Trigonometric series and set theory

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This is a short historical survey concerning the interactions between the theory of trigonometric series and descriptive set theory. We concentrate here on the area related to problems of uniqueness for trigonometric series. Detailed historical and bibliographical references can be found in the books and survey papers listed at the end.

(A) Trigonometric and Fourier Series

A **trigonometric series** is an expression of the form

$$s \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, x \in \mathbb{T}, c_n \in \mathbb{C}.$$

A **Fourier series** is an expression of the form

$$s \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}, f \in L^1(\mathbb{T}),$$

where $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx$. (We identify here the unit circle $\mathbb{T}$ with the interval $[0, 2\pi]$ with 0 and 2π identified.)

(B) Riemann, Heine and Cantor

Riemann in his Habilitationsschrift (1854) initiated the study of the structure of functions that can be represented by trigonometric series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Riemann

This work suggests three general problems:

- (**The Uniqueness Problem**) Is such an expansion unique, whenever it exists?

- (**The Characterization Problem**) Can one characterize the functions that admit a trigonometric expansion?

- (**The Coefficient Problem**) How does one "compute" the coefficients of the expansion from the function?

I will concentrate on the Uniqueness Problem but here are some comments on the other two problems.

- (The Characterization Problem) Even for continuous functions, although there are many well-known sufficient criteria for the expansion in a trigonometric series, one can argue (on the basis of a result that I will mention later – see Section (F)) than no reasonable exact criteria can be found.

- (The Coefficient Problem) If an integrable function can be represented by a trigonometric series, then the coefficients are its Fourier coefficients (de la Vallée-Poussin). However there are everywhere convergent series, like
$$\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log n},$$
whose sum is not integrable. Denjoy from 1941 to 1949 wrote a 700 (!) page book describing a general procedure for computing the coefficients.

Heine suggested to Cantor to study the Uniqueness Problem, who obtained the following two results. As it is well-known, it was through his work on trigonometric series that Cantor was led to the creation of set theory.

Heine

Cantor

Theorem 1 (Cantor, 1870) *If* $\sum_{n=-\infty}^{\infty} c_n e^{inx} = 0, \forall x \in \mathbb{T}$*, then* $c_n = 0, \forall n$*. Thus trigonometric series expansions are unique.*

Theorem 2 (Cantor, 1872) *If* $\sum_{n=-\infty}^{\infty} c_n e^{inx} = 0, \forall x \in \mathbb{T}$*, except on a countable closed set of finite Cantor-Bendixson rank, then* $c_n = 0, \forall n$*.*

These results were extended later on by Lebesgue, Bernstein and W.H. Young.


Lebesgue


Bernstein


W.H. Young

Theorem 3 (Lebesgue, 1903) *If* $\sum_{n=-\infty}^{\infty} c_n e^{inx} = 0, \forall x \in \mathbb{T}$*, except on a countable closed set, then* $c_n = 0, \forall n$*.*

Theorem 4 (Bernstein, 1908, W.H.Young, 1909) *If* $\sum_{n=-\infty}^{\infty} c_n e^{inx} = 0, \forall x \in \mathbb{T}$*, except on a countable set, then* $c_n = 0, \forall n$*.*

(C) Sets of Uniqueness

A Borel set $A \subseteq \mathbb{T}$ is called a **set of uniqueness** if $\sum_{n=-\infty}^{\infty} c_n e^{inx} = 0, \forall x \notin A$, implies $c_n = 0, \forall n$. Otherwise it is called a **set of multiplicity**.

We denote by $\mathcal{U}$ the class of Borel sets of uniqueness and by $\mathcal{M}$ the class of Borel sets of multiplicity. Thus

$$\text{countable} \subseteq \mathcal{U} \subseteq \text{(Lebesgue) null}.$$

The following two questions come immediately to mind.

- Is $\mathcal{U} =$ countable?
- Is $\mathcal{U} =$ (Lebesgue) null?

(D) The Russian and Polish Schools (mid 1910s - mid 1930s)

The structure of sets of uniqueness was investigated intensely during the 1910's and 1920's by the Russian school of Luzin, Menshov and Bari, and the Polish school of Rajchman, Zygmund and Marcinkiewicz.

Luzin

Menshov

Bari

Zygmund

Marcinkiewicz

Here are some of the main results obtained in that period.

Theorem 5 (Bari, Zygmund, 1923) *The union of countably many closed sets of uniqueness is also a set of uniqueness.*

Given real numbers $\xi_1, \xi_2, \dots$, with $0 < \xi_n < 1/2$, denote by $E_{\xi_1,\xi_2,\dots}$ the Cantor-type set (in $\mathbb{T}$) constructed with successive ratios of dissection $\xi_1, \xi_2, \dots$ (the ratio of dissection is the ratio of the length of one of the intervals that are being kept to the length of the whole interval). We also let $E_\xi = E_{\xi,\xi,\dots}$. In particular, $E_{1/3}$ is the usual Cantor set.

