Chapter 9: Distinct distances variants

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After two rather technical chapters which completed the proof of the distinct distances theorem, we move to a short simple chapter. This chapter presents the best known bounds for a couple of open variants of the distinct distances problem.

1 Subsets with no repeated distances

Given a set $\mathcal{P}$ of points in $\mathbb{R}^2$, let $\mathsf{subset}(\mathcal{P})$ denote the size of the largest subset $\mathcal{P}' \subset \mathcal{P}$ such that every distance is spanned by the points of $\mathcal{P}'$ at most once. That is, there are no points $a, b, c, d \in \mathcal{P}'$ such that $|ab| = |cd| > 0$ (including cases where $a = c$). Figure 1 depicts a set of 25 points and a subset of four points that span every distance at most once.

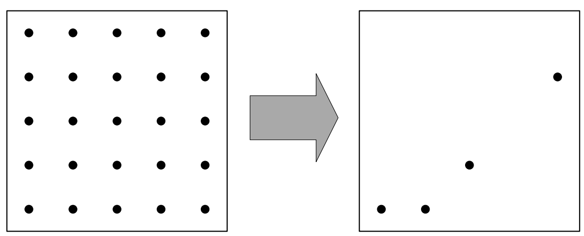


Figure 1: A set of 25 points and a subset of four points that span every distance at most once. No subset of five points has this property.

Let $\mathsf{subset}(n) = \min_{|\mathcal{P}|=n} \mathsf{subset}(\mathcal{P})$. In other words, $\mathsf{subset}(n)$ is the maximum number satisfying the property that every set of n points in the plane contains a subset of $\mathsf{subset}(n)$ points that do not span any distance more than once. We are interested in the asymptotic value of $\mathsf{subset}(n)$.

Let $\mathcal{P}$ be a point set that spans d distinct distances. Notice that if all of the distances that are spanned by the points of a subset $\mathcal{P}' \subset \mathcal{P}$ are unique, then the number of distances that are spanned by $\mathcal{P}$ must be at least $\binom{|P'|}{2}$. That is, $|\mathcal{P}'| = O\left(\sqrt{d}\right)$. Let $\mathcal{L}$ be a $\sqrt{n} \times \sqrt{n}$ integer lattice. In Chapter 1 we proved that $\mathcal{L}$ determines $\Theta(n/\sqrt{\log n})$ distinct distances. Therefore, we have $\mathsf{subset}(n) \le \mathsf{subset}(\mathcal{L}) = O\left(\sqrt{n}/\log^{1/4} n\right)$. This is the current best upper bound for $\mathsf{subset}(n)$.

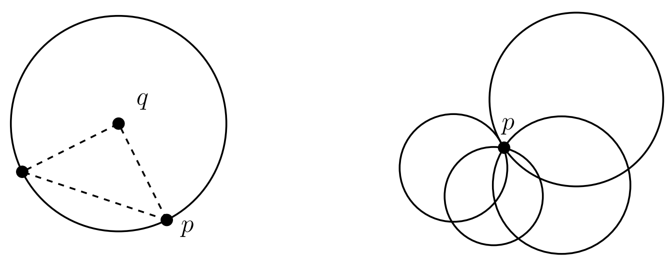
The following is the current best lower bound for $\mathsf{subset}(n)$.

Theorem 1.1 (Charalambides [1]). $\mathsf{subset}(n) = \Omega\left(n^{1/3}/\log^{1/3} n\right)$.

The proof of Theorem 1.1 combines ideas from the probabilistic method with results that were obtained by the polynomial method. For this proof, we require a bound on the maximum number of isosceles triangles that can be determined by a set of n points in $\mathbb{R}^2$. The current best bound, by Pach and Tardos [3], is $O(n^{2.137})$. The following weaker bound suffices for proving Theorem 1.1.

Claim 1.2. *Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^2$. Then $\mathcal{P}$ determines $O(n^{7/3})$ isosceles triangles (that is, triangles whose three vertices are points of $\mathcal{P}$).*

Proof. We consider a point $p \in \mathcal{P}$ and bound the number of isosceles triangles in which p is incident to the base edge. Given another point $q \in \mathcal{P}$, we consider triangles where q is the vertex not incident to the base edge, and p is one of the other two vertices. The third vertex of such a triangle must be incident to the circle centered at q and incident to p (for example, see Figure 2(a)). We denote this circle as C_q, and set $\mathcal{C} = \{C_q : q \in \mathcal{P} \setminus \{p\}\}$. The number of isosceles triangles in which p is incident to the base edge is exactly $I(\mathcal{P} \setminus \{p\}, \mathcal{C})$.



