Chapter 7: Lines in $\mathbb{R}^3$

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In the previous chapter we studied how the distinct distances problem can be reduced to a problem involving intersections of lines in $\mathbb{R}^3$ (namely, using the Elekes-Sharir-Guth-Katz framework). In the current chapter and in the one following it, we solve this problem, thus completing the proof of the Guth-Katz distinct distances theorem.

1 From intersections to incidences

We begin by recalling where we stand after Chapter 6. To solve the distinct distances theorem, we consider a set $\mathcal{P}$ of n points in $\mathbb{R}^2$. Given two points $a, b \in \mathbb{R}^2$, we define the following line in $\mathbb{R}^3$.

$$\ell_{ab} = \left(\frac{a_x + b_x}{2}, \frac{a_y + b_y}{2}, 0\right) + t\left(\frac{b_y - a_y}{2}, \frac{a_x - b_x}{2}, 1\right), \qquad \text{for } t \in \mathbb{R}. \tag{1}$$

The line ℓ_{ab} is the set of parameterizations of the rotations of $\mathbb{R}^2$ that take a to b. We consider the set of n^2 lines

$$\mathcal{L} = \{\ell_{ab} : a, b \in \mathcal{P}\}. \tag{2}$$

To prove that $\mathcal{P}$ determines $\Omega(n/\log n)$ distinct distances, it suffices to prove that the number of pairs of intersecting lines in $\mathcal{L}$ is $O(n^3 \log n)$. Let N_k denote the number of points in $\mathbb{R}^3$ that are incident to exactly k lines of $\mathcal{L}$, and let $N_{\geq k}$ denote the number of points that are incident to at least k such lines. The number of pairs of intersecting lines can be expressed as

$$\sum_{k=2}^{n^2} N_k \binom{k}{2} = \sum_{k=2}^{n^2} (N_{\geq k} - N_{\geq k+1}) \binom{k}{2} = \sum_{k=2}^{n^2} N_{\geq k} \left(\binom{k}{2} - \binom{k-1}{2}\right)$$
$$= \sum_{k=2}^{n^2} N_{\geq k}(k-1). \tag{3}$$

To obtain the desired bound of $O(n^3 \log n)$, for any $2 \leq k \leq n^2$ we will prove that

$$N_{\geq k} = O\left(\frac{n^3}{k^2}\right). \tag{4}$$

Combining (3) and (4) implies that the number of pairs of intersecting lines is

$$\sum_{k=2}^{n^2} N_{\geq k}(k-1) = \sum_{k=2}^{n^2} O\left(\frac{n^3}{k}\right) = O\left(n^3 \log n\right).$$

That is, to prove Guth and Katz's bound on the number of distinct distances [3], it suffices to prove (4).

2 The case of $k \geq 3$

Our goal in this chapter is to prove (4) for the case of $k \geq 3$. The case of $k = 2$ is postponed to the next chapter.

Although (4) is an incidence bound, it looks somewhat different than the incidence bounds that we proved in previous chapters. In some cases, such a bound is equivalent to a "standard" incidence bound. We now show such an equivalence for the case of the Szemerédi-Trotter bound. Let $M_{\geq k}$ denote the maximum number of points in $\mathbb{R}^2$ that can be incident to at least k lines of a set of n lines. For example, consider an $n/4 \times n/4$ integer lattice and cover it with $n/4$ horizontal lines, $n/4$ vertical lines, and $n/2$ lines with slope 1. This configuration implies that $M_{\geq 3} \geq n^2/16$.

Claim 2.1. *The Szemerédi-Trotter bound implies the bound $M_{\geq k} = O\left(\frac{n^2}{k^3} + \frac{n}{k}\right)$ for every $k \geq 2$. In the other direction, $M_{\geq k} = O\left(\frac{n^2}{k^3} + \frac{n}{k}\right)$ implies the Szemerédi-Trotter theorem up to an extra $n \log n$ term.*[1]

Proof. We first show that the Szemerédi-Trotter bound implies the asserted bound for $M_{\geq k}$, for every $k \geq 2$. When k is a constant, the claim $M_{\geq k} = O(n^2/k^3)$ trivially holds. Thus, we may assume that k is larger than the constant in the $O(\cdot)$-notation of the Szemerédi-Trotter bound. Consider a set $\mathcal{L}$ of n lines in $\mathbb{R}^2$ and a fixed value of k. Let $\mathcal{P}$ denote the set of points of $\mathbb{R}^2$ that are incident to at least k lines of $\mathcal{L}$, and set $m_k = |\mathcal{P}|$. By definition, we have $I(\mathcal{P}, \mathcal{L}) \geq m_k k$. On the other hand, the Szemerédi-Trotter bound implies $I(\mathcal{P}, \mathcal{L}) = O(m_k^{2/3} n^{2/3} + n + m_k)$. Combining these two bounds yields $m_k k = O(m_k^{2/3} n^{2/3} + n + m_k)$. Since k is larger than the constant in the $O(\cdot)$-notation, we disqualify the case of $m_k k = O(m_k)$ and remain with $m_k k = O(m_k^{2/3} n^{2/3} + n)$. This immediately implies $m_k = O\left(\frac{n^2}{k^3} + \frac{n}{k}\right)$, as required.

