Top Fifteen Results

I thought that the reader of these Selecta might find it useful for me to point out what I regard as my most significant research contributions. I've picked an order although on another day, the order would be slightly different. I suspect most people's top three would be among my top 5. Some of the items are actually collections of several separate results – for example, singular eigenvalue perturbation theory.

Of course, my impact is not only through the papers highlighted in the list below. Most significant of the other contributions, of course, are my books – especially Reed–Simon (I often get into an email exchange from someone outside mathematical physics who mentions that they first learned Functional Analysis from volume one) and, I expect, as time passes, the Comprehensive Course five volume set will have a large impact. Also the OPUC books, Trace Ideal and Functional Integration Books which are standard references within their areas. One measure of the combined impact is that I have over 60,000 Google Scholar citations and my Google Scholar h-index is 103 (that is 103 publications with at least 103 citations each).

Besides the research in my books and papers, my publications have introduced a lot of now standard terms including: hypercontractive, ultracontractive, infrared bounds, CLR inequality, Kato's inequality, diamagnetic inequalities, weak trace ideal, Thouless formula, Aubry duality, Kotani theory, almost Matthieu equation, Maryland model, ten martini problem, Berry's phase, Wonderland theorem, OPUC, OPRL, Verblunsky coefficient, CMV matrix, clock spacing of zeros. I have made numerous conjectures that have stimulated further research including publicizing a conjecture of Mark Kac which I dubbed the ten martini problem. I have also publicized some breakthroughs by others such as Enss, Mourre, Berry's Phase, Kotani theory and Remling's reflectionless theorem. And I have direct impact on my PhD. students, postdocs and some number of long term visitors.

Here is the list of contributions:

1. **Continuous Symmetry Breaking** [64, 65, 67, 68, xiv] Phase transitions accompanying breaking of a continuous non-abelian symmetry are an important element of nature – not only in ferromagnetism but also in models of particle physics. The only method known for a rigorous mathematical proof of this is the method of infrared bounds that appeared first in my paper with Frölich and Spencer [65]. Phase transitions are viewed as a Bose condensation of spin waves. We applied the method to the classicial Heisenberg ferromagnet (and by symmetry also the anti–ferromagnet). This was extended to the quantum anti–ferromagnet by Dyson, Lieb and me [68]. It is remarkable that 40 years later the quantum ferromagnet (which we mistakenly announced and later withdrew) is still open.

2. **Statistical Mechanical Methods in QFT** [32, 33, 34, 37, 40, 47, 48 49, 50, IV] In early 1972, there was a revolution in constructive quantum field

theory due to Guerra's realization of the power of the Euclidean quantum field theory of Symanzik and Nelson. Symanzik had realized that this set up some formal analogies between QFT and statistical mechanics. It was Guerra, Rosen and me [33] who discovered that one could discretize Euclidean space–time and so directly approximate QFT by lattice spin models and thereby carry over powerful tools (mainly correlation inequalities) to EQFT. With Griffiths [47], I introduced a second level of approximation whereby Φ^4 theories could be approximated by spin $\frac{1}{2}$ models and thus get GHS and Lee–Yang theorems in EQFT.

3. **Thomas Fermi and other Quasiclassical Limits** [39, 53, 73, 175] The Thomas–Fermi approximation for atomic and molecular physics goes back to the earliest days of quantum mechanics. In 1972–3, Lieb and I realized that for the total binding energy and total electron density, TF theory is exact in the large Z limit [53]. Before our work it wasn't even known that the equations had solutions in the case of molecules. TF theory is a quasi–classical limit. My paper [73] on quasiclassical bounds stimulated the work of Cwikel and of Lieb on the CLR bounds.

4. **Sum Rules and Operator Methods in Orthogonal Polynomials** [280, 281, 282, 288, 290, 294, 296, 301, 302, 303, 306, 312, 315, 317, 318, 319, 323, 324, 327, 329, 332, 336, XIII, XIIV, XV] Starting about 2000, I shifted a part of my research focus to the spectral theory of orthogonal polynomials. My most significant work was joint with Killip [281]. We found an analog of Szegő's OPUC result for OPRL that gave spectral theory necessary and sufficient conditions for the Jacobi matrix to be a Hilbert Schmidt perturbation of the free Jacobi matrix and using results from [280] settled a conjecture of Nevai. We also studied Szegő asympototics. Extending some of this to periodic and finte gap situations was a major theme of my work over the following ten years including joint work with Zlatoš, Damanik, Killip, Chritiansen, Zinchenko and Frank.