Theorem 6 (Menshov, 1916) *There is a closed null set of multiplicity. In fact, $E_{\xi_1,\xi_2,\dots}$, with $\xi_n = \frac{(n+1)}{2(n+2)}$, is such a set.*

Theorem 7 (Bari, Rajchman, 1921-1923) *There are perfect (nonempty) sets of uniqueness. In fact, $E_{1/3}$ is such a set.*

It follows that

$$\text{countable} \subsetneqq \mathcal{U} \subsetneqq \text{null}.$$

Thus by the 1920's it had become clear that the class of (even closed) sets of uniqueness is hard to delineate in terms of classical notions of smallness in analysis.

Bari's memoir in 1927 stated some classical problems on sets of uniqueness.

- **(The Characterization Problem) Find necessary and sufficient conditions for a perfect set to be a set of uniqueness.**

  It appears that the intended meaning was to ask for geometric, analytic (or, as we will see later, even number theoretic) structural properties of a given perfect set, expressed "explicitly" in terms of some standard description of it, that will determine whether it is a set of uniqueness or multiplicity.

- **(The Union Problem) Is the union of two Borel sets of uniqueness also a set of uniqueness?**

  The first open case is that of two G_δ sets.

- **(The Category Problem) Is every Borel set of uniqueness of the first category (meager)?**

(E) Thin sets in harmonic analysis (early 1950s - mid 1970s)

During that period there was an explosion of research into the structure of thin sets in harmonic analysis, including the study of closed sets of uniqueness.

Piatetski-Shapiro

Piateski-Shapiro in 1952-54 introduced functional analysis methods into the subject of uniqueness. This has become the standard language of the subject since then.

We denote by $A(\mathbb{T})$ the Banach algebra of functions with absolutely convergent Fourier series. This is of course the same as $\ell^1(\mathbb{Z})$. Its dual is the space $\ell^\infty(\mathbb{Z})$, which in this context is called the space of **pseudomeasures** and denoted by **PM**. Its predual is the space $c_0(\mathbb{Z})$, which in this context is called the space of **pseudofunctions** and denoted by **PF**.

The **support** of a pseudomeasure S is the complement of the largest open set on which S vanishes, i.e., annihilates all functions in $A(\mathbb{T})$ supported by it.

Piatetski-Shapiro's reformulation of the concept of closed set of uniqueness is the following.

Theorem 8 (Piatetski-Shapiro, 1952) *A closed set E is a set of uniqueness iff it does not contain the support a non-0 pseudofunction.*

At this point it is time to introduce an important variation of the concept of set of uniqueness, which really goes back to Menshov's work. His example of a null closed set of multiplicity was witnessed by the Fourier-Stieltjes series of a (probability Borel) measure. Such a set is called a set of strict multiplicity.

A Borel set is called a set of **extended uniqueness** if it satisfies uniqueness for Fourier-Stieltjes series of measures. Otherwise it is called a set of **strict multiplicity**. The class of Borel sets of extended uniqueness is denoted by $\mathcal{U}_\mathbf{0}$ and the class of Borel sets of strict multiplicity is denoted by $\mathcal{M}_\mathbf{0}$.

A **Rajchman measure** is a measure whose Fourier-Stieltjes coefficients converge to 0, i.e., form a pseudofunction. Lebesgue measure is of this form and a Rajchman measure is thought of as a measure with "large support". However, Menshov showed that there are singular Rajchman measures. In terms of Rajchman measures, the sets of extended uniqueness are exactly those that are null for all such measures.

Theorem 9 (Piatetski-Shapiro, 1954) *There is a closed set of extended uniqueness but not of uniqueness.*

This result of Piatetski-Shapiro was amplified in the work of Körner in the early 1970's, who solved a major problem at that time by constructing a particular kind of closed thin set, called a *Helson set*, which is of multiplicity. As Helson in 1954 had already shown that these sets are of extended uniqueness, this also implied Theorem 9. This result of Körner was one of the last major results of that period. Its original proof was extremely complicated and despite a major simplification by Kaufman it remains a subtle and difficult result.

During that period there was also a major advance in the characterization problem.

A real number is called a **Pisot** (or **Pisot-Vijayaraghavan**) number if it is an algebraic integer > 1 all of whose conjugates have absolute value < 1. Examples include the integers > 1 and the golden mean.

Intuitively, these are numbers whose powers approach integers. We now have the following striking results.

