\tau_j(F(z)) = \lambda_j^\kappa \tau_j(z), \qquad z \in D_j.$$

The main part of the proof is to show that $\tau_1 = \tau_2$. To see this, we use subharmonicity and represent τ_j as the difference of a harmonic function and the logarithmic potential of some finite positive measure μ_j on J. Then for $G = \partial(\tau_1 - \tau_2)$, we have

$$G(z) = \text{analytic function} + \int \frac{d(\mu_1 - \mu_2)(\zeta)}{z - \zeta}.$$

If we fix $z \in D$, then for large n we have

$$G(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{G(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \sum_{\operatorname{rank} X = n} \int_{\partial D_X} \frac{G(\zeta)}{\zeta - z} d\zeta. \tag{2.24}$$

Since $\tau_j(z) = o(g(z))$ as $z \to J$, we have $|G(z)| = o(\delta(z)^{-1} g(z))$, where $\delta(z) = \operatorname{dist}(z, J)$, and so the absolute value of the sum in (2.24) does not exceed

$$o(1) \sum_{\operatorname{rank} X = n} \int_{\partial D_X} \frac{g(\zeta)|d\zeta|}{\delta(\zeta)} \asymp o(1) \sum_{\operatorname{rank} X = n} \omega(X) = o(1).$$

This shows that G is an analytic and $\tau_1 - \tau_2$ is a harmonic function in D. Considering the zero set, we have $\tau_1 = \tau_2$ in D if J is not a subset of an analytic curve. Otherwise, we can assume that $J \subset \mathbb{R}$ and using reflection conclude that $\partial(\tau_1 - \tau_2) = 0$ on J.

Finally we extend the function $\tau = \tau_1 = \tau_2$ to a continuous function on $\mathbb{C}$ satisfying $\tau(Az) = |A'|^\kappa \tau(z)$ for every affine conformal map A in the group $\mathcal{G}$ generated by F_1 and F_2. Then we have $\tau = 0$ on $\cup\{AJ : A \in \mathcal{G}\}$. It is easy to see that the latter set has an everywhere dense projection in at least one direction. Thus we have $\dim J \geq 1$ and the application of Theorem 2.1 completes the proof.

Let Ω be a simply connected domain with compact boundary and $\phi : \mathbb{D} \to \Omega$ be the Riemann map. Our aim is to describe the structure of harmonic measure in terms of ϕ. We will compare the integral means spectrum $\beta(t)$ of ϕ, see Introduction, and the packing spectrum $\pi(t)$ of harmonic measure. The parameter t_* has the same meaning as in Section 1.

3.1. Theorem. *If $t \le t_*$, then*

$$\beta(t) = \pi(t) + t - 1. \tag{3.1}$$

Remarks and Examples.

1) The relation (3.1), for all $t \in \mathbb{R}$, is almost obvious for quasidiscs. Consider the partition of $\partial\mathbb{D}$ into arcs I of equal length ε. The corresponding partition $\{\phi(I)\}$ of $\partial\mathbb{D}$ satisfies

$$\sum \delta(\phi(I))^t \asymp \sum_{(I)} \varepsilon^t |\phi'(z_I)|^t \asymp \varepsilon^{t-1} \int_{|z|=1-\varepsilon} |\phi'|^t.$$

The geometry of quasidiscs allows then to construct a cover and a packing of the boundary with discs B satisfying $\omega(B) \asymp \varepsilon$ such that the same estimate holds for $\sum \delta(B)^t$.

Even for John domains, it is possible that $\beta(t) \neq \pi(t) + t - 1$. Consider, for instance, a domain with a V-shaped boundary. One can easily modify this example to obtain a Jordan domain. The idea is to "screen" the sets $X \subset \partial\Omega$ where the concentration of harmonic measure is small with the sets Y of large concentration, see Figure 4. One should in fact repeat the construction indicated in the picture on an infinite sequence of scales tending to zero.

2) By (3.1), we have $t_* = \beta(t_*) + 1 \ge 1$, and therefore

$$\beta(1) = \pi(1).$$

This special case of Theorem 3.1 was established in [CJ]. The argument in [CJ] is based on the representation of the integral means of order one as a length of the corresponding level set. In fact, their proof implies

$$\beta(t) = \tilde{\pi}(t) + t - 1, \qquad \text{for } t = 1, \tag{3.2}$$

see (1.18). This statement also follows from the combination of Theorem 3.1 and (1.19). In general, (3.2) is not true for $t \neq 1$.

3) One can combine Theorem 3.1 with estimates of the integral means to obtain certain estimates of harmonic measure. For instance, the inequality $|\phi'(z)| \ge \text{const}(1 - |z|)$ implies that $\beta'(-\infty) \ge -1$ or $\pi'(-\infty) \ge -2$, and hence

$$\alpha_{\min} \ge .$$

Figure 4

A less trivial example is the following (see [P1, Section 5.1]). The identity

$$\frac{d}{dr}\left(r\frac{d}{dr}I_t\right) = t^2 r \int_{|z|=r} |\phi'|^t \left(\frac{|\phi"|}{|\phi'|}\right)^2$$

for the integral means I_t of order t, together with the bound $|\phi"|/|\phi'| \leq \text{const}(1-|z|)^{-1}$ imply that

$$\frac{d^2}{dr^2} I \leq \text{const}\ \ t^2 \frac{1}{(1-r^2)} I,$$

and

$$I \leq \left(\frac{1}{1-r}\right)^{Ct^2}.$$

Hence

$$\beta(t) \leq Ct^2$$

for some universal constant $C \geq 0$, and by Proposition 1.1 we have

$$\begin{aligned} f(\alpha) &\leq \inf_t [\alpha\pi(t) + t] \\ &\leq \inf_t [\alpha(1 - t + Ct^2) + t] = \alpha - \frac{(\alpha-1)^2}{4C}. \end{aligned}$$

This proves, in particular, that $\underline{\dim}\,\omega = \dim\,\omega = 1$, see (1.15), and moreover that there is a universal upper bound for the transition parameter (2.19), cf. [M1].

4) The inequality

$$\beta(t) \geq \pi(t) + t - 1$$

holds for all $t \in \mathbb{R}$. By definition, Ω is a *Hölder* domain if the Riemann map is Hölder continuous. In this case, we have $\beta'(+\infty) < 1$ and $\alpha_{\max} < \infty$. By (1.3), it follows that

$$t_* = M(J) \qquad \text{for every Hölder domain.} \tag{3.3}$$

This fact seems to have been known only for John domains, cf. [P2, Sect. 10.5].

We outline now the proof of Theorem 3.1. It is somehow easier to relate $\beta(t)$ directly to $f(\alpha)$ rather than to $\pi(t)$ and then apply the Legendre transform. We will need the following auxiliary function $d(a)$ defined in terms of the distribution function:

$$d(a) = \overline{\lim_{a_1 \to a}}\ \overline{\lim_{r \to 1}} \frac{\log \frac{\lambda_{a_1}}{1-r}}{\log \frac{1}{1-r}}, \quad a \in \mathbb{R},$$

where

$$\lambda_a = \begin{cases} m\{\zeta \in \partial\mathbb{D} : |\phi'(r\zeta)| > (1-r)^{-a}\}, & a > 0, \\ m\{\zeta \in \partial\mathbb{D} : |\phi'(r\zeta)| < (1-r)^{|a|}\}, & a < 0. \end{cases}$$

By the distortion theorem and a simple large deviations argument, we have

$$\beta(t) - t + 1 = \sup_a [d(a) - (1-a)t]. \tag{3.4}$$

It follows that the spectrum $d(a)$ has a maximum point at $a = 0$, $d(0) = 1$, and satisfies the concavity condition at this point:

$$d(\eta a) \geq \eta d(a) + 1 - \eta, \qquad \forall a \in \mathbb{R}, \qquad \forall \eta \in (0,1).$$

Therefore, $d(a)$ is strictly increasing on $[a_-, 1]$, and strictly decreasing on $[1, a_+]$, where a_- is the minimum and a_+ is the maximum of the set $\{d(a) \neq -\infty\}$. Thus we can find an r arbitrarily close to 1 such that there are $\approx (1-r)^{-d(a)}$ arcs I_j of the circle $\{|z| = r\}$ satisfying $|I_j| = 1 - r$ and

$$|\phi'| \approx \left(\frac{1}{1-r}\right)^{a \pm 0} \qquad \text{on} \quad I_j.$$

If we consider the discs

$$B_j = B(\phi(z_j), C\delta_j),$$

where z_j denotes the center of I_j, $\delta_j = (1-r)|\phi'(z_j)|$, and C is a large absolute constant, then we have

$$\omega B_j \geq \text{const}\ (1-r),$$

and therefore

$$d(a) \le \frac{f^+(\alpha)}{\alpha}, \qquad \left(\alpha \equiv \frac{1}{1-a}\right). \tag{3.5}$$

By (3.4) and Proposition 1.1, this implies the inequality "$\le$" in (3.1) for $t \le t_*$.

The opposite inequality (for all $t \in \mathbb{R}$) is a consequence of the following statement. If α is a growth point of $f^+(\alpha)$, or $f^-(\alpha)$, then

$$d(a) \ge \frac{f^\pm(\alpha)}{\alpha}, \qquad \text{respectively},$$

where a is such that $\alpha = (1-r)^{-1}$. The proof of the latter statement is based on Lemma 0.4.