5. **Singular Eigenvalue Perturbation Theory** [6, 7, 10, 11, 17, 19, 20, 28, 70, 72, 80, 100, 101, 102, 104, 105, 111, 115, 122, 156, 161, 162, 163, 174, 177] This is a collection of disparate results that have in common that they all look at eigenvalue perturbations that go beyond the classical Kato-Rellich perturbation theory that describes isolated eigenvalues under (relatively) bounded perturbations and also goes beyond Kato's results on asymptotic series and on spectral concentration. [7] has an exhaustive study of the analytic structure under coupling constant variation for the anharmonic oscillator that was the basis for a proof (with Loffel, Martin and Wightman) of Padé [6] and (with Graffi and Greechi) of Borel [10] summability of eigenvalue perturbation series. In [20], I proved the convergence of time dependent perturbation series for autoionizing states of atoms by using complex scaling to reduce it to the Kato-Rellich theory. In [105] Harrell and I used complex scaling and ODE asymptotics to prove the Oppenheimer formula for Stark widths and the Bender–Wu formula for the asymptotics of the coefficients of the divergent anharmonic oscillator perturbation series. In [162], I used large deviations

in path integrals to obtain leading asymptotics in multi–dimensional double wells. I have a number of papers on point eigenvalues emerging from continuous spectrum under coupling constant variation, most notably my work on weak coupling in one and two dimensions [70]

6. **Geometric Methods in Quantum Mechanics (including Berry's Phase)** [171, 172, 204, 205, 228, 229] I was an early advocate of the use of methods from algebraic topology and differential geometry in condensed matter physics. Avron, Seiler and I [171] realized that the integer invariants found by Thouless, Kohmoto, Nightingale and den Nijs were Chern classes and proved that these were the only invariants associated to energy bands. In what is my most cited paper in the physics literature I realized that the quantity I dubbed Berry's phase was a holonomy [172]. With Avron, Sadun and Seegert, I found the proper analog of Berry's phase in Fermi systems [205]. As an outgrowth of some of this work, Avron, Seiler and I found a supersymmetic approach to the study of pairs of projections [229].

7. **Generic Singular Continuous Spectrum** [233, 234, 235, 236, 242, 243, 246, 247, 248, 250, 251] I like to joke that I spent the first part of my career showing singular continuous spectrum never occurs (see point 9 below!) and the second part showing it always occurs. Motivated by some results of delRio and of Gordon, I found that singular continuous spectrum is generic in a Baire category sense. A general strategy was presented in [234] that for example showed that a Baire G_δ of decaying random potentials with decay rate $n^{-\alpha}; 0 < \alpha < \frac{1}{2}$ have purely singular continuous spectrum. With delRio and Markarov [235], I showed for rank one perturbations with Lebesgue generic dense point spectrum have Baire generic purely singular continuous spectrum. With Jitomirskaya [236], I proved for certain almost periodic models with positive Lyaponov exponent, Baire genric points in the hull have purely singular continuous spectrum.

8. **Almost Periodic and Random Schrödinger Operators** [146, 147, 148, 149, 152, 155, 166, 168, 169, 170, 180, 181, 187, 188, 189, 190, 192, 194, 198, 211, 216, 232, 245, 257, 263, 265, X, xxvi] This is also a huge catchall – two areas in which I was an early (although, not always the earliest) worker. I'll focus on some of the high points of my research. At the same time as Johnson–Moser, Avron and I worked out some of the most basic properties of almost periodic discrete Schrödiner operators [147, 149] including the first proof of the Thouless formula and the first examples where Liouville frequencies implies singular continuous spectrum. I extended Kotani's theory which he developed for continuum Schrödinger operators to the discrete case [168]. For random Schrödinger operators, I proved with Wolff [189] a fundamental criterion for localization. With Taylor [188], I proved that the passage from potential randomness to density of states is smoothing, and, in particular for the original Anderson model (which has a discontinuous potential density), the density of states is C^∞.

9. **Absence of Singular Spectrum in N-Body Quantum Systems** [131, 132] In "normal" quantum systems, discrete eigenfunctions are bound states and a.c. spectrum can be associated to scattering states. That there was no singular continuous spectrum was called "the no-goo hypothesis" by my advisor, Arthur Wightman, since there was no physical interpretation. For N–body quantum systems ($N \geq 3$) this was notoriously hard. When the potentials were analytic under scaling, Balslev–Combes proved the absence in 1971. Mourre developed techniques to prove this when $N = 3$ for a class of not necessarily smooth $V's$ but his work wasn't understood or appreciated. Perry, Sigal and I [132] figured out Mourre's paper and the non–trivial extension to general N. This was the first proof, for example, for N–body systems with C_0^∞ two-body potentials. This work was subsumed in the work of Sigal–Sofer and Graf on strong asymptotic completeness.