Salem

Theorem 10 (Salem, 1944) *The Pisot numbers form a closed set of algebraic integers.*

Theorem 11 (Salem-Zygmund, 1955) *The set E_ξ is a set of uniqueness iff $1/\xi$ is a Pisot number.*

(F) Applications of descriptive set theory (mid 1980s - mid 1990s)

We have seen that the problems of uniqueness have involved ideas from many subjects, such as classical real analysis, modern harmonic analysis, functional analysis, number theory, etc. Although set theory owes its origin to Cantor's work on the uniqueness problem, relatively little contact existed between set theory and the study of sets of uniqueness until the 1980's, when ideas from a basic area of set theory, called *descriptive set theory*, were brought to bear in the study of this subject. This is interesting since descriptive set theory was also originally developed in the Russian and Polish schools during the same period 1915-1935.

Luzin's school was concerned with what was then called the *theory of real functions* and there was at that time a distinction between the so-called *metric theory* (differentiation, integration, trigonometric series, etc.) and the *descriptive theory* (called today descriptive set theory). Strangely enough, according to Kolmogorov, who was a member of that school, Luzin divided his students to those that would study the metric theory and those that would study the descriptive one. (Kolmogorov actually violated this rule and worked on both.) In the following years the subjects drifted apart, the first one practiced by analysts and the second one by logicians. They were brought back together in the 1980's in the study of sets of uniqueness.

Descriptive set theory is the study of definable sets and functions in Polish (complete, separable metric) spaces, like the Euclidean spaces, separable Hilbert space and more generally separable Banach spaces, second countable locally compact groups, etc.

In this theory sets are classified in hierarchies according to the complexity of their definitions and the structure of sets at each level of these hierarchies is systematically studied.

Of particular importance are the Borel and projective sets. The Borel sets are obtained from the open sets by applying repeatedly countable Boolean operations and the projective sets are obtained from the Borel sets by the operations of complementation and projection.

These classes of sets are ramified in natural hierarchies as follows:

$$\underbrace{\begin{matrix} \text{open} & F_\sigma & F_{\sigma\delta} & \cdots \\ \text{closed} & G_\delta & G_{\delta\sigma} & \cdots \end{matrix}}_{\textbf{Borel}} \qquad \underbrace{\begin{matrix} \text{A} & \text{PCA} & \cdots \\ \text{CA} & \text{CPCA} & \cdots \end{matrix}}_{\textbf{Projective}}$$

(Here F_σ is the collection of all countable unions of closed sets, G_δ the collection of all countable intersections of open sets, etc., A (= *analytic sets*) is the collection of all projections of Borel sets, CA (= *co-analytic sets*) the collection of complements of analytic sets, etc.)

Intuitively, sets whose membership is characterized in "effective" terms, even allowing countable operations, are Borel.

Here are a couple of examples of co-analytic non-Borel sets in analysis:

- In the space $C(\mathbb{T})$, the set of differentiable functions is co-analytic but not Borel (Mazurkiewicz, 1936)

- In the space $C(\mathbb{T})$, the set of functions that can be expanded in a trigonometric series is co-analytic but not Borel (Ajtai-Kechris, 1987).

In the 1980's and 1990's methods of descriptive set theory were combined with previous work in analysis to further the study of sets of uniqueness. This was primarily developed in series of papers (and unpublished work) of the following authors: Debs-Saint Raymond, Kaufman, Kechris-Louveau, Kechris-Louveau-Woodin, Solovay. Detailed history of these developments can be found in the references [C], [KS], [KL1], [KL2], [L].

The main point is that descriptive set theory allows one to develop a "global" theory of closed sets of uniqueness, with many applications to the classical theory.

The appropriate space here is the compact metric space $K(\mathbb{T})$ of closed subsets of the circle, with the usual Hausdorff metric. One studies the structure of the following two subsets of this space:

- $U = \{E \in K(\mathbb{T}) : E \text{ is a set of uniqueness}\}$
- $U_0 = \{E \in K(\mathbb{T}) : E \text{ is a set of extended uniqueness}\}$

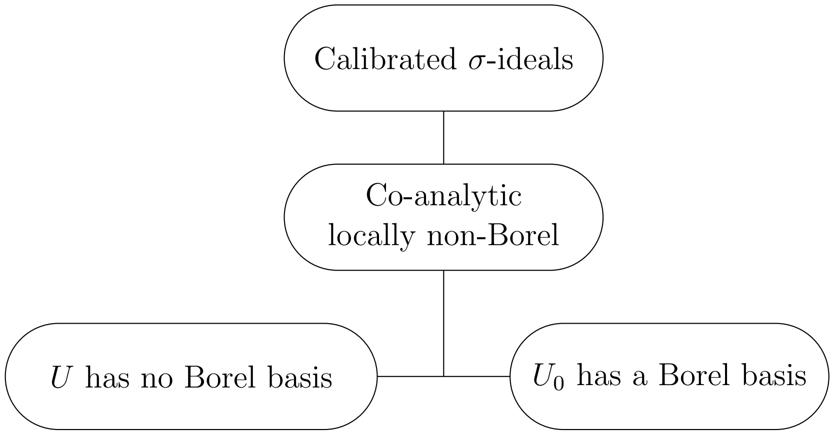
The global theory can be encapsulated in the following main theorem, whose proof is contained in a series of papers (and unpublished work) of the above authors (detailed references can be found in [KL]). It states three basic principles that describe the structure of the classes U, U_0.