Dynamical Interpretation. In the case of regular fractals one can interpret the relation

$$d(a) = \frac{f(\alpha)}{\alpha}, \qquad \left(\alpha \equiv \frac{1}{1-a}\right),$$

which was used in the proof of Theorem 3.1, as a statement about the radial behaviour of the derivative of the conformal map. For a univalent function ϕ, we consider the following "Hausdorff" version of the spectrum $d(a)$:

$$\tilde{d}(a) = \begin{cases} \dim\left\{\zeta \in \partial\mathbb{D} : \overline{\lim_{r\to 1}} \dfrac{\log|\phi'(r\zeta)|}{|\log(1-r)|} \ge a\right\}, & a \ge 0, \\ \dim\left\{\zeta \in \partial\mathbb{D} : \underline{\lim_{r\to 1}} \dfrac{\log|\phi'(r\zeta)|}{|\log(1-r)|} \le a\right\}, & a \le 0. \end{cases} \tag{3.6}$$

3.2. Proposition. *If ϕ is a conformal map onto a domain with a regular fractal boundary, then*

$$d(a) = \tilde{d}(a) = \frac{f(\alpha)}{\alpha} =$$
$$\dim\left\{\zeta \in \partial\mathbb{D} : \lim_{r\to 1} \frac{\log|\phi'(r\zeta)|}{|\log(1-r)|} = a\right\},$$

where $\alpha = (1-a)^{-1}$.

We will explain this result in the case of a Jordan domain Ω. Let $\phi : \mathbb{D} \to \Omega$ be the Riemann map. If we transplant the dynamics F from a neighborhood of $\partial\Omega$ in $\overline{\Omega}$ to a neighborhood of $\partial\mathbb{D}$ in $\overline{\mathbb{D}}$, and symmetrize it with respect to $\partial\mathbb{D}$, then we obtain an expanding conformal dynamics B on $\partial\mathbb{D}$ such that ϕ conjugates the systems $(\partial\mathbb{D}, B)$ and $(\partial\Omega, F)$. With respect to $(\partial\mathbb{D}, B)$, the Gibbs measure corresponding to the function
$\Theta = -\log|B'|$ is equivalent to the Lebesgue measure on $\partial\mathbb{D}$. Since harmonic measure is the image of the latter, we have the following equation for the packing spectrum $\pi(t)$, (see (2.12):

$$P(\pi\Theta + t\Psi) = 0, \qquad (\Psi = -\log|F' \circ \phi|).$$

Let $\sigma_{(t)}$ denote the Gibbs measure on $\partial\mathbb{D}$ with potential $\pi(t)\Theta + t\Psi$, and $\mu_{(t)} = \phi_*\sigma_{(t)}$. By an argument in the proof of Theorem 2.1, we have

$$\dim \mu_{(t)} = \frac{h_{\sigma_{(t)}}}{\int \Psi \, d\sigma_{(t)}} = f(\alpha),$$

where

$$\alpha = -\frac{1}{\pi'(t)} \overset{(2.14)}{=} \frac{\int \Theta \, d\sigma_{(t)}}{\int \Psi \, d\sigma_{(t)}},$$

and therefore

$$\dim \sigma_{(t)} = -\frac{h_{\sigma_{(t)}}}{\int \Theta \, d\sigma_{(t)}} = \frac{f(\alpha)}{\alpha}.$$

By the ergodic theorem, we have

$$\lim_{r\to 1} \frac{\log|\phi'(r\zeta)|}{|\log(1-r)|} = \frac{\int(\Psi - \Theta)\, d\sigma_{(t)}}{-\int \Theta \, d\sigma_{(t)}} = 1 - \frac{1}{\alpha} = a$$

for $\sigma_{(t)}$-a.e. $\zeta \in \partial\mathbb{D}$. In particular, this holds on a set of dimension $f(\alpha)/\alpha$. Combining this fact with (3.5), we complete the proof.

The formulae

$$\dim \sigma_{(t)} = \frac{f(\alpha)}{\alpha}, \qquad \dim \mu_{(t)} = f(\alpha) \tag{3.7}$$

provide the following characterization of the *boundary distortion.*

Consider the graph Γ of the packing spectrum $\pi(t)$. By (1.14) and (3.7), the tangent to Γ corresponding to t intersects the axes at the points $(0, \dim\sigma_{(t)})$ and $(\dim\mu_{(t)}, 0)$. We see that the coordinates of the intersection points describe the dimensions of the sets on $\partial\mathbb{D}$ and $\partial\Omega$ corresponding to each other under the conformal map. This motivates the following definition. Let Γ_- denote the part of the graph Γ lying in the halfplane $\{t < 0\}$. For $0 \le p \le 1$, we consider the tangent to Γ_- passing through the point $(0, p)$, and define $q_-(p)$ as the coordinate of the point at which the tangent intersects the t-axis. If there is no such tangent we consider, instead of it, the straight line parallel to the asymptote of Γ at $-\infty$ passing through $(0, p)$.

For $p \in [0, p_*]$, where $p_* = t_*|\pi'(t_*)|$, we define the function $q_+(p)$ in a similar fashion by considering the part $\Gamma_+ = \Gamma \cap \{t > t_*\}$, and the asymptote at $= \infty$. For $p \in [p_*, 1]$, we set $q_+(p) = t_*$. Finally, for $p \in [p_*, 1]$, we define the function $\sigma(p)$ by considering $\Gamma \cap \{0 \le t \le t_*\}$, and define $\sigma(p) = t_*$ for $p \in [0, p_*]$.

3.3. Proposition. *If ϕ is a conformal map onto a domain with a regular fractal boundary, then*

$$\begin{aligned} q_-(p) &= \inf\{\dim \phi E : \ E \subset \partial\mathbb{D}, \ \dim E = p\}; \\ q_+(p) &= \sup\{\dim \phi E : \ E \subset \partial\mathbb{D}, \ \dim E = p\}; \\ \sigma(p) &= \inf\{\dim \phi E : \ E \subset \partial\mathbb{D}, \ \dim(\partial\mathbb{D} \setminus E) = p\}. \end{aligned}$$

We sketch the graphs of these three functions in Figure 5. In general we have a loop which consists of a real analytic arc and two tangent straight line segments.

Figure 5

4. Universal spectra

We define the *universal spectra* $F(\alpha)$ and $\Pi(t)$ by the formulae

$$F(\alpha) = \sup_{\Omega} \ \tilde{f}^{+}(\alpha), \qquad \alpha \geq 0, \tag{4.1}$$

and

$$\Pi(t) = \sup_{\Omega} \ \tilde{\pi}(t), \qquad t \in \mathbb{R}, \tag{4.2}$$

where the suprema are taken over all plane domains Ω with compact boundaries. (See Section 1 for the definitions of $\tilde{f}^{+}$ and $\tilde{\pi}(t)$.) In this section and in the next one we provide some estimates and discuss some properties of the universal spectra.

Let us first mention that the universal bounds for the box dimension spectrum $f(\alpha)$ and the packing spectrum $\pi(t)$ are trivial — they are the same as the bounds for arbitrary measures in the complex plane:

$$\sup_\Omega f(\alpha) = \begin{cases} \alpha, & 0 \le \alpha \le 2, \\ 2, & \alpha \ge 2, \end{cases}$$

$$\sup_\Omega \pi(t) = \begin{cases} +\infty, & t < 0, \\ 1 - \frac{t}{2}, & 0 \le t \le 2, \\ 0, & t \ge 2. \end{cases}$$

Indeed, consider the collection of N very small closed discs B which are distributed almost uniformly inside the unit disc (the distances between the discs are $\ge \text{const}\, N^{-1/2}$). We can choose the diameters of the discs so that the harmonic measure of the domain $\hat{\mathbb{C}} \setminus \cup B$ is also almost equidistributed: $\omega_\infty B \asymp N^{-1}$. Then we iterate this construction and obtain a domain such that similar estimates hold on a sequence of scales tending to zero. This example does not work for the Hausdorff spectrum: the Hausdorff dimension of the boundary is very small. We still have

$$F(\alpha) = \alpha, \qquad \alpha \in [0, 1],$$

and

$$\Pi(t) = \begin{cases} +\infty, & t < 0, \\ 0, & t \ge 2, \end{cases}$$

but the bounds are nontrivial for $\alpha > 1$ and $t \in (0, 2)$.

The following theorem shows that the universal spectra have some features similar to those in the regular fractal case. We discuss this relation in the next section.

4.1. Theorem. *1) The universal spectra $F(\alpha)$ and $\Pi(t)$ satisfy the following Legendre type relations:*

$$F(\alpha) = \inf_{0 \le t \le 2} [\alpha \Pi(t) + t], \qquad \alpha \ge 1,$$

$$\Pi(t) = \sup_{\alpha \ge 1} \left[\frac{F(\alpha) - t}{\alpha} \right], \qquad t \in [0, 2].$$

2) We have the same universal bounds $F(\alpha)$ if we consider $\overline{\lim}$ instead of $\underline{\lim}$ in the definition (1.4) of the Hausdorff dimension spectrum, and the same bounds $\Pi(t)$ if we consider $\underline{\lim}$ instead of $\overline{\lim}$ in the definition (1.18) of $\tilde{\pi}(t)$.

Let us now turn to the simply connected case and define the universal spectra $F_{\rm sc}(\alpha)$ and $\Pi_{\rm sc}(t)$ by the same formulae (4.1), (4.2) but with suprema now being taken over the class (SC) of all simply connected domains with compact boundaries. In contrast to the general case, the universal bounds for the box counting and the Hausdorff spectra coincide for simply connected domains. The interval of parameters α such that $F_{\rm sc}(\alpha)$ is nontrivial is now $(1/2, +\infty)$. Of course,

$$\Pi_{\rm sc}(t) = 0 \qquad \text{for} \quad t \ge 2.$$

The universal covering (or packing) spectrum can be expressed in terms of the *universal integral means spectrum*

$$B(t) = \sup_{\Omega \in (\mathrm{SC})} \beta(t).$$

4.2. Theorem. *1) We have*

$$\begin{aligned} F_{\mathrm{sc}}(\alpha) &= \sup_{\Omega \in (\mathrm{SC})} f(\alpha), \\ \Pi_{\mathrm{sc}}(t) &= \sup_{\Omega \in (\mathrm{SC})} \pi(t) = B(t) - t + 1. \end{aligned}$$

2) The spectra $F_{\mathrm{sc}}(\alpha)$ and $\Pi_{\mathrm{sc}}(t)$ satisfy the Legendre type relations. 3) The uviversal bounds do not change if we take $\overline{\lim}$ or $\underline{\lim}$ in the definitions of the corresponding spectra.