10. **Schrödinger Operators with Magnetic Fields, especially diamagnetic inequalities** [66, 76, 86, 87, 88, 89, 90, 91, 92, 98, 117, 130, 151, 183, X, xxi] Fix a continuum Schrödinger operator with potential V and let $H(\mathbf{a})$ be the operator with a magnetic vector potential $\mathbf{a}$. If $\exp(-tH)(x, y)$ is the integral kernel of the semigroup, then the diamagnetic inequality says that $|\exp(-tH(\mathbf{a})(x, y)| \leq \exp(-tH)(x, y)$. It appeared first in [76] with some regularity conditions on $\mathbf{a}$ and for the general case in [98]. Avron, Herbst and I [89, 90, 91, 92] have a systematic study of the mathematical physics of quantum systems in magnetic field.

11. **Inverse Problems** [230, 239, 240, 241, 260, 261, 264, 267, 268, 271, 272, 273, 275, 289] For Jacobi matrices, there are two "standard" methods for going from the spectral measure to the Jacobi parameters: one can form the OPRL and look at their recursion coefficients or, alternatively, one can form the m–function and look at its continued fraction expansion. The celebrated Gel'fand–Levitan inverse spectral method is the analog of the OP approach for the continuum case. In [271], I found the analog of the continued fraction approach (developed further in [272], jointly with Gesztesy). With Gesztesy (and sometimes others), I have a series of papers on various aspects of one dimensional inverse problems including the development of the xi function [241] and that the Dirichlet spectrum for $[0, 1], [0, a]$ and $[a, 1]$ plus a non-degeneracy condition determines the potential.

12. **Eigenfunction Behavior for Schrödinger Operators** [xxi, 36, 43, 46, 51, 59, 95, 119, 120, 133, 134, 135, 253, 263, X] Again, this is a catchall category. My Schrödinger Semigroups article [xxi], which is partly a review article, has become a standard reference on eigenfunctions of Schrödinger operators. I have a series (some jointly with others) [43, 46, 51, 95, 133, 134] on pointwise bounds on decay of eigenfunctions. Last and I [263] use eigenfucntions to study a.c. spectrum in great generality.

13. **Hypercontractivity and Ultracontractivity** [16, 24, 69, 173] In fundamental 1966 work, Ed Nelson proved results about the boundedness from below of some QFT cutoff Hamiltonians using L^p properties of the semigroup of the free field

Hamiltonian and then Segal showed that these bounds also implied self–adjointness. In 1970, Hoegh Krohn and I abstracted these ideas and invented the name "hypercontractive semigroup" for the L^p semigroup. In 1983, Davies and I [173] discovered that for many finite dimensional (but not quantum field theory) situations a stronger set of L^p properties held and was useful and we dubbed semigroups with this stronger property "ultracontractive". Both notions have become industries.

14. **Quadratic Form Methods** [13, 25, 30, 81, 82, 96, 98, 116, I] While the theory of semibounded self–adjoint operators as quadratic forms had been developed, especially by Kato and Nelson, when I entered graduate school, Schrödinger operators were almost always studied in terms of operator domains and operator perturbations. My thesis, published as a book, [I] took a class of two body potentials which were form bounded perturbations but not necessarily operator bounded perturbations and developed all the currently available theory to this class. Since then, many works on the subject, including many of mine, have used form methods. In particular, [98] discussed a form analog of essential self–adjointness on C_0^∞.

15. **Self-Adjointness Methods** [13, 14, 16, 24, 35, 38, 98] If one considers potentials, V, with no restriction on sign, then essential self-adjointness of $-\Delta + V$ on $C_0^\infty(\mathbb{R}^\nu)$ holds if $V \in L^p$ with $p \geq 2$ and $p > \nu/2$. For any $p \in [2, \nu/2)$ there exist V's in L^p where essential self-adjointness fails. In [24], I discovered that for positive V's one gets self–adjointness for more singular possibilities than for negative V's. Using hypercontrative methods and a trick of Konrady, I proved essential self–adjointness if $V \geq 0$ and $V \in L^2(\mathbb{R}^\nu, e^{-|x|^2} d^\nu x)$. I conjectured that for $V \geq 0$, L^2_{loc} sufficed. Motivated by my paper, Kato wrote a famous paper (where Kato's inequality first appeared) proving my conjecture and making [24] obsolete. In [98], I found a semigroup proof of my conjecture and proved a form analog of that conjecture.