Next, we show that the Szemerédi-Trotter bound is implied by $M_{\geq k} = O\left(\frac{n^2}{k^3} + \frac{n}{k}\right)$. Consider a set $\mathcal{P}$ of m points and a set $\mathcal{L}$ of n lines. Let $\hat{m}_i$ denote the number of points of $\mathcal{P}$ that are incident to more than 2^{i-1} lines of $\mathcal{L}$, and to at most 2^i such lines. We set $r = \left\lceil \log\left(n^{2/3}/m^{1/3}\right)\right\rceil$. Since $\hat{m}_i \leq m$ obviously holds for every i, we

[1] This extra term can be removed with a slightly longer argument, which we do not include here.

have

$$\begin{aligned}
I(\mathcal{P},\mathcal{L}) \le \sum_{i\ge 0}\hat{m}_i 2^i &\le \sum_{i=0}^{r} m2^i + \sum_{i=r+1}^{\log n} M_{\ge 2^{i-1}} 2^i \\
&= O\left(m^{2/3}n^{2/3} + m + \sum_{i=r+1}^{\log n}\left(\frac{n^2}{2^{2i}} + n\right)\right) \\
&= O\left(m^{2/3}n^{2/3} + m + n\log n\right). \qquad \square
\end{aligned}$$

Claim 2.1 hints that we might be able to derive a standard-looking point-line incidence bound in $\mathbb{R}^3$, and then show that this bound implies (4). The following claim points out a problem with this approach.

Claim 2.2. *For any m and n, the maximum number of incidences between a set of m points and a set of n lines in $\mathbb{R}^2$ is identical to the maximum number of incidences between a set of m points and a set of n lines in $\mathbb{R}^3$.*

Proof. Consider a set $\mathcal{P}$ of m points and a set $\mathcal{L}$ of n lines, both in $\mathbb{R}^2$, such that $\mathcal{P}\times\mathcal{L}$ contains X incidences. By choosing an arbitrary plane h in $\mathbb{R}^3$ and placing the same point-line configuration in h, we get X incidences in $\mathbb{R}^3$. That is, the maximum number of point-line incidences in $\mathbb{R}^3$ is at least as large as the one in $\mathbb{R}^2$.

For the other direction, consider a set $\mathcal{P}$ of m points and a set $\mathcal{L}$ of n lines, both in $\mathbb{R}^3$, such that $\mathcal{P}\times\mathcal{L}$ contains Y incidences. We perform a generic rotation of $\mathbb{R}^3$ and then project $\mathcal{P}$ and $\mathcal{L}$ onto the x_1x_2-plane. Due to the rotation, no two points of $\mathcal{P}$ are projected to the same point of $\mathbb{R}^2$, no line of $\mathcal{L}$ is projected to a single point of $\mathbb{R}^2$, and no two lines of $\mathcal{L}$ are projected to the same line in $\mathbb{R}^2$. This yields a planar point-line configuration in $\mathbb{R}^2$ with at least Y incidences. Thus, the maximum number of point-line incidences in $\mathbb{R}^2$ is at least as large as the one in $\mathbb{R}^3$. $\square$

Recall that (4) should hold with respect to n^2 lines. In this case, Claims 2.1 and 2.2 imply the bound $M_{\ge k} = O\left(\frac{n^4}{k^3} + \frac{n^2}{k}\right)$, which is weaker than the one in (4). Moreover, this weaker bound is tight for general sets of lines in $\mathbb{R}^3$. Fortunately, since our set of lines $\mathcal{L}$ is constructed according to (2), it has additional properties.

Lemma 2.3. *Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^2$, and let $\mathcal{L}$ be the set of n^2 lines that is defined in* (2)*. Then every plane in $\mathbb{R}^3$ contains at most n lines of $\mathcal{L}$, and every point of $\mathbb{R}^3$ is incident to at most n lines of $\mathcal{L}$.*

Proof. For any $a\in\mathcal{P}$, we set

$$\mathcal{L}_a = \{\ell_{ab} : b\in\mathcal{P}\}.$$

Consider two lines $\ell_{ab},\ell_{ac}\in\mathcal{L}_a$. Recall, from Chapter 6, that every point of ℓ_{ab} parameterizes a rotation of $\mathbb{R}^2$ that takes a to b, and similarly for the points of ℓ_{ac}. Since no rotation can take a to both b and c, the lines ℓ_{ab} and ℓ_{ac} do not intersect. By examining (1), we notice that it is impossible for two lines of $\mathcal{L}_a$ to have the same slope. Since lines in $\mathcal{L}_a$ do not intersect and are not parallel, no plane contains more than one such line. By summing this up over every $a \in \mathcal{P}$, we get that no plane contains more than n lines of $\mathcal{L}$ and that every point of $\mathbb{R}^3$ is incident to at most n lines of $\mathcal{L}$. □

The cases of a plane containing many lines and of a point incident to many lines are not the only counterexamples to (4). However, the other problematic constructions are only relevant to the case of $k = 2$. When studying the case of $k \geq 3$, relying on Lemma 2.3 suffices.