In Figure 6 we sketch the graphs of the universal spectra, see estimates below.

FIGURE 6

Estimates. We first describe the behavior of the universal dimension spectrum as $\alpha \to \infty$. The following result was obtained in [JM].

4.3. Theorem.

$$2 - F(\alpha) \asymp \frac{1}{\alpha}, \qquad 2 - F_{\rm sc}(\alpha) \asymp \frac{1}{\alpha} \qquad as \quad \alpha \to \infty.$$

Taking the Legendre transform, we have

$$\Pi(t),\ \Pi_{\rm sc}(t) \asymp (2-t)^2 \qquad \text{as} \quad t \to 2- .$$

Thus we have a "smooth" phase transition in the universal packing spectrum at $t = 2$.

To see that $2 - F_{\rm sc}(\alpha) \le \text{const}\,\alpha^{-1}$, it is enough to consider, for instance, the snowflake of dimension $2 - \alpha^{-1}$ constructed from a polygon P_0 with 4 equal sides, see Section 2.

In the opposite direction, we will give a simple argument which proves a *weaker* estimate

$$2 - f^{+}(\alpha) \ge \text{const}\, \frac{1}{\alpha \log \alpha} \tag{4.3}$$

in the *simply connected* case.

Let Ω be a bounded simply connected domain and $a \in \Omega$. Suppose $|\zeta - a| \ge 1$, $r \le 1/2$. The following estimate of harmonic measure is a consequence of Lemma 0.3:

$$\omega_a B(\zeta, r) \le \exp\left\{ -c_1 \int_r^1 \frac{dt}{d(\zeta, t)} \right\},$$

where

$$d(\zeta, t) = \max\{\delta(z) :\ z \in \Omega, |\zeta - z| = t\}, \quad \delta(z) \equiv \text{dist}(z, \partial\Omega),$$

and $c_1 > 0$ is an absolute constant. The idea, suggested in [CJ], is to compare the function

$$\zeta \mapsto \int_r^1 \frac{dt}{d(\zeta, t)}$$

with the Marcinkiewicz integrals I_κ, $\kappa \in (0, 1)$,

$$I_\kappa(\zeta) \stackrel{\text{def}}{=} \int_\Omega \frac{\delta(z)^\kappa}{|z - \zeta|^{2+\kappa}}\ dm_2(z), \qquad \zeta \in \mathbb{C} \setminus \Omega.$$

It is well known that I_κ satisfy the BMO-type inequalities

$$m_2\{I_\kappa > \lambda\} \le m_2(\Omega) e^{-c_2 \kappa \lambda}, \qquad \lambda > 1, \tag{4.4}$$

with an absolute constant $c_2 > 0$. The integral $I_\kappa(\zeta)$ has a trivial lower bound in terms of the function $d(t) \equiv d(\zeta, t)$:

$$I_\kappa(\zeta) \ge \text{const} \int_0^\infty \frac{d^{1+\kappa}(t)\ dt}{t^{2+\kappa}}.$$

We have

$$\log r| = \int_r^1 \frac{dt}{t} \le \left(\int_r^1 \frac{d^{1+\kappa}(t)\ dt}{t^{2+\kappa}}\right)^{\frac{1}{2+\kappa}} \left(\int_r^1 \frac{dt}{d(t)}\right)^{\frac{1+\kappa}{2+\kappa}}$$
$$\le \text{const}\ I_\kappa(\zeta)^{\frac{1}{2+\kappa}} \left(\int_r^1 \frac{dt}{d(t)}\right)^{\frac{1+\kappa}{2+\kappa}},$$

and therefore,

$$\omega_a B(\zeta, r) \le \exp\{-c_3 I_\kappa^{-\frac{1}{1+\kappa}} |\log r|^{\frac{2+\kappa}{1+\kappa}}\}.$$

It follows that if

$$\omega_a B(\zeta_0, r) \ge r^\alpha, \tag{4.5}$$

then

$$I_\kappa(\zeta) \ge \text{const}\,\alpha^{-1-\alpha} |\log r|, \qquad \eta \in B(\zeta_0, r).$$

Combining this estimate with (4.4), we obtain the following inequality for the number N of discs satisfying (4.5):

$$Nr^2 \le \text{const}\ \ r^{c_4 \kappa \alpha^{-1-\kappa}}.$$

The choice $\kappa \asymp (\log \alpha)^{-1}$ gives (4.3).

An interesting property of the universal spectrum $\Pi_{\text{sc}}(t)$ is the existence of a negative phase transition point. The following result was established in [CM].

4.4. Theorem. *There is an absolute constant $K \ge 4$ such that*

$$F_{\text{sc}}\left(\frac{1}{2} + \varepsilon\right) \sim K\varepsilon \qquad \textit{as} \quad \varepsilon \to 0. \tag{4.6}$$

Applying the Legendre transform, we have

$$\begin{cases} \Pi_{\text{sc}}(t) & = -2t, \qquad t \le t_0 \stackrel{\text{def}}{=} -K/2, \\ \Pi_{\text{sc}}(t) & > -2t, \qquad t > t_0. \end{cases}$$

In terms of the universal integral means spectrum $B(t)$, we have

$$B(t) = |t| - 1 \quad \text{for} \quad t \le t_0.$$

Let us describe the idea of the proof. Since the function $F_{\text{sc}}(\alpha)$ is concave (this is a part of Theorem 4.2), to prove (4.6) for some K, it is sufficient to check that

$$f^+\left(\frac{1}{2} + \varepsilon\right) \le (\text{abs. const})\ \varepsilon \tag{4.7}$$

for every simply connected domain Ω. Suppose $\infty \in \Omega$, $\operatorname{diam} \partial\Omega = 1$, and let

$$\omega_\infty B(z, \delta) \geq \delta^{1/2+\varepsilon}.$$

By Lemma 0.4, there is an arc $l \subset \partial\Omega \cap \partial B(z, \delta)$ such that the extremal distance $\lambda_\Omega(l, L)$ in Ω between l and, say, $L = \{|z|\}$ satisfies the inequality

$$\lambda_\Omega(l, L) \leq \left(\frac{1}{2} + 2\varepsilon\right) \log \frac{1}{\delta}. \tag{4.8}$$

Let $\delta = e^{-nA}$ for some $A \gg 1$. We consider the annuli

$$R_\nu(z) = \{\zeta : \ \frac{1}{2} e^{-\nu A} < |\zeta - z| < 2e^{(1-\nu)A}\}, \qquad 1 \leq \nu \leq n.$$

They all have the same extremal distance $\lambda \asymp A$ between the inner and the outer boundaries $\partial_\pm$. We define $X_\nu(z)$, the "excess" of extremal length in $\Omega \cap R_\nu(z)$, by the equations

$$X_\nu(z) = \lambda_\nu(z) - \lambda,$$
$$\lambda_\nu = \inf\{\lambda_\Omega(l_+, l_-) : \ l_\pm \text{ arcs on } \partial_\pm \cap \Omega\}.$$

Thus $X_\nu(z) \geq 0$ and $X_\nu(z)$ is zero if and only if the part of $\partial\Omega$ in $R_\nu(z)$ lies on a straight line segment with endpoint z. It follows from (4.8) and the subadditivity of the extremal length that

$$\sum_{\nu=1}^{n-1} X_\nu(z) \leq \varepsilon n. \tag{4.9}$$

In other words, $X_\nu(z)$ is small on the average. The smallness of $X_\nu(z)$ means that the boundary $\partial\Omega$ passes though a very thin sector in $R_\nu(z)$. More precisely, if

$$X_\nu(z) \leq e^{-\sigma k A} \tag{4.10}$$

(σ is an absolute constant), then the angle of the sector is $\ll e^{-kA}$, and we have

$$X_{\nu+1}(z'), \ldots, X_{\nu+k}(z') \geq \text{const}\, A \tag{4.11}$$

for every point $z' \in \partial\Omega \cap R_\nu(z)$. By a combinatorial argument, one can show that the maximal number of points $z \in \partial\Omega$ separated by $\delta = e^{-nA}$, satisfying (4.9) and also satisfying (4.11) every time we have (4.10), does not exceed $\text{const}\, e^{c\varepsilon n}$, which proves (4.7).

Determining the value of the constant K in Theorem 4.4 is an interesting and perhaps difficult problem. It is conjectured (J.Brennan) that $K = 2$, and consequently, $t_0 = -2$. In fact, it is not ruled out that the universal integral means spectrum $B(t)$ is an even function. (Recall that $B(t)$ has a positive phase transition point at $t = 2$.)

The estimate $K \geq 4$ is obtained by considering the following class of fractal sets (*"dandelions").* We start with a simply connected domain Ω_0 such that $\infty \in \Omega$ and the boundary $\Gamma_0 = \partial\Omega_0$ consists of a finite number of straight line segments. Let $b, a_1, \ldots, a_m$ be the extreme points of Γ_0, i.e. the points at which Ω_0 makes a full angle. We assume that $b = 0, \{x : x < 0\} \subset \Omega_0$, and that the segment L of Γ_0 with endpoint b lies on a real axis. For a fixed small number $\kappa > 0$ we denote l_j the segment of length κ lying on Γ_0 and having a_j as an endpoint. We define the polygon Γ_1 by replacing each l_j with a rescaled copy of Γ_0 so that under rescaling the segment L corresponds to l_j. The polygon Γ_1 has m^2 extreme points other than b. To obtain Γ_2 we repeat the above procedure with scale κ^2. Proceeding with this construction we define polygons $\Gamma_3, \Gamma_4, \ldots$ which converge to some fractal set $\Gamma = \Gamma(\kappa)$. (Observe that if κ is small enough, then no intersections occur at any step of the construction.)