Below a subset $I \subseteq K(\mathbb{T})$ is **hereditary** if $E, F \in K(\mathbb{T}), E \subseteq F, F \in I$ implies $E \in I$. It is a σ-**ideal** if it is hereditary and $E_n \in I, \forall n \in \mathbb{N}$, and $E = \bigcup_n E_n \in K(\mathbb{T})$ implies that $E \in I$. It is **calibrated** if it satisfies the following stronger property: If $F \in K(\mathbb{T})$, $F_n \in I, \forall n \in \mathbb{N}$, and $K(F \setminus \bigcup_n F_n) \subseteq I$, then $F \in I$, where for any $G \subseteq \mathbb{T}$, $K(G)$ is the collection of all compact subsets of G. Finally, a σ-ideal I has a **Borel basis** if there is Borel hereditary subset B of I such that every E in I is a countable union of sets in B.

Theorem 12 *a) (Stability property) The sets U, U_0 are calibrated σ-ideals.*

b) (Descriptive complexity, I) Both U, U_0 are co-analytic and locally non-Borel, i.e., for every closed set E not in U (resp., not in U_0) the set $U \cap K(E)$ (resp., $U_0 \cap K(E)$) is not Borel.

c) (Descriptive complexity, II) The σ-ideal U_0 admits a Borel basis but the σ-ideal U does not.



This theory has numerous applications both in the solution of classical problems and also in understanding and proving in a new way previously established results.

- (Characterization Problem) One can argue that this has a negative solution in a strong sense, since not only there is no "explicit" characterization of closed (as well as perfect) sets of uniqueness of the sought after type but also there is no way to characterize such sets in terms of decomposing them into a countable number of explicitly characterizable components.

- Every known until the early 1980's closed set of uniqueness could be written as a union of a countable sequence of simpler uniqueness sets, belonging to a class denoted by U'. This is a Borel class, so the non-basis theorem shows that there are many new kinds of U-sets. This result can therefore be viewed as a powerful new existence theorem. For example, it answers a question of Piatetski-Shapiro, on the existence of U-sets not expressible as countable unions of so-called $H^{(n)}$-sets (with varying n).

- (Category Problem) This is solved affirmatively in a strong sense, as it follows that every Borel set of *extended uniqueness* is of the first category. Equivalently this means that every Borel set of the second category supports a Rajchman measure. This in turn has several applications, including in particular a unified, simple way of proving some well-known results in the theory, originally established by various techniques and constructions.

- Menshov's Theorem says that there are (Lebesgue) null sets that support Rajchman measures. This is now clear as it well-known that there are comeager null sets. Thus Menshov's result is seen as a consequence of the orthogonality between measure and category.

  Ivashev-Musatov and Kaufman have extended Menshov's Theorem to show that for any function h there are h-Hausdorff measure 0 closed sets that support Rajchman measures. The same argument as above applies.

- A problem of Kahane-Salem (1964) asks whether the set of non-normal numbers supports a Rajchman measure. This was solved affirmatively by Lyons (1986). Again this follows from the fact that the set of non-normal numbers is comeager. The same argument applies to show that the set of Liouville numbers supports a Rajchman measure, a result proved by Bluhm (2000).

- (The Union Problem) This is still open, even for the union of two G_δ sets. It is mostly believed that there is a counterexample. The preceding theory however implies that from a counterexample one obtains a closed set with properties similar to those obtained by Körner (Helson sets of multiplicity). Thus conceivably Körner's Theorem could be useful in the construction of such a counterexample.

There are several further applications of descriptive set theoretic methods also to other aspects of the subject, e.g., Lyons' important characterization of Rajchman measures by their null sets is seen as following from a general descriptive set theoretic result of Mokobodzki about analytic classes of measures. Also such ideas have been applied by S. Kahane to the solution of some old problems about other types of thin sets in harmonic analysis.

(G) Conclusion

This is where we stand now. Despite the progress made over the last 150 years many mysteries remain. Here are for example some intriguing problems that are still open:

- Where is the dividing line in the Characterization Problem?

  E_ξ: characterizable

$E_{\xi_1,\xi_2,\ldots}$: ???

E in general: uncharacterizable

- The Union Problem for G_δ sets and for arbitrary Borel sets.

- (The Interior Problem, Bari 1927) Is the concept of set of uniqueness determined by the closed sets, i.e., does a Borel set of multiplicity contain a closed set of multiplicity?

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