Theorem 2.4. *Let $\mathcal{P}$ be a set of m points and let $\mathcal{L}$ be a set of n lines, both in $\mathbb{R}^3$, such that every plane contains at most q lines of $\mathcal{L}$ (where $1 \leq q \leq n$) and every point of $\mathcal{P}$ is incident to at least three lines of $\mathcal{L}$. Then*

$$I(\mathcal{P}, \mathcal{L}) = O\left(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3} + n\right).$$

Notice that, unlike the previous incidence bounds that we derived, the bound of Theorem 2.4 does not include an m term. Usually this term cannot be removed from the bound. For example, when considering point-line incidences in $\mathbb{R}^2$, we can take a single line and place m points on it, where m is as large as we like. In the current scenario, we cannot obtain a similar construction due to the requirement of having at least three lines through every point.

Corollary 2.5. *Let $\mathcal{L}$ be a set of n^2 lines in $\mathbb{R}^3$, such that every plane contains $O(n)$ lines of $\mathcal{L}$. Then for any $3 \leq k \leq n^2$, the number of points in $\mathbb{R}^3$ that are incident to at least k lines of $\mathcal{L}$ is $O\left(\frac{n^3}{k^2} + \frac{n^2}{k}\right)$.*

Proof. We imitate the proof of Claim 2.1. Let $\mathcal{P}$ denote the set of points of $\mathbb{R}^3$ that are incident to at least k lines of $\mathcal{L}$, and set $m = |\mathcal{P}|$. By definition, we have $I(\mathcal{P}, \mathcal{L}) \geq mk$. On the other hand, by Theorem 2.4 with $q = O(n)$, we have $I(\mathcal{P}, \mathcal{L}) = O\left(m^{1/2}n^{3/2} + n^2\right)$. Combining these two bounds yields $mk = O\left(m^{1/2}n^{3/2} + n^2\right)$, which in turn implies $m = O\left(\frac{n^3}{k^2} + \frac{n^2}{k}\right)$. □

By Lemma 2.3, it suffices to consider values of k that are at most n. In this range of k, the bound of Claim 2.5 is $O\left(\frac{n^3}{k^2}\right)$, which indeed implies (4) for $k \geq 3$. It remains to prove Theorem 2.4. Before doing that, we consider constructions that show that the theorem is tight in some cases. First, notice that when there is no restriction on the lines (that is, when $q = n$), we obtain the Szemerédi-Trotter bound, as expected.

Claim 2.6. *Consider any m, n, and q, such that $q \leq n$ and $n/q \leq m$. Then there exist a set $\mathcal{P}$ of m points and a set $\mathcal{L}$ of n lines, both in $\mathbb{R}^3$, such that no plane contains more than q lines of $\mathcal{L}$, every point of $\mathcal{P}$ is incident to at least three lines of $\mathcal{L}$, and*

$$I(\mathcal{P}, \mathcal{L}) = \Theta\left(m^{2/3}n^{1/3}q^{1/3} + n\right).$$

Proof. We consider n/q parallel planes, each containing q lines and mq/n points. There are planar point-line constructions that show that obtain Szemerédi-Trotter bound $\Theta\left(M^{2/3}N^{1/3}+N\right)$ for every number of points and lines, and with every point incident to at least three lines. Thus, we can arrange the points and lines in each of the n/q parallel planes so that the number of incidences in it is

$$\Theta\left(\left(\frac{mq}{n}\right)^{2/3} q^{2/3}+q\right)=\Theta\left(\frac{m^{2/3}q^{4/3}}{n^{2/3}}+q\right).$$

This in turn implies

$$I(\mathcal{P},\mathcal{L})=\Theta\left(\frac{n}{q}\left(\frac{m^{2/3}q^{4/3}}{n^{2/3}}+q\right)\right)=\Theta\left(m^{2/3}n^{1/3}q^{1/3}+n\right).$$

□

Claim 2.7. *Consider m and n that satisfy $m\geq 4\sqrt{n}$ and $m\leq n^{3/2}$. Then there exist a set $\mathcal{P}$ of m points and a set $\mathcal{L}$ of n lines, both in $\mathbb{R}^3$, such that every plane contains $O(\sqrt{n})$ lines of $\mathcal{L}$, every point of $\mathcal{P}$ is incident to at least three lines of $\mathcal{L}$, and*

$$I(\mathcal{P},\mathcal{L})=\Theta\left(m^{1/2}n^{3/4}\right).$$

Proof. Set $k=m^{1/2}/(2n^{1/4})$ and $\ell=\sqrt{2}n^{3/8}/m^{1/4}$, and let

$$\mathcal{P}=\left\{(r,s,t)\in\mathbb{N}^3 : 1\leq r\leq k \ \text{ and } \ 1\leq s,t\leq 2k\ell\right\}.$$

As the set of lines, we take

$$\mathcal{L}=\left\{\mathbf{V}(y-ax-b,z-cx-d) : 1\leq a,c\leq \ell \ \text{ and } \ 1\leq b,d\leq k\ell\right\}.$$

First, notice that we indeed have

$$|\mathcal{P}|=k(2k\ell)^2=4k^3\ell^2=4\cdot\frac{m^{3/2}}{8n^{3/4}}\cdot\frac{2n^{3/4}}{m^{1/2}}=m,$$
$$|\mathcal{L}|=k^2\ell^4=\frac{m}{4n^{1/2}}\cdot\frac{4n^{3/2}}{m}=n.$$