We estimate the dimension spectrum of the harmonic measure on Γ in terms of the following conformal invariant (*"reduced extremal length").* Let Ω be a simply commected domain and $a, b \in \mathbb{C}$. For $\varepsilon > 0$ let λ_ε denote the extemal length of the family of all curves joining the ε-neighborhoods of a and b in Ω, and $\tilde{\lambda}_\varepsilon$ the extremal length of the corresponding family in $\mathbb{C}$. Define

$$\beta(\Omega; a, b) = \lim_{\varepsilon \to 0} \exp\{2\pi(\tilde{\lambda}_\varepsilon - \lambda_\varepsilon)\};$$

the existence of the limit is a standard property of the extremal length. Suppose now we have $m + 1$ distinct points $b, a_1, \ldots, a_n \in \partial\Omega$. Denote

$$\beta_j = \beta(\Omega; a_j, b).$$

For any $p > 2$ we can choose the initial polygon Γ_0 (actually we take a union of two perpendicular segments) in the dandelions construction such that

$$\sum \beta_j^p > 1, \qquad \text{(with respect to } \Omega_0).$$

Then for a fixed small $\varepsilon > 0$ and the dandelion $\Gamma(\kappa)$ with $\kappa \ll 1$, we have

$$f^+\left(\frac{1}{2} + \varepsilon\right) \geq 2p\varepsilon$$

which shows that $K \geq 4$.

In fact it can be shown that K is exactly twice the minimal value of p such that

$$\sum_{j=1}^{m} \beta_j^p \leq 1$$

for any m and every configuration $(\Omega; \{a_j\}, b)$. In particular, Brennan's conjecture is equivalent to the statement

$$\sum_{j=1}^{m} \beta_j^2 \leq 1 \tag{4.12}$$

In [CM] it is shown that (4.12) is always true for $m = 2$.

Another notable parameter in the universal dimension spectrum is $\alpha = 1$. As we explained in Section 3, we have

$$\alpha - F_{sc}(\alpha) \asymp (\alpha - 1)^2 \qquad \text{as} \quad \alpha \to 1, \tag{4.13}$$

which is one order stronger than the statement $\dim \omega = 1$ (for every simply connected domain). We conjecture that in the non-simply connected case we also have

$$\alpha - F(\alpha) \asymp (\alpha - 1)^2 \qquad \text{as} \quad \alpha \to 1+.$$

The corresponding statement concerning the dimension of harmonic measure,

$$\dim\ \omega \leq 1,$$

is a theorem of Jones and Wolff [JW].

We conclude this section with two open questions:

(1) Is it true that $F(\alpha) = F_{\mathrm{sc}}(\alpha) \quad \text{for} \quad \alpha \geq 1$?
(2) Is it true that the functions $F_{\mathrm{sc}}(\alpha)$ and $F(\alpha)$ are smooth or even real analytic on the intervals $(1/2, \infty)$ and $(1, \infty)$ respectively?

5. Fractal approximation. Applications

We define

$$F^{\mathrm{fr}}(\alpha) = \sup_{\Omega \in (\mathrm{Cantor})} f(\alpha),$$

where the supremum is taken over domains such that the boundary is a Cantor set. Similarly, we define $\Pi^{\mathrm{fr}}(t)$.

5.1. Theorem.

$$F(\alpha) = F^{\mathrm{fr}}(\alpha), \qquad \Pi(t) = \Pi^{\mathrm{fr}}(t).$$

The same is true for simply connected domains. For instance,

$$\Pi_{\mathrm{sc}}(t) = \sup_{\Omega \in (\mathrm{snowflakes})} \pi(t). \tag{5.1}$$

These results imply, in particular, Theorems 5.1 and 5.2 in the previous section.

In the simply connected case, the relation (5.1) is stated in [CJ] for $t = 1$. We will outline the proof for the non-simply connected case, and the argument will be based on an idea from [JW]. In fact, the main result of [JW], the inequality $\dim \omega \leq 1$, can be thought of as a statement in fractal approximation:

$$\sup_{\Omega} \dim\ \omega = \sup_{\Omega \in (\mathrm{Cantor})} \dim\ \omega \underset{(\mathrm{Thm.2.3})}{\leq} 1.$$

Let $N_\alpha(\delta)$ denote the maximal number of disjoint closed discs $\Delta_j \subset \mathbb{D}$ of radius δ such that

$$\omega_\infty(\Delta_j, \hat{\mathbb{C}} \setminus \cup\Delta_j) \geq \delta^\alpha.$$

We define

$$\Phi(\alpha) = \lim_{\eta\to 0} \overline{\lim_{\delta\to 0}} \frac{\log N_{\alpha+\eta}(\delta)}{|\log\delta|}.$$

The first formula of Theorem 5.1 is a consequence of the following two inequalities:

$$F(\alpha) \leq \Phi(\alpha), \tag{5.2}$$

$$\Phi(\alpha) \leq F^{\mathrm{fr}}(\alpha). \tag{5.3}$$

To prove (5.2), we fix $\alpha > 0$ and $\eta > 0$, and consider some domain Ω such that $\infty \in \Omega$, $\partial\Omega \subset \mathbb{D}$. Let $\underline{\alpha}(z)$ denote the lower pointwise dimension (see (1.4)) of the harmonic measure $\omega = \omega_\infty$ of Ω, and

$$A = \{z \in \partial\Omega : \ \underline{\alpha}(z) \leq \alpha\}.$$

Fix $\varepsilon > 0$. By the covering lemma, there is a finite multiplicity cover of the set A with discs B_j of radii $\delta_j \leq \delta(\varepsilon)$, $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\omega B_j \geq \delta_j^{\alpha+\eta}. \tag{5.4}$$

Given integer numbers m, k, let us estimate the number $N(m,k)$ of B_j's satisfying

$$\begin{cases} \delta_j \asymp \delta \equiv 2^{-n}, \\ \kappa_j \equiv \operatorname{cap}(\partial\Omega \cap B_j) \asymp \kappa \equiv 2^{-(m+k)}. \end{cases} \tag{5.5}$$

Let $M = M(\varepsilon)$ denote the constant in Lemma 0.2. We can select $N \asymp M^{-2}N(m,k)$ discs Δ from the collection $\{B_j\}$ satisfying (5.4), (5.5) such that the centers of Δ's are separated by $M\delta$.

For each Δ, let $\tilde{\Delta}$ be the closed disc concentric with Δ of radius

$$\tilde{\delta} = \delta\left(\frac{\kappa}{\delta}\right)^\varepsilon = 2^{-(m+k\varepsilon)}.$$

By Lemma 0.2, the harmonic measure of $\tilde{\Delta}$ with respect to the domain $\tilde{\mathbb{C}} \setminus \cup\tilde{\Delta}$ is $\geq$ const $\tilde{\delta}^{\alpha+\eta}$, and therefore, by the choice of $\delta(\varepsilon)$, we have

$$N \leq \text{const}\left(\frac{1}{\tilde{\delta}}\right)^{\Phi(\alpha+\eta)+\varepsilon}.$$

We can now estimate the Hausdorff dimension of the set A. For $p > \Phi + \varepsilon$, $\Phi \equiv \Phi(\alpha+\eta)$, we have

$$\begin{aligned} \sum \delta_j^p &\leq \ \text{const} \sum \kappa_j^p \\ &\asymp \sum_{m \geq |\log\delta(\varepsilon)|} \sum_{k\geq 0} 2^{-(m+k)p} N(m,k) \\ &\leq \ \text{const} \sum_{(m)} 2^{-m(p-\Phi-\varepsilon)} \sum_{(k)} 2^{-k(p-\varepsilon\Phi-\varepsilon^2)} \\ &= \ o(1) \qquad \text{as} \ \ \varepsilon \to 0, \end{aligned}$$

and $\dim A \le \Phi(\alpha+\eta)+\varepsilon$. Since ε and η are arbitrary, we have

$$\tilde{f}^+(\alpha) \le \Phi(\alpha),$$

and hence (5.2).

To prove (5.3), we first fix positive numbers $\eta \ll 1$ and $\kappa \ll \eta$, and take an arbitrarily small δ such that there are $\delta^{-\Phi(\alpha)+0}$ closed δ-discs $B \subset \mathbb{D}$ satisfying

$$\omega_\infty(B, \hat{\mathbb{C}} \setminus \cap B) \ge \delta^\alpha$$

and such that the centers are separated by 4δ. Next we take $\asymp \delta^{-\kappa}$ discs of radius δ^η such that the centers are equidistributed on the circle $\{|z| = 1/2\}$. We replace each of the latter discs with a rescaled copy of the configuration $(\mathbb{D}, \{B\})$. Then we have

$$N \asymp \left(\frac{1}{\delta}\right)^{\kappa+\Phi(\alpha)-0} \ge \text{const} \left(\frac{1}{\tilde{\delta}}\right)^{\frac{\Phi(\alpha)}{1+\eta}}$$

closed discs Δ_j of radius $\tilde{\delta} = \delta^{1+\eta}$. By Harnack's inequality,

$$\omega_j \equiv \omega_\infty(\Delta_j, \hat{\mathbb{C}} \setminus \cup_{j=1}^N \Delta_j) \ge \text{const}\ \ \delta^{\alpha+\kappa} \asymp \tilde{\delta}^{\frac{\alpha+\kappa}{1+\eta}}.$$

Let J be the selfsimilar Cantor set corresponding to the initial configuration $(\mathbb{D}, \{\Delta_j\})$. $\blacksquare$
We have

$$\operatorname{cap} J \ge c_1 > 0,$$

where the constant c_1 depends only on κ. The analysis of the Cantor construction shows that for every cylinder set $X \subset J$, $X = (x_1, \ldots, x_n) \in \{1, \ldots, N\}^n$, we have

$$\omega_\infty(X, \hat{\mathbb{C}} \setminus J) \ge c_2^n \prod_{\nu=1}^{n} \omega_{x_\nu},$$

where c_2 depends only on c_1. Since δ is arbitrarily small, we have

$$F^{\text{fr}}(\alpha) \ge \frac{\Phi(\alpha)}{1+\eta},$$

and since η is arbitrary, we obtain (5.3).