(a) (b)

Figure 2: (a) To obtain an isosceles triangle with p incident to the base edge but not q, the third vertex must be on the circle centered at q and incident to p. (b) Circles that intersect at p lead to an incidence graph with no $K_{2,2}$ (when p is not one of the points in this graph).

To bound $I(\mathcal{P} \setminus \{p\}, \mathcal{C})$, we first notice that no two circles of $\mathcal{C}$ are identical, since each circle has a different center. In a general point-circle configurations, the incidence graph might contain a $K_{2,2}$. However, since all of the circles of $\mathcal{C}$ are incident to p, the incidence graph of $\mathcal{P} \setminus \{p\} \times \mathcal{C}$ does not contain a $K_{2,2}$ (an example is depicted in Figure 2(b)). Theorem 2.1 of Chapter 3 implies $I(\mathcal{P}\setminus\{p\}, \mathcal{C}) = O\left(n^{4/3}\right)$. By summing this bound over every $p \in \mathcal{P}$, we obtain that the number of isosceles triangles that are determined by $\mathcal{P}$ is $O(n^{7/3})$. $\square$

We are now ready to derive an upper bound for $\mathsf{subset}(n)$.

Proof of Theorem 1.1. Consider a set $\mathcal{P}$ of n points in $\mathbb{R}^2$. Similarly to the Elekes-Sharir reduction, we define the set

$$Q_1 = \left\{(a, b, c, d) \in \mathcal{P}^4 \;\middle|\; |ab| = |cd| > 0\right\},$$

such that every quadruple of Q_1 consists of four distinct points. In Chapters 6–8, we proved that $|Q_1| = O(n^3 \log n)$ (which also applies when Q_1 is allowed to contain quadruples where not all four points are distinct). Let Q_2 be the set of isosceles triangles that are spanned by points of $\mathcal{P}$ (including equilateral triangles). By Claim 1.2, we have $|Q_2| = O(n^{7/3})$.

Let $\mathcal{P}' \subset \mathcal{P}$ be a subset that is obtained by choosing every point of $\mathcal{P}$ independently with probability $0 < p < 1$ that will be determined below. We have $\mathbb{E}[|\mathcal{P}'|] = pn$. Let $Q_1' \subset Q_1$ be the set of quadruples of Q_1 that contain only points of $\mathcal{P}'$. Every quadruple of Q_1 is in Q_1' with a probability of p^4, so $\mathbb{E}[|Q_1'|] \le \alpha p^4 n^3 \log n$ for a sufficiently large constant α. Let Q_2' be the set of triangles of Q_2 that contain only points of $\mathcal{P}'$. We have $\mathbb{E}[|Q_2'|] \le \alpha p^3 n^{7/3}$, for sufficiently large α. Notice that the points of $\mathcal{P}'$ span every distance at most once if and only if $|Q_1'| = |Q_2'| = 0$. By linearity of expectation, we have

$$\mathbb{E}\left[|\mathcal{P}'| - |Q_1'| - |Q_2'|\right] \ge pn - \alpha p^4 n^3 \log n - \alpha p^3 n^{7/3}.$$

By setting $p = 1/(2\alpha n^2 \log n)^{1/3}$, for sufficiently large n we obtain

$$\mathbb{E}\left[|\mathcal{P}'| - |Q_1'| - |Q_2'|\right] = \frac{n^{1/3}}{2^{1/3}\alpha^{1/3}\log^{1/3} n} - \frac{n^{1/3}}{2^{4/3}\alpha^{1/3}\log^{1/3} n} - \frac{n^{1/3}}{2\log n} > \frac{n^{1/3}}{3\alpha^{1/3}\log^{1/3} n}.$$