Notice that for any line $\gamma\in\mathcal{L}$ and $1\leq r\leq k$, there exists a unique point in $\mathcal{P}$ that is incident to γ and whose x-coordinate is r. That is, every line of $\mathcal{L}$ is incident to exactly k points of $\mathcal{P}$. Thus, we have

$$I(\mathcal{P},\mathcal{L})=k\cdot n=\frac{m^{1/2}}{2n^{1/4}}\cdot n=\frac{m^{1/2}n^{3/4}}{2}.$$

To avoid checking whether every point of $\mathcal{P}$ is incident to at least three lines of $\mathcal{L}$, we simply add to $\mathcal{L}$ every line that is parallel to one of the axes and is incident to points of $\mathcal{P}$. This increases the number of lines by $4k^2\ell+4k^2\ell^2\leq 8k^2\ell^2$, which is negligible compared to the size of $\mathcal{L}$.

It remains to verify that every plane contains $O(\sqrt{n})$ lines of $\mathcal{L}$. We first consider a plane h that is defined by an equation of the form $y = sx + t$ (that is, a plane that contains lines that are parallel to the z-axis). In this case h contains lines of $\mathcal{L}$ that have $a = s$ and $b = t$, and also lines that are parallel to the z-axis and were added in the previous paragraph. There are $k\ell^2 = O(\sqrt{n})$ lines that satisfy $a = s$ and $b = t$, and only $k\ell$ lines that are parallel to the z-axis.

Next, consider a plane h that is not defined by an equation of the form $y = sx+t$. Then for every choice of a, b, the plane $h' = Z(y - ax - b)$ intersects h in a unique line. That is, for every choice of a, b as in the definition of $\mathcal{L}$, there at most one line of $\mathcal{L}$ with these parameters that is contained in h. Thus, h contains $k\ell^2 = O(\sqrt{n})$ lines of $\mathcal{L}$. □

Claims 2.6 and 2.7 might give the impression that Theorem 2.4 is tight. However, it is suspected that the bound of the theorem is far from tight when q is small (specifically, when q is asymptotically smaller than $\sqrt{n}$). This seems to be a challenging open problem.

We now prove Theorem 2.4, following the analysis in [1, 4] (the proof is originally by Guth and Katz [3]).

Proof of Theorem 2.4. We prove the theorem by induction on $m+n$. Specifically, we prove that there exists a sufficiently large constant α such that

$$I(\mathcal{P}, \mathcal{L}) \le \alpha \left(m^{1/2} n^{3/4} + m^{2/3} n^{1/3} q^{1/3} + n \right).$$

For the induction basis, the bound holds for small $m + n$ (e.g., for $m + n \le 100$) by taking α to be sufficiently large.

From our weak incidence bound (Lemma 2.3 of Chapter 3), we have the bound

$$I(\mathcal{P}, \mathcal{L}) = O(m\sqrt{n} + n). \tag{5}$$

If $m = O(\sqrt{n})$, then (5) implies $I(\mathcal{P}, \mathcal{L}) = O(n)$. By taking α to be sufficiently large, we obtain $I(\mathcal{P}, \mathcal{L}) \le \alpha n$. We may thus assume that $m = \Omega(\sqrt{n})$.

By the polynomial partitioning theorem (Theorem 1.1 of Chapter 3), there exists an r-partitioning polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ for $\mathcal{P}$, such that $\deg f = O(r)$. The exact value of r will be determined below.

Let c denote the number of cells (i.e., connected components of $\mathbb{R}^3 \setminus Z(f)$). By Warren's theorem (see Theorem 1.2 of Chapter 3), we have $c = O(r^3)$. We set $\mathcal{P}_0 = Z(f) \cap \mathcal{P}$, and similarly denote by $\mathcal{L}_0$ the set of lines of $\mathcal{L}$ that are fully contained in $Z(f)$. For $1 \le i \le c$, let $\mathcal{P}_i$ denote the set of points that are contained in the i-th cell, and let $\mathcal{L}_i$ denote the set of lines of $\mathcal{L}$ that intersect the i-th cell. Notice that

$$I(\mathcal{P}, \mathcal{L}) = I(\mathcal{P}_0, \mathcal{L}_0) + I(\mathcal{P}_0, \mathcal{L} \setminus \mathcal{L}_0) + \sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{L}_i).$$

By Bézout's theorem, every line $\ell \in \mathcal{L} \setminus \mathcal{L}_0$ intersects $Z(f)$ in $O(r)$ points.[2] This

[2] To use the planar version of Bézout's theorem, as described in Chapter 2, consider a generic plane h that fully contains ℓ. Then, the variety $\gamma = Z(f) \cap h$ is of dimension at most one and of degree $O(r)$. We can then apply the theorem in h to bound $\gamma \cap \ell$.

immediately implies

$$I(\mathcal{P}_0, \mathcal{L} \setminus \mathcal{L}_0) = O(nr).$$

Let $\beta > 0$ be a sufficiently small constant. We now split the analysis into two cases, according to whether $m \le \beta n^{3/2}$ or not. First, we set different values for r in each case.

$$r = \begin{cases} \frac{m^{1/2}}{n^{1/4}}, & \text{when } m \le \beta n^{3/2}, \\ \beta n^{1/2}, & \text{when } m > \beta n^{3/2}. \end{cases} \tag{6}$$

Since we assume that $m = \Omega(\sqrt{n})$, we indeed have $r \ge 1$.