Finally, we observe that the argument used in the proof of (5.2) also gives the estimate

$$\Pi(t) \le \sup_\alpha \frac{\Phi(\alpha)-t}{\alpha},$$

and that the latter implies the second formula of Theorem 5.1:

$$\begin{aligned} \Phi(t) \le \sup_\alpha \frac{\Phi(\alpha)-t}{\alpha} &\underset{(5.3)}{\le} \sup_\alpha \frac{F^{\text{fr}}(\alpha)-t}{\alpha} \\ &\underset{(\text{Thm.}2.1)}{=} \Pi^{\text{fr}}(t) \le \Pi(t). \end{aligned}$$

Description of integral means spectra. The methods of fractal approximation can be used to describe all posible dimension and packing spectra of harmonic measures in the complex plane in terms of the universal spectra. We will state the corresponding result for the packing spectrum in the simply connected case.

5.2. Theorem. *Let $\pi(t)$, $t \in \mathbb{R}$, be a convex function. Then there exists a simply connected domain Ω such that $\pi(t)$ is the packing spectrum of the harmonic measure of Ω if and only if $\pi(t)$ satisfies the following two conditions:*

(1) $1 - t \leq \pi(t) \leq \Pi_{\mathrm{sc}}(t)$;
(2) *the asymptotes of the graph $y = \pi(t)$ intersect the y-axis in the segment $[0, 1]$.*

It is clear that the conditions are necessary. To prove the sufficiency, consider the Legendre transform of $\pi(t)$:

$$\hat{f}(\alpha) = \inf_t [\alpha \pi(t) + t].$$

Then (1) and (2) imply that $\hat{f}(\alpha) \leq F_{\mathrm{sc}}(\alpha)$, $\hat{f}(1) = 1$, and $\hat{f}(\alpha)$ is either ≥ 0 or $= -\infty$. We want to construct a domain such that $\hat{f}(\alpha)$ is the concave envelope of the dimension spectrum. Clearly, it is sufficient to do this for the functions $\hat{f} = \hat{f}_{\alpha_0, q_0}$ such that

$$\begin{cases} \hat{f}(1) = 1, \qquad \hat{f}(\alpha_0) = q_0, \\ \hat{f} \ \text{ is linear on } \ [1, \alpha_0], \\ \hat{f} = -\infty \ \text{ on } \mathbb{R} \setminus [1, \alpha_0], \end{cases}$$

where $\alpha_0 \geq 1/2$ and $q_0 \in [0, F_{sc}(\alpha_0)]$ are fixed parameters. The idea is to use the following version of the dandelion construction. We start with a snowflake domain such that the dimension spectrum satisfies $f(\alpha) \geq q_0$. Let ϕ denote the corresponding Riemann map. Then we construct a certain "selfsimilar" Cantor type set $E \subset \partial \mathbb{D}$ of dimension q_0/α_0 such that

$$|\phi'(r\zeta)| \approx \left(\frac{1}{1-r} \right)^{1 - 1/\alpha_0}, \qquad \zeta \in E,$$

cf. Proposition 3.2. We will get a domain with the desired properties if we take the ϕ-image of the saw-like domain

$$\{ z \in \mathbb{D} : \ \operatorname{dist}(z, E) < \rho(1 - |z|) \},$$

where $\rho = \rho(x)$ is a function slowly tending to zero as $x \to 0$.

We can restate Theorem 5.2 in terms of the integral means spectrum as follows.

Let $\beta(t)$, $t \in \mathbb{R}$, be a convex function such that

(1) $0 \leq \beta(t) \leq B(t)$,
(2) $|\beta'(t \pm 0)| \leq |t|^{-1}(1 + \beta(t))$.

Figure 7

Then there exists a bounded univalent function ϕ such that $\beta(t)$ is the integral means spectrum of ϕ'.

We will indicate now some applications.

Suppose there are certain conditions on one or several parameters in the integral means spectrum $\beta(t)$ of a univalent function ϕ. Theorem 5.2 can be used to describe the whole set of restrictions we have for the integral means spectrum.

Example 1. Let Hölder(η) denote the class of Hölder continuous univalent functions ϕ with Hölder exponent η, $0 < \eta \leq 1$. Denote

$$B_\eta(t) = \sup\{\beta(t) : \quad \phi \in \text{Hölder}(\eta)\}.$$

Observe that we have the restriction $\beta'(+\infty) \leq 1-\eta$. Let $y = \tau_\eta(t)$ be the equation of the tangent to the graph $y = B(t)$ with slope $1-\eta$. Then

$$B_\eta(t) = \begin{cases} B(t), & t \leq t_\eta, \\ \tau_\eta(t), & t \geq t_\eta, \end{cases}$$

where t_η corresponds to the tangent point, see Figure 7a.

Similarly, we can characterize the influence of the boundary size, namely the Minkowski dimension, on the behavior of the integral means.

Example 2. For $M \in [1,2]$, denote

$$B^M(t) = \sup\{\beta : \ M(\partial\Omega) \le M\}.$$

Let $y = \tau^M(t)$ be the equation of the tangent to the graph $y = B(t)$ such that $\tau^M(M) = M - 1$. Then

$$B^M(t) = \begin{cases} B(t), & t \le t^M, \\ \tau^M(t), & t^M \le t \le M, \\ t-1, & t \ge M, \end{cases}$$

where t^M corresponds to the tangent point, see Figure 7b.

For a Hölder domain, the dimension $M(\partial\Omega)$ is the solution to the equation $\beta(t) = t - 1$, see (3.3). Therefore, we have

$$B_\eta(t) = B^M(t) \qquad \text{for } \ t \le M, \tag{5.6}$$

where $M = F_{\mathrm{sc}}(1/\eta)$. The equation (5.6) implies the following characterization of the universal dimension spectrum.

Corollary.

$$F_{\mathrm{sc}}(\alpha) = \sup\{M(\partial\Omega) : \ \ \Omega \in \textit{Hölder}(1/\alpha)\}, \quad \alpha \ge 1.$$

In particular, by Theorem 4.3 we have

$$\Omega \in \text{Hölder}(\eta) \qquad \Longrightarrow \qquad \dim \partial\Omega \le M(\partial\Omega) \le 2 - C\eta,$$

where $C > 0$ is an absolute constant.

The growth of the integral means of order one is closely related to the growth of the *coefficients* $\{a_n\}$ of a univalent

$$\phi(z) = \sum_{n \ge 0} a_n z^n.$$

Denote

$$\gamma(\phi) = \varlimsup_{n \to \infty} \frac{\log |a_n|}{\log n}.$$

It was shown in [CJ] that

$$\gamma \stackrel{\text{def}}{=} \sup\{\gamma(\phi) : \ \phi \text{ is a bounded univalent function}\} = B(1) - 1.$$

Applying the argument in the proof of this theorem and the relation (5.6) for $t = 1$, one can describe the dependence of the growth of coefficients on the size of the boundary. The following result was obtained in [MP]. Denote

$$\gamma^M = \sup\{\gamma(\phi) : \ M(\partial\Omega) \le M\}.$$

5.3. Theorem. *There is a number $M_c \in (1,2)$ such that*

$$\begin{cases} \gamma^M = \gamma, & \text{for } M \geq M_c, \\ \gamma^M < \gamma, & \text{for } 1 \leq M < M_c. \end{cases}$$

More precisely,

$$M_c = 1 + \frac{B(1)}{1 - B'(1+)},$$

and for $1 \leq M < M_c$, we have

$$\gamma^M = \frac{M-1}{F_{\mathrm{sc}}^{-1}(M)} - 1.$$

In particular, by (4.13),

$$\gamma^M = -1 + (M-1) - (M-1)^2 + \mathrm{O}((M-1)^3) \quad \text{as} \quad M \to 1+.$$

The universal spectra $B_\eta(t)$ and $B^M(t)$ have some phase transition points. Here is another example.

Example 3. For a fixed parameter $\beta_1 \in (0, B(1))$ consider the graph of the function

$$\sup\{\beta(t) : \ \phi \text{ satisfies } \beta(1) \leq \beta_1\},$$

see Figure 8. Then there are *five* phase transition points: $t_0, t_-, 1, t_+, 2$, where $t_0 = -K/2$ (see Theorem 4.4), and $t_\pm$ correspond to the tangets to the graph of $B(t)$ passing though the point $(1, \beta_1)$.

We will now discuss some other applications of the concept of the universal integral means spectrum.

Unbounded unvalent functions. First we consider *arbitrary* univalent functions. Denote

$$\check{B}(t) = \sup \beta(t),$$

where the supremum is taken over all univalent functions in the unit disc.