Thus, there exists a subset $\mathcal{P}' \subset \mathcal{P}$ for which $|\mathcal{P}'| - |Q_1'| - |Q_2'| > \frac{n^{1/3}}{3(\alpha_1 \log n)^{1/3}}$. Let $\mathcal{P}''$ be a subset of $\mathcal{P}'$ that is obtained by removing from $\mathcal{P}'$ a point from every element of Q_1' and Q_2'. The subset $\mathcal{P}''$ does not span any repeated distances and contains $\Omega(n^{1/3}/\log^{1/3} n)$ points of $\mathcal{P}$. □

2 Point sets with few distinct distances

Characterizing the sets of n points in $\mathbb{R}^2$ that determine a small number of distinct distances (say $O(n/\sqrt{\log n})$) appears to be one of the most difficult open problems concerning distinct distances. Erdős asked whether every such point set "has lattice structure" [2]. In Chapter 1 we saw that a $\sqrt{n} \times \sqrt{n}$ integer lattice determines $O(n/\sqrt{\log n})$ distinct distances. The same holds for various other types of $\sqrt{n} \times \sqrt{n}$ lattices (e.g., see [4]). As a first step, Erdős [2] suggested to determine whether every near-optimal point set contains $\Omega(\sqrt{n})$ points on a line, and thus most of it can be covered by a small number of lines. Since this also appears to be quite difficult, Erdős asked whether there exists a line with $\Omega(n^{\varepsilon})$ points of the set. Embarrassingly, even this weaker variant remains open after several decades.

We now present the (rather weak) current best bound. To derive this bound, we require the following straightforward generalization of Theorem 2.1 of Chapter 3 (proving this generalization was a question in the first homework assignment).

Theorem 2.1. *Let $\mathcal{P}$ be a set of m points and let Γ be a set of n distinct irreducible algebraic curves of degree at most k, both in $\mathbb{R}^2$. If the incidence graph of $\mathcal{P} \times \Gamma$ contains no copy of $K_{s,t}$, then*

$$I(\mathcal{P}, \Gamma) = O_{s,k}\left(m^{\frac{s}{2s-1}} n^{\frac{2s-2}{2s-1}} t^{\frac{1}{2s-1}} + m + n\right).$$

Claim 2.2. *Let $\mathcal{P}$ be a set of n points in $\mathbb{R}^2$, such that $\mathcal{P}$ determines $O(n/\sqrt{\log n})$ distinct distances. Then there exists a line that contains $\Omega(\log n)$ points of $\mathcal{P}$.*

Proof. Denote the set of distinct distances that are spanned by $\mathcal{P}$ as D. Let $\mathcal{C}$ be the set of circles that is obtained by drawing $|D| = O(n/\sqrt{\log n})$ circles around every point of $\mathcal{P}$, with the radii being the distances in D. Notice that $|\mathcal{C}| = O(n^2/\sqrt{\log n})$. Since the circles around any point $p \in \mathcal{P}$ contain all of the points of $\mathcal{P} \setminus \{p\}$, we have

$$I(\mathcal{P}, \mathcal{C}) = n(n-1). \tag{1}$$

Let x denote the maximum number of points of $\mathcal{P}$ that are on a common line. Given two points $p, q \in \mathcal{P}$, a circle $C \in \mathcal{C}$ can contain both p and q only if the perpendicular bisector of p and q is incident to the center of C. This implies that the incidence graph of $\mathcal{P} \times \mathcal{C}$ contains no copy of $K_{2,x+1}$. Thus, by Theorem 2.1 we have

$$I(\mathcal{P}, \mathcal{C}) = O\left(n^{2/3}\left(\frac{n^2}{\sqrt{\log n}}\right)^{2/3} x^{1/3} + \frac{n^2}{\sqrt{\log n}}\right) = O\left(\frac{n^2 x^{1/3}}{\log^{1/3} n} + \frac{n^2}{\sqrt{\log n}}\right).$$

Combining this with (1) implies the assertion of the claim. □

References

[1] M. Charalambides, A note on distinct distance subsets, *Journal of Geometry*, **104** (2013), 439–442.

[2] P. Erdős, On some metric and combinatorial geometric problems, *Discrete Math.* **60** (1986), 147–153.

[3] J. Pach and G. Tardos, Isosceles triangles determined by a planar point set, *Graphs and Combinatorics* **18** (2002), 769–779.

[4] A. Sheffer, Point sets with few distinct distances, blog post, `https://adamsheffer.wordpress.com/2014/07/16/point-sets-with-few-distinct-distances/`.