We begin with the case where $m \le \beta n^{3/2}$, and study $\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{L}_i)$. For every $1 \le i \le c$, we set $m_i = |\mathcal{P}_i|$ and $|n_i| = |\mathcal{L}_i|$. By definition, for every such i we have $m_i \le m/r^3$. By applying (5) in each cell, we obtain

$$\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{L}_i) = \sum_{i=1}^{c} O(m_i\sqrt{n_i} + n_i) = O\left(\frac{m}{r^3}\sum_{i=1}^{c}\sqrt{n_i} + \sum_{i=1}^{c} n_i\right). \tag{7}$$

Since every line of $\mathcal{L} \setminus \mathcal{L}_0$ intersects $Z(f)$ in $O(r)$ points, such a line intersects $O(r)$ cells. That is, we have $\sum_{i=1}^{c} n_i = O(nr)$. By the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{c}\sqrt{n_i} \le \left(\sum_{i=1}^{c} n_i\right)^{1/2}\left(\sum_{i=1}^{c} 1\right)^{1/2} = O\left((nr)^{1/2}\cdot r^{3/2}\right) = O\left(n^{1/2}r^2\right).$$

By combining this with (7), we obtain

$$\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{L}_i) = O\left(\frac{m\sqrt{n}}{r} + nr\right).$$

By (6), since we are in the case of $m \le \beta n^{3/2}$, we have

$$\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{L}_i) + I(\mathcal{P}_0, \mathcal{L} \setminus \mathcal{L}_0) = O\left(\frac{m\sqrt{n}}{r} + nr\right) = O_\beta\left(m^{1/2}n^{3/4}\right).$$

We next move to the case of $m > \beta n^{3/2}$. In this case, we have

$$I(\mathcal{P}_0, \mathcal{L} \setminus \mathcal{L}_0) = O\left(nr\right) = O\left(n^{3/2}\right) = O\left(m^{1/2}n^{3/4}\right).$$

Handling the incidences inside of the cells as in the previous case would yield the term $\frac{m\sqrt{n}}{r} = m$, which would ruin the induction step. Instead, we notice that in this case $\sum_{i=1}^{c} n_i = O(nr) = O(n^{3/2})$. Since there are $O(r^3) = O(n^{3/2})$ cells, we expect an average cell to contain $O(1)$ lines of $\mathcal{L}$. Specifically, for a constant μ, the number of cells containing more than μ lines of $\mathcal{L}$ is $O(n^{3/2}/\mu)$. Let $\mathcal{P}_\mu$ denote the points of $\mathcal{P}$ that are contained in a cell that is intersected by at least μ lines of $\mathcal{L}$. By taking μ to be sufficiently large, we obtain that $|\mathcal{P}_\mu| \le \frac{m}{r^3}\cdot O(n^{3/2}/\mu) = O(m/\mu) \le m/9$.

Consider a cell C that is intersected by at most μ lines of $\mathcal{L}$. Since every point of $\mathcal{P}$ is incident to at least three of lines, and there are $O(\mu^2)$ pairs of intersecting lines in C, the number of incidences in C is $O(\mu^3) = O_\mu(1)$. This in turn implies that the total number of incidences in such cells is $O(c) = O\left(n^{3/2}\right) = O\left(m^{1/2}n^{3/4}\right)$.

Since $|\mathcal{P}_\mu| \le m/9$, the induction hypothesis implies

$$I(\mathcal{P}_\mu, \mathcal{L}) \le \frac{\alpha}{3}\left(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3}\right) + \alpha n.$$

At this point we conclude the separation into two cases. By combining the bounds for both cases, taking α to be sufficiently large with respect to the constants in the $O(\cdot)$-notations, and taking β to be sufficiently small, we get

$$\sum_{i=1}^{c} I(\mathcal{P}_i, \mathcal{L}_i) + I(\mathcal{P}_0, \mathcal{L} \setminus \mathcal{L}_0) \le \frac{\alpha}{2}\left(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3}\right) + \alpha n. \tag{8}$$

To handle the analysis of $I(\mathcal{P}_0, \mathcal{L}_0)$, we first need to present some additional tools. Thus, we present this case as a separate lemma, and postpone the proof of this lemma to Section 3.

Lemma 2.8.

$$I(\mathcal{P}_0, \mathcal{L}_0) \le \frac{\alpha}{2}\left(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3}\right).$$

Combining Lemma 2.8 with (8) completes the induction step, and the proof of the theorem. □

3 Proving Lemma 2.8

We first introduce some tools that will be required for proving Lemma 2.8. The following lemma can be seen as a variant of Bézout's theorem for surfaces in $\mathbb{R}^3$ (for a proof, see for example [2]).

Lemma 3.1. *Consider two polynomials $f, g \in \mathbb{R}[x_1, x_2, x_3]$ of degrees k_f and k_g. If f and g have no common factors, then $Z(f) \cap Z(g)$ contains at most $k_f k_g$ lines.*

Given a variety $U \subset \mathbb{R}^3$, we say that a line $\ell \subset \mathbb{R}^3$ is a *singular line* of U if ℓ is fully contained in U_{sing} (recall that U_{sing} is the set of singular points of U). Lemma 3.1 leads to the following bound on the number of singular lines that a variety can have.