5.4. Theorem. $\check{B}(t) = \max\{B(t), 3t-1\}$.

Of course, the term $3t-1$ comes from the Koebe function $\phi(z) = z(1-z)^{-2}$. The phase transition point $\check{t}$ (see Figure 9) is the solution to the equation

$$\beta(t) = 3t - 1.$$

It is clear that $1/3 < \check{t} < 2/5$, cf. [FM].

To prove Theorem 5.4 we can assume that $\phi = 1/\psi$, where ψ is a bounded univalent function. Then

$$\int_{|z|=r} |\phi'|^t = \int_{|z|=r} \left| \frac{\psi'}{\psi^2} \right|^t, \tag{5.7}$$

FIGURE 8 FIGURE 9

and we must take into consideration the sets where $|\psi'|$ is large and the sets where $|\psi|$ is small. For $b \in [0,2]$ and $a \in [-1,1]$, we define

$$m(a,b) = -\varlimsup_{r\to 1} \frac{\log |\{z: \ |z| = r, |\psi(z)| \asymp (1-r)^b, \psi'(z)| \asymp (1-r)^{-a}\}|}{|\log(1-r)|}.$$

If we use a trivial estimate

$$\omega_{\psi(0)}\Delta \leq \text{const } (1-r)^{b/2} \tag{5.8}$$

for the harmonic measure of the disc $\Delta = B(0,(1-r)^b)$ with respect to the domain $\Omega = \psi\mathbb{D}$, and rescale $\partial\Omega \cap \Delta$ to the unit disc, then we obtain the inequality

$$m(a,b) \geq 1 - (1-a-b)F_{\text{sc}}\left(\frac{1-b/2}{1-a-b}\right). \tag{5.9}$$

Then it follows from (5.7) that

$$\beta(t) \leq \sup_{(a,b)} [(a+2b)t - m(a,b)].$$

The computation of the two-dimensional Legendre transform completes the proof of the theorem: for $l \in [0,2]$, we have

$$\text{(5.10)}\quad \sup\left[(a+2b)t-1+(1-a-b)F\left(\frac{1-b/l}{1-a-b}\right)\right]=\max\{B(t),(1+l)t-1\},$$

where the supermum is over the set $\{-1 \leq a \leq 1,\ 0 \leq b \leq l\}$.

Similar considerations can be used to characterize the integral means spectrum of univalent functions with *restricted growth.* For $l \in [0,2]$, let (S_l) denote the class of univalent functions ϕ satisfying

$$\phi(z)=O\left(\frac{1}{(1-|z|)^l}\right).$$

The function

$$\text{(5.11)}\qquad \phi(z)=(1-z)^{-l}$$

with

$$\beta(t)=\max\{0,(1+l)t-1\}$$

is now a natural candidate for the extremal growth.

If we use the estimate

$$\omega\Delta \leq \text{const } (1-r)^{b/l\ -0}$$

instead of (5.8), then we get (5.9) with b/l instead of $b/2$. Applying (5.10), we have

$$\sup_{\phi\in(S_l)}\beta(t)=\max\{B(t),(1+l)t-1\}.$$

The special case $t=1$ implies the corresponding result for the coefficients. The function (5.11) satisfies

$$\gamma(\phi)\equiv\overline{\lim_{n\to\infty}}\frac{\log|a_n|}{\log n}=l-1.$$

Littlewood and Paley [LP] observed that

$$\sup_{\phi\in(S_l)}\gamma(\phi)=l-1$$

for $l \in [1/2,2]$. This was improved to $l \geq .497$ in [Ba].

5.5. Theorem.

$$\sup_{\phi\in(S_l)} \gamma(\phi) = \begin{cases} l-1\ , & l \geq B(1), \\ \gamma = B(1)-1, & l \leq B(1). \end{cases}$$

Finally, we consider the coefficients problem for *symmetric* univalent functions. For $m \in \mathbb{N}$, let $S^{[m]}$ denote the class of m-fold symmetric univalent functions ϕ:

$$\phi(e^{\frac{2\pi i}{m}} z) = \phi(z), \qquad \phi(0) = 0.$$

Every such function can be represented in the form

$$\phi(z) = [\psi(z^m)]^{1/m},$$

where ψ is univalent. Reasoning as in the proof of Theorem 5.4, we show that

$$\sup_{\phi\in S^{[m]}} \beta(t) = \max\{B(t), (1+\frac{2}{m})t - 1\}.$$

Combining this fact for $t = 1$ with the argument in [CJ], we obtain the following result.

5.6. Theorem.

$$\sup_{\phi\in S^{[m]}} \gamma(\phi) = \begin{cases} \frac{2}{m} - 1 & m < \frac{2}{B(1)}, \\ \gamma = B(1)-1, & m > \frac{2}{B(1)}. \end{cases}$$

Known estimates of $B(1)$ show that we have the first case for $m = 1, 2, 3, 4$, and the second case for $m \geq 12$. The exponent $2/m - 1$ corresponds to the symmetrized Koebe function. Littlewood and Paley [LP] observed that the coefficients of this function have the maximal order of growth for $m \leq 3$, but later Littlewood [L] proved that this is not true for some very large m.

Exceptional sets.

We will also state some results concerning exceptional sets for the radial behavior of ϕ' and for the boundary distortion, cf. [M2]. The point is that, by fractal approximation, we have the same relations between the universal bounds as in the case of regular fractals, see Propositions 3.2 and 3.3.

For a univalent function ϕ, we consider the spectrum $\tilde{d}(a)$ defined in (3.6) in terms of the Hausdorff dimension of the sets where $|\phi'|$ or $|\phi'|^{-1}$ has the radial growth greater than $(1-r)^{-|a|}$. Taking the supremum over all univalent functions, we define

$$D(a) = \sup_{\phi} \tilde{d}(a).$$

FIGURE 10

5.7. Proposition. *For $a \in [-1,1]$ and $\alpha = (1-a)^{-1}$, we have*

$$D(a) = \frac{F_{\mathrm{sc}}(\alpha)}{\alpha}.$$

We also note that the spectrum $D(a)$ does not depend on whether we take $\underline{\lim}$ or $\overline{\lim}$ in the definition (3.5) if $\tilde{d}(a)$. The estimates of the dimension spectrum $F_{\mathrm{sc}}(\alpha)$ established in Section 4 imply the corresponding properties of $D(a)$, for instance: $D'(1-) = -2,\ D'(-1+) = K/2$, where K is the constant in Theorem 4.4.

Finally we consider the functions

$$Q(p) = \sup_{\phi} q_{-}(p), \qquad \Sigma(p) = \sup_{\phi} \sigma(p), \qquad (0 \le p \le 1),$$

where

$$\begin{aligned} q_{-}(p) &= \inf\{\dim e :\ e \subset \partial\Omega,\ \dim \phi^{-1} e = p\}, \\ \sigma(p) &= \inf\{\dim e :\ e \subset \partial\Omega,\ \dim(\partial\Omega \setminus \phi^{-1} e) = p\}, \end{aligned}$$

and

$$\phi^{-1} e \equiv \{\zeta \in \partial\mathbb{D} :\ \lim_{r\to 1} \phi(r\zeta) \ \text{ exists and } \ \in e\},$$

FIGURE 11

cf. Proposition 3.3.

5.8. Proposition. *The spectra $Q(p)$ and $\Sigma(p)$ can be obtained from the graph of $\Pi_{\mathrm{sc}}(t)$ by the procedure indicated in Figure 11.*

6. Two examples

In this final section of the paper we return to the multifractal analysis of particular harmonic measures and consider two examples of (non-regular) fractal sets.

Polynomial Julia sets. Let F be an arbitrary polynomial of degree d and $\omega = \omega_\infty$ be the harmonic measure of the basin Ω at ∞. We will characterize the negative part of the packing spectrum $\pi(t)$, $t < 0$, and the corresponding parts of the dimension spectra $f(\alpha), \tilde{f}(\alpha)$ for $\alpha < \alpha_- \equiv |\pi'(0-)|^{-1}$, see Section 1 for definitions. It turns out that the Hausdorff dimension spectrum $\tilde{f}(\alpha)$ behaves exactly as in the hyperbolic case but $\pi(t)$ and $f(\alpha)$ can have phase transitions. The simplest example is provided by Chebychev's polynomials $\pm P_d$, where P_d is defined by the functional equation

$$P_d(z + z^{-1}) = z^d + z^{-d},$$

e.g., $P_2(z) = z^2 - 2$, $P_3(z) = z^3 - 3z$, etc. The Julia set of $\pm P_d$ is the segment $[-2, 2]$, (because the map $z \mapsto z + z^{-1}$ conjugates z^d and P_d), and the integral means spectrum

$$\beta(t) = \max\{-t - 1, 0\}$$

has a phase transition at $t_c = -1$. The polynomials $\pm P_d$ and z^d are known to be the only polynomials (up to conjugation) satisfying dim $\omega = \dim J$, see [Z]. The following results were obtained in [MS2], see also [MS1].

6.1. Theorem. *Let F be a polynomial such that* dim $\omega < \dim J$. *Then either*

(1) *$\pi(t)$ is a strictly convex, real analytic function on $(-\infty, 0)$, or*
(2) *there is a point $t_c < 0$ such that*

$$\pi'(t_c-) < \pi'(t_c+).$$

In the latter case, $\pi(t)$ is strictly convex, real analytic on $[t_c, 0)$, and linear $(= \pi'(t_c-)t))$ on $(-\infty, t_c]$.