Corollary 3.2. *Let $U \subset \mathbb{R}^3$ be a hypersurface of degree k_U. Then U has less than k_U^2 singular lines.*

Proof. Since U is a hypersurface, there exists a polynomial f of degree k_U such that $\langle f \rangle = \mathbf{I}(U)$. Without loss of generality, we assume that $f_1 = \frac{\partial f}{\partial x_1}$ is not identically zero. By definition, f is square-free and thus has no common factors with f_1. Since f_1 vanishes on every singular point of U, every singular line of U must be fully contained in $Z(f) \cap Z(f_1)$. By Lemma 3.1, there are at most $k_U(k_U - 1)$ such lines. □

Let $U \subset \mathbb{R}^3$ be a variety, let $p \in U$ be a regular point of U, and let ℓ_1, ℓ_2, ℓ_3 be three lines that are fully contained in U and incident to p. If these three lines are not coplanar, then the tangent plane of U at p is not well defined, contradicting p being a regular point. Thus, ℓ_1, ℓ_2, ℓ_3 must be coplanar. More specifically, they must all be contained in the tangent plane to U at p.

The *Hessian matrix* of a polynomial $f \in \mathbb{R}[x_1, x_2, x_3]$ is defined as

$$H_f = \begin{pmatrix} \frac{\partial f}{\partial x_1^2} & \frac{\partial f}{\partial x_1 \partial x_2} & \frac{\partial f}{\partial x_1 \partial x_3} \\ \frac{\partial f}{\partial x_2 \partial x_1} & \frac{\partial f}{\partial x_2^2} & \frac{\partial f}{\partial x_2 \partial x_3} \\ \frac{\partial f}{\partial x_3 \partial x_1} & \frac{\partial f}{\partial x_3 \partial x_2} & \frac{\partial f}{\partial x_3^2} \end{pmatrix}.$$

Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$, and let $f \in \mathbb{R}[x_1, x_2, x_3]$. For every $i \in \{1,2,3\}$, we define the polynomial $\phi_i(f) : \mathbb{R}^3 \to \mathbb{R}$ as

$$\phi_i(f)(p) = (\nabla f(p) \times e_i)^T H_f(p) (\nabla f(p) \times e_i),$$

where $\times$ marks the cross product. If $k = \deg f$ then $\deg \phi_i(f) \le (k-1) + (k-2) + (k-1) = 3k - 4$. We say that a point $p \in Z(f)$ is a *flat point* of $Z(f)$ if p is a regular point of $Z(f)$ and $p \in \mathbf{V}(\phi_1(f), \phi_2(f), \phi_3(f))$.

Lemma 3.3. *Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a polynomial such that $\langle f \rangle = \mathbf{I}(Z(f))$, let p be a regular point of $Z(f)$, and let ℓ_1, ℓ_2, ℓ_3 be three lines that are incident to p and fully contained in $Z(f)$. Then p is a flat point of $Z(f)$.*

Proof. Consider the second-order Taylor expansion of f at p:

$$\begin{aligned} q(a) &= f(p) + \nabla f(p) \cdot (a-p) + \frac{1}{2}(a-p)^T H_f(p)(a-p) \\ &= \nabla f(p) \cdot (a-p) + \frac{1}{2}(a-p)^T H_f(p)(a-p). \end{aligned} \tag{9}$$

For $i \in \{1,2,3\}$, let $v_i \in \mathbb{R}^3$ be a vector in the direction of ℓ_i. For some $\varepsilon > 0$ and $i \in \{1,2,3\}$, set $a = p + \varepsilon v_i$. Since $\ell_i \subset U$, we have

$$0 = \varepsilon \nabla f(p) \cdot v_i + \frac{\varepsilon^2}{2} v_i^T H_f(p) v_i + O(\varepsilon^3)$$

(where the term $O(\varepsilon^3)$ is the error term in Taylor's theorem). Since this equation holds for arbitrarily small ε, the only solution for this equation is

$$\nabla f(p) \cdot v_i = v_i^T H_f(p) v_i = 0.$$

By combining this with (9), we notice that q vanishes on the entire line ℓ_i, for every $i \in \{1,2,3\}$. Let h be the tangent plane to $Z(f)$ at p, and recall that $\ell_1, \ell_2, \ell_3 \subset h$ (since p is a regular point of $Z(f)$). Let ℓ' be a line in h that intersects the lines ℓ_1, ℓ_2, ℓ_3 in three distinct points. This implies that ℓ' intersects $Z(q)$ in at least three points. Since q is a quadratic polynomial, by Bézout's theorem we have that $\ell' \subset Z(q)$. Since we can take ℓ' to be almost any line in h, we get that $h \subset Z(q)$.