6.2. Theorem. *1) In the first case of Theorem 6.1, we have*

$$f(\alpha) = \tilde{f}(\alpha) = \inf_{t<0} [\alpha\pi(t) + t], \qquad 0 < \alpha < \alpha_-.$$

2) In the second case, the function $\pi|(t_c, 0)$ extends to a strictly convex, real analytic function $\tilde{\pi}(t)$ on $(-\infty, 0)$, and

$$f(\alpha) = \inf_{t<0} [\alpha\pi(t) + t],$$
$$\tilde{f}(\alpha) = \inf_{t<0} [\alpha\tilde{\pi}(t) + t],$$

for $\alpha \in (0, \alpha_-)$. If we denote $\tilde{\alpha}_{\min} = |\tilde{\pi}'(-\infty)|^{-1}$, $\alpha_c = |\pi'(t_c+)|^{-1}$, then

$$0 < \alpha_{\min} = |\pi'(t_c-)|^{-1} < \tilde{\alpha}_{\min} < \alpha_c < \alpha_-,$$

the function $\tilde{f}$ is strictly concave and real analytic on $(\tilde{\alpha}_{\min}, \alpha_-)$, $f = \tilde{f}$ on $[\alpha_c, \alpha_-]$, and f is linear on $[\alpha_{\min}, \alpha_c]$, $f(\alpha_{\min}) = 0$, see Figure 12.

In the simply connected case we have $t_c < -1$ because $|\pi'(-\infty)| \leq 2$.

The phase transition phenomenon for the negative part of the packing spectrum has the following dynamical interpretation. For a periodic point b, $F^p b = b$, let us denote

$$\mu(b) = |(F^p)'(b)|^{1/p}.$$

The multipliers $\mu(b)$ characterize the behavior of harmonic measure near the periodic points — the concentration of ω is greater near points with larger multiplies. We will see that a necessary condition for the phase transition case in Theorem 6.1 is the existence of a fixedpoint a and $\delta > 0$ such that

$$b \in \mathrm{Per}F, \quad b \neq a \implies \mu(b) \leq \mu(a) - \delta.$$

Figure 12

For $t < t_c$, the distinguished point a provides a predominant contribution to the quantities $L(t;\varepsilon)$ in the definition (1.8) of $\pi(t)$. Observe that we must have

$$F^{-1}a \setminus \{a\} \subset \text{Crit } F \stackrel{\text{def}}{=} \{c : \ F' = 0\}, \tag{6.1}$$

because otherwise the structure of the Julia set at the preimage points would be the same as at a.

It is easy to see that no quadratic polynomial satisfies (6.1), and so $z^2 - 2$ is the only example with a phase transition. For degree $d = 3$, we have a one-parameter family of polynomials

$$F_c(z) = z^3 - 3c^2 z + 2(c^3 - c), \qquad c \in \mathbb{C},$$

satisfying (6.1). (In fact, every F satisfying (6.1) is conjugate one of F_c's.) In Figure 13 we show the "Mandelbrot set" $\{c : \ J_c \text{ is connected}\}$ for this family, and also the region

$$G = \{c : |F_c'(a)| > |F_c'(b)| \text{ for every fixedpoint } b \neq a\}$$

(the fixedpoints of F_c are $a = -2c$ and $b = c \pm 1$). G is the exterior domain bounded by the outer curve in the picture. It follows that for every cubic polynomial with

FIGURE 13

connected Julia set, except for $\pm P_3$, the negative packing spectrum is real analytic. On the other hand, there are many cubic polynomials with disconnected Julia set for which we have the phase transition case.

The critically finite degree 4 polynomial

$$F(z) = (z-c)^3(z+3c) - 3c, \qquad c = \frac{1+i\sqrt{2}}{3},$$

belongs to the phase transition case and has connected Julia set, see Figure 14a. One can see that the point $a = -3c$ responsible for the phase transition is more "exposed" than any other tip point s in the picture. The Julia set has the same structure at all tips except a. Figure 14b is the blow up at point a.

It is helpful to compare Figure 14 with the Julia sets of the polynomials

$$F(z) = z^2 + i, \qquad \text{(Figure 15a)},$$

$$F(z) = (z-c)^2(z+2c) - 2c, \quad c = \sqrt{\frac{3+i\sqrt{7}}{8}}, \qquad \text{(Figure 15b)}.$$

These polynomials are also critically finite but have real analytic spectrum. The Julia set of z^2+i has tip points all similar to each other. The Julia set in Figure 15b

FIGURE 14

does have a distinquished point $a = -2c$ but the corresponding tip is less exposed than the other tips.

We describe now some ideas in the proof of Theorems 6.1, 6.2.

Let Ω be a large disc containing J_F such that $F^{-1}\Omega \subset \Omega$. For $t < 0$ we consider the Perron-Frobenius operator L_t acting on the space $C \equiv C(\bar{\Omega})$ of continuous functions:

$$L_t f(z) = \sum_{w \in F^{-1} z} f(w)|F'(w)|^{-t}$$

(the preimages are counted with multiplicities). Using the subharmonicity of the function $z \mapsto L_t^n 1(z)$, we express the packing spectrum $\pi(t)$ in terms of the spectral radius $r(L_t, C)$ of L_t:

$$r(L_t, C) = d^{\pi(t)}.$$

Following [R3], we also consider the action of the Perron-Frobenius operator in the Sobolev spaces

$$W^{1,p} \equiv W^{1,p}(\Omega) = \{f : \|f\|_{1,p} \overset{\text{def}}{=} \|f\|_p + \|\nabla f\|_p < \infty\}.$$

Recall that $W^{1,p}(\Omega) \subset C(\bar{\Omega})$ for $p > 2$. Straightforward computation shows that

$$t < -2(1 - 2/p) \implies L_t W^{1,p} \subset W^{1,p}.$$

FIGURE 15

Thus for any $t < 0$, we can find $p > 2$ such that L_t acts on $W^{1,p}$. In this case, the operator $L_t : W^{1,p} \to W^{1,p}$ has the following important property (*quasicompactness):*

$$r_{\mathrm{ess}}(L_t, W^{1,p}) < r(L_t, W^{1,p}), \tag{6.2}$$

where r_{ess} denotes the essential radius. The inequality (6.2) means that for some $r < r(L_t, W^{1,p})$ the part of the spectrum of L_t lying outside of the disc $\{|z| \le r\}$ consists of a finite number of eigenvalues and all these eigenvalues have finite multiplicity. We also have

$$r(L_t, W^{1,p}) = r(L_t, C). \tag{6.3}$$

Both (6.2) and (6.3) follow from the estimate (cf. [IM])

$$\|L_t^n f\|_{1,p} \le d^{nw(p,t)+o(n)} \|f\|_{1,p} + C_n \|f\|_\infty,$$

where

$$w(p,t) \overset{\mathrm{def}}{=} \frac{1}{p'} \pi(p'(1+t-2/p)) < \pi(t), \qquad p' \equiv \frac{p}{p-1}.$$

Further analysis shows that the number $\lambda_t = r(L_t, C)$ is the eigenvalue of the operator $L_t : W^{1,p} \to W^{1,p}$. It can be verified that this eigenvalue is simple (and hence the packing spectrum is real analytic) if there is a positive measure ν such that

$$(6.4) \qquad L_t^* \nu = \lambda_t \nu,$$

where L_t^* is the dual operator of $L_t : C(\bar{\Omega}) \to C(\bar{\Omega})$, and such that

$$\operatorname{supp} \nu = J.$$

There always exists a measure satisfying (6.4), (*"conformal measure"),*but to make sure that $\operatorname{supp} \nu = J$, we need to require that F is not conjugate to $\pm P_d$ and does not have a fixpoint a satisfying (6.1). If such a point a exists, we continue our analysis by introducing a function G satisfying the homological equation

$$(6.5) \qquad |F'|^{-t} = G \, \frac{H \circ F}{F}$$

with $H = |z-a|^{-\alpha t}$ for some $\alpha > 0$. We can choose α such that the above argument applies to the Perron-Frobenius operator L_G,

$$L_G f(z) = \sum_{w \in F^{-1} z} f(w) G(w),$$

and gives the real analyticity of the spectral radius $\tilde{\lambda}(t) = r(L_G)$ as a function of t. It then follows from (6.5) that

$$\pi(t) = \max\{\tilde{\pi}(t), -t \log_d \mu(a)\},$$

where

$$\tilde{\pi}(t) = \log_d \tilde{\lambda}(t),$$

and we have a phase transition case if and only if

$$|\tilde{\pi}'(-\infty)| < \log_d \mu(a).$$

Remarks.

1)It is an interesting question whether Theorem 6.1 can be extended to some part of the *positive* spectrum. Namely, is it true that $\pi(t)$ is always real analytic on the interval $[0, t_1]$ for some positive t_1 depending on F? Can one take $t_1 = t_*$? There is a partial result concerning the point $t = 0$. In [PUZ], it is shown that for an arbitrary polynomial $F(z) = z^n + \ldots$ with connected, non-smooth Julia set, the second derivative $\pi"(0)$ exists and is positive. Moreover, one has the same limit theorems for harmonic measure as in the regular fractal case (see Remark 2 following Theorem 2.3). One can also derive these facts from the lacunary series representation

$$\log |\phi'(z)| = -\sum_{k=0}^{\infty} g(z^{d^k})$$

for the derivative of the Riemann map $\phi : \{|z| > 1\} \to \Omega$, where

$$g(z) = \log \left| \frac{F'(\phi(z))}{dz^{d-1}} \right|,$$

and from the estimate

$$\sum_{k=1}^{\infty} \left[\sum_{I \in \mathcal{G}_k} \|g - g(z_I)\|^2_{L^2(I)} \right]^{1/2} < \infty,$$

where $\mathcal{G}_k$ denotes the collection of all d-adic intervals of $\partial\mathbb{D}$, cf. [IL]. The latter estimate is a consequence of some general properties of univalent functions.

2)One can also try to describe the behavior of the positive part of the packing spectrum in certain special cases. For instance, it is shown in [MS1] that the positive spectrum is real analytic in the critically finite case, and this should also be true for general subhyperbolic polynomials, see [CJY]. In the parabolic case, one can show that $\pi(t)$ is real analytic for $t < t_* = \dim J$ and is $\equiv 0$ for $t \geq t_*$, cf. [DU], [ADU].

Random snowflakes. We will consider the following class of random fractals. Fix an unbounded simply connected domain $G \subset \mathbb{C}_+ \equiv \{\mathrm{Re} z > 0\}$ such that $\partial G \cap \mathbb{C}_+$ is a polygon of diameter $\asymp 1$ and $\partial G \cap \mathbb{C}_+ \subset \mathbb{D}$. For $\delta > 0$, denote

$$G_\delta = \tau(\delta G), \qquad \tau(z) = \frac{i+z}{i-z},$$

and let

$$\psi_\delta : \ \mathbb{D}_- \equiv \{|z| > 1\} \to G_\delta$$

be the Riemann map such that $\psi_\delta(\infty) = \infty$, $\psi'_\delta(\infty) > 0$. Observe that

$$\psi'_\delta(\infty) - 1 \asymp \delta^2. \tag{6.6}$$

Let $\{\delta_n\}$ be a given sequence of positive numbers such that

$$\sum \delta_n^2 < \infty, \tag{6.7}$$

and $\{\zeta_n\}$ be a sequence of independent random variables uniformly distributed on $\partial\mathbb{D}$. We define

$$\phi_n(z) = \zeta_n \psi_{\delta_n}(\bar{\zeta}_n z),$$

and

$$\Phi_n = \Phi_{n-1} \circ \phi_n = \phi_1 \circ \cdots \circ \phi_n,$$

see Figure 16. By (6.6) and (6.7), there is a limit univalent function

$$\Phi(z) = \lim_{n \to \infty} \Phi_n(z), \qquad z \in \mathbb{D}_-.$$

FIGURE 16

We will study the integral means spectrum of Φ, and the properties of the harmonic measure $\omega = \omega_\infty$ on the *"random snowflake"* $\partial(\Phi\mathbb{D}_-)$.

Let $\beta_0(t)$ denote the integral means spectrum of the functions ψ_δ:

$$\beta_(t) = \begin{cases} \frac{t}{t_-} - 1, & t \le t_-, \\ 0\ , & t_- \le t \le t_+, \\ \frac{t}{t_+} - 1, & t \ge t_+, \end{cases}$$

where $t_+ = (1-\theta_{\min})^{-1},\ t_- = -(\theta_{\max} - 1)^{-1}$, and $\theta_{\min}$ and $\theta_{\max}$ are the minimal and the maximal angles of the domain G.

6.3. Theorem. *The conformal map Φ almost surely satisfies the following.*

(1) *If $\delta_n = 1/n$, then there is a constant $c > 0$ (depending on G) such that*

$$\beta(t) \ge ct^2, \qquad |t| \le 1.$$

Moreover, ω is singular with respect to the Hausdorff measure Λ_h, $h(t) = t\exp\{c\sqrt{(}\log 1/t\ \ \log\log\log 1/t)\}$, cf. [M1].

(2) *If $\sum \delta_n < \infty$, then $\beta(t) \equiv \beta_0(t)$.*

(3) *If* $\sum \delta_n = \infty$ *but* $\sum \delta_n^{-2} n^{-3} < \infty$, *then* $\beta(t) \equiv 0$ *for* $t \leq 1$.

(4) ω *is singular with respect to* Λ_1 *if and only if*

$$\sum \delta_n = \infty, \quad \textit{and} \quad \sum \delta_n^{-2} n^{-3} = \infty.$$

In the last two statements we assume that the sequence $\{\delta_n\}$ is sufficiently regular. If, for example, $\delta_n = n^{-1} \log^\alpha n$, $\alpha \in \mathbb{R}$, then $\omega \perp \Lambda_1$ iff $-1 \leq \alpha \leq 1/2$.

The case $\delta_n \asymp 1/n$ is essentially the only one when Φ has a non-trivial integral means spectrum. Figure 17 corresponds to the choice of G such that $\partial G \cap \bar{\mathbb{D}}$ is a semicircle.

FIGURE 17

The sequence of conformal maps Φ_n represents a descrete time Loëwner chain. The proof of the theorem depends on the estimates for the *inverse* Loëwner chain

$$\Psi_k \equiv \Psi_{k,n} \stackrel{\text{def}}{=} \Phi_k^{-1} \circ \Phi_n = \phi_k \circ \cdots \circ \phi_n,$$

where $n \gg 1$ is a fixed number. We will give some examples of such estimates.

Let us first consider the case $\sum \delta_n < \infty$. To prove that $\beta(t) \le \beta_0(t)$ it is enough to show that the expectation $E \int_{\partial\mathbb{D}} |\Phi'|^t$ is finite for $t \in (t_-, t_+)$. Denote

$$X_k = E \int_{\partial\mathbb{D}} |\Psi_k'|^t.$$

Thus we have $X_n = 1$, and we must prove $X_0 = O(1)$ as $n \to \infty$. We have

$$X_{k-1} - X_k = E \int_{\partial\mathbb{D}} |\Psi_k'|^t V_k(|\Psi_k|),$$

where

$$V_k(s) = \int_{|z|=s} (|\psi_{\delta_k}'|^t - 1) \le \text{const} \ \ \delta_k.$$

Therefore,

$$X_{k-1} \le (1 + C\delta_k) X_k$$

and we are done.

To prove the opposite inequality $\beta(t) \ge \beta_0(t)$ we need to estimate the integral means from below. We will show how to get such estimates in the case $\delta_n = 1/n$. Actually, we will condider the integrals

$$X_k = E \int_{|z|=r} \log^2 |\Psi_k'|$$

and show that

$$X_0 \ge \text{const} \log \frac{1}{r-1}.$$

Since

$$\int_{|z|=s} \log^2 \left| \frac{\psi_{\delta_k}'}{\psi_{\delta_k}'(\infty)} \right| \asymp \frac{\delta_k^4}{(\delta_k + s - 1)^3}, \qquad (1 \le s \le 2),$$

reasoning as above we have

$$X_0 \asymp \sum_{k=1}^{n} \delta_k^4 A_k,$$

where

$$\begin{aligned} A_k &= E \int_{|z|=r} (\delta_k + |\Psi_k| - 1)^{-3} \\ &\ge \left[\delta_k + E \left(\int_{|z|=r} (|\Psi_k(z)| - 1) \right) \right]^{-3} \\ &\ge \text{const} \left[\delta_k + (r-1) + \sum_{\nu=k}^{\infty} \delta_\nu^2 \right]^{-3}. \end{aligned}$$

The latter inequality can be justified as follows. Denote

$$a_\nu = E \int_{|z|=r} (|\Psi_\nu| - 1).$$

Then

$$a_{\nu-1} = E \int_{|z|=r} \eta_\nu(|\Psi_\nu|),$$

where

$$\eta_\nu(s) = \int_{|z|=s} (|\psi_{\delta_\nu}| - 1) \le (s-1) + \text{const}\, \delta_\nu^2,$$

and therefore,

$$a_{\nu-1} - a_\nu \le \text{const}\, \delta_\nu^2.$$

It follows that

$$\begin{aligned} X_0 &\ge \text{const} \sum_{k=1}^{n} \frac{1}{k^4} \frac{1}{[(r-1)+k^{-1}]^3} \\ &\ge \text{const} \sum_{k=1}^{1/(r-1)} \frac{1}{k} \asymp \log \frac{1}{r-1}. \end{aligned}$$

Let us finally show how to estimate the intergal means in the case $\sum \delta_k^{-2} k^{-3} < \infty$. Reasoning as above we have

$$E \int_{\partial\mathbb{D}} \log^2 |\Psi_k'| \asymp \sum_1^n \delta_k^4 A_k,$$

where

$$A_k = E \int_{\partial\mathbb{D}} (\delta_k + |\Psi_k| - 1)^{-3}.$$

Denote also

$$\begin{aligned} B_k &= E \int_{\partial\mathbb{D}} (\delta_k + |\Psi_k| - 1)^{-2}, \\ D_k &= E \int_{\partial\mathbb{D}} (\delta_k + |\Psi_k| - 1)^{-1}, \\ G_k &= E \int_{\partial\mathbb{D}} \log \frac{1}{\delta_k + |\Psi_k| - 1}. \end{aligned}$$

Since for $\psi = \psi_\delta$ and $s \in (1,2)$,

$$\begin{aligned} &\log \frac{1}{\delta + s - 1} - \int_{|z|=s} \log \frac{1}{\delta + |\psi| - 1} \ge \text{const} \frac{\delta^2}{\delta + s - 1}, \\ &\frac{1}{(\delta + s - 1)^m} - \int_{|z|=s} \frac{1}{(\delta + |\psi| - 1)^m} \ge \text{const} \frac{\delta^2}{(\delta + s - 1)^{1+m}}, \qquad (m \ge 2), \end{aligned}$$

we have

$$\Delta B_k \asymp \delta_k^2 A_k,$$
$$\Delta D_k \asymp \delta_k^2 B_k,$$
$$\Delta G_k \asymp \delta_k^2 D_k.$$

Observe that $G_k \leq \text{const} \log k$. If we assume that the sequence $\{\delta_n\}$ is sufficiently regular, then we obtain the following:

$$\sum_1^n \delta_k^4 A_k \asymp \sum_1^n \delta_k^2 \Delta B_k$$
$$\asymp O(1) + \sum_1^n \frac{\Delta G_k}{\delta_k^2 k^2} \leq O(1) + \text{const} \sum_1^\infty \frac{1}{\delta_k^2 k^3} < \infty.$$

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