A quick recap: Since p is a regular point of $Z(f)$ and there are three lines that are incident to p and fully contained in $Z(f)$, the second-order Taylor expansion of f at p (which we denote as q) vanishes on the tangent plane to U at p. By definition, the gradient $\nabla f(p)$ is orthogonal to the tangent plane of $Z(f)$ at p. Thus, q must vanish on every point $a \in \mathbb{R}^3$ that satisfies $(p-a) \times \nabla f(p) = 0$. Specifically, we have

$$q(p + \nabla f(p) \times e_i) = 0, \qquad \text{for every } i \in \{1,2,3\}. \tag{10}$$

Consider again the definition of q in (9) and notice that $\nabla f(p) \cdot (p - (p + \nabla f(p) \times e_i))$ is identically zero. Combining this with (10) immediately implies $\phi_i(f)(p) = 0$ for every $i \in \{1,2,3\}$, which completes the proof of the lemma. (A converse statement also holds: A generic point $p \in Z(f)$ is flat if and only if q vanishes on the tangent plane to $Z(f)$ at p. We do not prove this statement here.) □

One consequence of Lemma 3.3 is that a plane contains only flat points. We say that a line ℓ that is contained in a variety $U \subset \mathbb{R}^3$ is a *flat line* of U if every point of ℓ is a flat point of U, possibly excluding finitely many singular points of U. We require the following property of flat lines, which can be proved by relying on Lemma 3.1 (for a proof, see for example [1]).

Lemma 3.4. *Let $f \in \mathbb{R}[x_1, x_2, x_3]$ be a polynomial of degree k such that $Z(f)$ contains no planes. Then u contains at most $k^2 - 4k$ flat lines.*

We are now ready to prove Lemma 2.8. We first recall the statement of the lemma.

Lemma 2.8.

$$I(\mathcal{P}_0, \mathcal{L}_0) \le \frac{\alpha}{2}\left(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3}\right).$$

Proof. Let $\mathcal{P}_0^-$ be the set of points of $\mathcal{P}_0$ that are incident to less than three lines of $\mathcal{L}_0$. By the assumption of Theorem 2.4, every point of $\mathcal{P}$ is incident to at least three lines of $\mathcal{L}$, so every point of $\mathcal{P}_0^-$ is incident to at least one line of $\mathcal{L} \setminus \mathcal{L}_0$. As we saw in the the main part of the proof of Theorem 2.4, every line of $\mathcal{L} \setminus \mathcal{L}_0$ intersects $Z(f)$ in $O(r)$ points. That is, $|\mathcal{P}_0^-| = O(nr)$, which in turn implies

$$I(\mathcal{P}_0^-, \mathcal{L}_0) = O(nr).$$

Consider a point $p \in Z(f)$ that is incident to three lines that are fully contained in $Z(f)$. By Lemma 3.3, p is either a singular point or a flat point.

Singular points. Let $\mathcal{P}_{\text{sing}}$ be the points of $\mathcal{P}_0 \setminus \mathcal{P}_0^-$ that are singular points of $Z(f)$. Let $\mathcal{L}_{\text{sing}}$ denote the lines of $\mathcal{L}_0$ that are singular lines of $Z(f)$. Corollary 3.2 implies $|\mathcal{L}_{\text{sing}}| = O(r^2)$. Recall our choice of r from (6). When $m \le \beta n^{3/2}$, we have

$$|\mathcal{L}_{\text{sing}}| = O\left(r^2\right) = O\left(m/n^{1/2}\right) = O(\beta n).$$

Similarly, when $m > \beta n^{3/2}$, we have

$$|\mathcal{L}_{\text{sing}}| = O\left(r^2\right) = O(\beta^2 n).$$

In either case, by taking β to be sufficiently small with respect to the constant in the $O(\cdot)$-notation, we have $|\mathcal{L}_{\text{sing}}| \le n/1000$. By the induction hypothesis (see the main part of the proof of Theorem 2.4), we have

$$I(\mathcal{P}_{\text{sing}}, \mathcal{L}_{\text{sing}}) \le \frac{\alpha}{10}\left(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3} + n\right).$$

Consider a line $\ell \in \mathcal{L}_0 \setminus \mathcal{L}_{\text{sing}}$. Since ℓ is not a singular line of $Z(f)$, at least one of the first partial derivatives of f does not vanish on it. Without loss of generality, assume that $f_1 = \frac{\partial f}{\partial x_1}$. By Bézout's theorem (in a generic plane that contains ℓ), $|\ell \cap Z(f_1)| = O(r)$. That is, any line of $\mathcal{L}_0 \setminus \mathcal{L}_{\text{sing}}$ is incident to $O(r)$ points of $\mathcal{P}_{\text{sing}}$. This immediately implies

$$I(\mathcal{P}_{\text{sing}}, \mathcal{L}_0 \setminus \mathcal{L}_{\text{sing}}) = O(nr).$$

Flat points. It remains to consider incidences with points of $\mathcal{P}_0$ that are regular points of $Z(f)$ and are incident to at least three lines of $\mathcal{L}_0$. Such a point $p \in \mathcal{P}_0 \setminus (\mathcal{P}_{\text{sing}} \cup \mathcal{P}_0^-)$ is contained in a single component of $Z(f)$, and the lines of $\mathcal{L}_0$ that are incident to it are fully contained in the same component. That is, p is a flat point. Let $\Pi_1, \ldots, \Pi_{c'}$ denote the components of $Z(f)$ that are planes, and let $\Omega_1, \ldots, \Omega_{c''}$ denote the other components of $Z(f)$.

We first consider the incidences in the planes $\Pi_1, \ldots, \Pi_{c'}$. Let $\mathcal{P}_{\Pi,i}$ denote the points of $\mathcal{P}_0 \setminus (\mathcal{P}_{\text{sing}} \cup \mathcal{P}_0^-)$ that are in Π_i and let $\mathcal{L}_{\Pi,i}$ denote the lines of $\mathcal{L}_0$ that are fully contained in Π_i. We set $m_{\Pi,i} = |\mathcal{P}_{\Pi,i}|$ and $n_{\Pi,i} = |\mathcal{L}_{\Pi,i}|$.

By applying the standard Szemerédi-Trotter bound in Π_i, we obtain $I(\mathcal{P}_{\Pi,i}, \mathcal{L}_{\Pi,i}) = O(m_{\Pi,i}^{2/3} n_{\Pi,i}^{2/3} + m_{\Pi,i} + n_{\Pi,i})$. Since every point of $\mathcal{P}_{\Pi,i}$ is incident to at least three lines of $\mathcal{L}_{\Pi,i}$, and there are less than $n_{\Pi,i}^2$ intersection points between the lines of $\mathcal{L}_{\Pi,i}$, we have $m_{\Pi,i} = O\left(n_{\Pi,i}^2\right)$. This implies $m_{\Pi,i} = O\left(m_{\Pi,i}^{2/3} n_{\Pi,i}^{2/3}\right)$, which in turn implies $I(\mathcal{P}_{\Pi,i}, \mathcal{L}_{\Pi,i}) = O(m_{\Pi,i}^{2/3} n_{\Pi,i}^{2/3} + n_{\Pi,i})$. Recalling that $n_i \le q$ for any $1 \le i \le c'$, and then applying Hölder's inequality (see Chapter 3), we obtain

$$\begin{aligned}
\sum_{i=1}^{c'} I(\mathcal{P}_{\Pi,i}, \mathcal{L}_{\Pi,i}) &= \sum_{i=1}^{c'} O\left(m_{\Pi,i}^{2/3} n_{\Pi,i}^{2/3} + n_{\Pi,i}\right) \\
&\le \sum_{i=1}^{c'} O\left(m_{\Pi,i}^{2/3} n_{\Pi,i}^{1/3} q^{1/3} + n_{\Pi,i}\right) \\
&= O\left(m^{2/3} n^{1/3} q^{1/3} + n\right).
\end{aligned}$$

Finally, consider the incidences in the remaining components $\Omega_1, \ldots, \Omega_{c''}$. Let $\mathcal{P}_\Omega$ denote the points of $\mathcal{P}_0 \setminus (\mathcal{P}_{\text{sing}} \cup \mathcal{P}_0^-)$ that are in the components $\Omega_1, \ldots, \Omega_{c''}$, and let $\mathcal{L}_\Omega$ denote the lines of $\mathcal{L}_0 \setminus \mathcal{L}_{\text{sing}}$ that are fully contained in one of these components. Let $\mathcal{L}_{\text{flat}}$ be the set of lines of $\mathcal{L}_\Omega$ that are flat lines of one of these components. By Lemma 3.4, $|\mathcal{L}_{\text{flat}}| = O(r^2)$. By repeating the above analysis for $\mathcal{L}_{\text{sing}}$ and taking β

to be sufficiently large, we have $|\mathcal{L}_{\text{flat}}| \le n/1000$. Thus, by the induction hypothesis we have

$$I(\mathcal{P}_\Omega, \mathcal{L}_{\text{flat}}) \le \frac{\alpha}{10}\left(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3} + n\right).$$

Consider a line $\ell \in \mathcal{L}_\Omega \setminus \mathcal{L}_{\text{flat}}$. Since ℓ is not a flat line of $Z(f)$, there exists $i \in \{1,2,3\}$ such that $\phi_i(f)$ does not vanish identically on ℓ. Recall that $\deg \phi_i(f) = 3\deg f - 4 = O(r)$. By Bézout's theorem (in a generic plane that contains ℓ), $|\ell \cap Z(\phi_i(f))| = O(r)$. That is, any line of $\mathcal{L}_0 \setminus \mathcal{L}_{\text{flat}}$ is incident to $O(r)$ points of $\mathcal{P}_{\text{flat}}$. This immediately implies

$$I(\mathcal{P}_{\text{flat}}, \mathcal{L}_0 \setminus \mathcal{L}_{\text{flat}}) = O(nr).$$

Wrapping up. By combining all of the above cases, we obtain

$$I(\mathcal{P}_0, \mathcal{L}_0) \le \frac{\alpha}{5}\left(m^{1/2}n^{3/4} + m^{2/3}n^{1/3}q^{1/3} + n\right) + O\left(nr + m^{2/3}n^{1/3}q^{1/3} + n\right).$$

The $O(nr)$ term can bounded by $O\left(m^{1/2}n^{3/4}\right)$, as explained in the main part of the proof of Theorem 2.4. Similarly, since we assume that $n = O\left(m^2\right)$, we have $n = n^{1/4}n^{3/4} = O\left(m^{1/2}n^{3/4}\right)$. Thus, by taking α to be sufficiently large with respect to the constants in the $O(\cdot)$-notations, we obtain the assertion of the lemma. □

References

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