# Chapter 2: The Sum-Product Problem

Adam Sheffer

April 7, 2016

## 1 Introduction

Consider two finite subsets $A, B \subset \mathcal{G}$, where $G$ is a field. We define the *product set* and *ratio set* of $A$ and $B$ as

$$AB = \{a \cdot b : a \in A \text{ and } b \in B\},$$

$$A/B = \{a/b : a \in A \text{ and } b \in B\}.$$

In this chapter we only consider the field $\mathbb{R}$. Just as with sum sets, a trivial upper bound for the size of a product set is $|AA| \le \frac{|A|^2+|A|}{2}$. If $A$ is obtained by choosing elements of $\mathbb{R}$ at random then we expect $|AA|$ to be very close to this upper bound, since the probability of $a_1a_2 = a_3a_4$ (with $a_1, a_2, a_3, a_4 \in A$) is very small. On the other hand, if $A = \{2^1, 2^2, \ldots, 2^n\}$ then $|AA| = |\{2^2, 2^3, 2^4, \ldots, 2^{2n}\}| = 2|A| - 1$. The same bound applies whenever $A$ is a geometric progression with $0 \notin A$. For every finite set $A \subset \mathbb{R}$ with $0 \notin A$ we have $|A+A| \ge 2|A|-1$ (this is immediately obtained by repeating the analogous proof for sum sets in Chapter 1). When $0 \in A$, it is possible to obtain $|AA| = 2|A| - 2$.

The *sum-product conjecture* suggests that no set has both a small sum set and a small product set.

**Conjecture 1.1 (Erdős and Szemerédi [5]).** *For any $\varepsilon > 0$ there exists $n_0$, such that any set $A$ of at least $n_0$ integers satisfies* $\max\{|A+A|, |AA|\} \ge c|A|^{2-\varepsilon}$ *(for some absolute constant $c$).*

Erdős and Szemerédi [5] also showed how to derive arbitrarily large sets $A$ with $\max\{|A + A|, |AA|\} = O(|A|^{2-c/\lg\lg n})$ (for some constant $c$). This example shows that Conjecture 1.1 is false without the extra $\varepsilon$ in the exponent. Erdős and Szemerédi also derived a lower bound of the form $\max\{|A+A|,|AA|\} \geq |A|^{1+\varepsilon}$ without finding the exact value of $\varepsilon$.

In the last two decades Conjecture 1.1 has been generalized to various fields and rings, as well as to variants concerning different expressions (for example, showing that $|AA+A|$ must be large). Table 1 surveys the progress in the work over $\mathbb{R}$.[1]

Table 1: Progress on the lower bound for $\max\{|A+A|,|AA|\}$ over the reals (the first several results apply only to sets of integers). The year stands for year of publication, and $\varepsilon$ applies for any sufficiently small $\varepsilon > 0$.

| Year | Bound | Author |
|---|---|---|
| 1983 | $\Omega\left(\vert A\vert^{1+\varepsilon}\right)$ | Erdős and Szemerédi [5] |
| 1997 | $\Omega\left(\vert A\vert^{1+\frac{1}{31}}\right)$ | Nathanson [8] |
| 1998 | $\Omega\left(\vert A\vert^{1+\frac{1}{15}}\right)$ | Ford [6] |
| 1997 | $\Omega\left(\vert A\vert^{1+\frac{1}{4}}\right)$ | Elekes [2] |
| 2005 | $\Omega\left(\vert A\vert^{1+\frac{3}{11}}\right)$ | Solymosi [10] |
| 2009 | $\Omega^*\left(\vert A\vert^{1+\frac{1}{3}}\right)$ | Solymosi [11] |
| 2016 | $\Omega^*\left(\vert A\vert^{1+\frac{1}{3}+\frac{5}{9813}}\right)$ | Konyagin and Shkredov [7] |

## 2 Elekes' Bound

Elekes' bound [2] is considered to be a milestone in the study of the problem for several reasons. Beyond providing the largest jump in the exponent so far, it also generalized the bound from the case of integers to the case of real numbers. Elekes achieved this by noting that the problem can be studied using geometry. All of the subsequent bounds also rely heavily on geometry, and thus also apply to the case of $\mathbb{R}$.

Given a point set $\mathcal{P}$ and a set of lines $\mathcal{L}$, both in $\mathbb{R}^2$, an *incidence* is a pair $(p,\ell) \in \mathcal{P} \times \mathcal{L}$ with the point $p$ being on the line $\ell$ (e.g., see Figure 1). We denote by $I(\mathcal{P},\mathcal{L})$ the number of incidences in $\mathcal{P} \times \mathcal{L}$. The following theorem is probably the most well known incidence result.

[1]Recall that the $\Omega^*(\cdot)$-notation hides polylogarithmic factors.

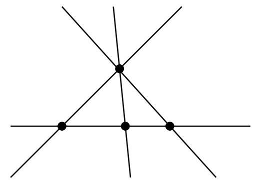

Figure 1: A configuration of four points, four lines, and nine incidences.

**Theorem 2.1 (The Szemerédi-Trotter theorem).** *Let $\mathcal{P}$ be a set of $m$ points and let $\mathcal{L}$ be a set of $n$ lines, both in $\mathbb{R}^2$. Then $I(\mathcal{P}, \mathcal{L}) = O(m^{2/3}n^{2/3} + m + n)$.*

This theorem is tight — for any $m$ and $n$ there are point-line constructions with $\Theta(m^{2/3}n^{2/3} + m + n)$ incidences. The main term in the bound is $m^{2/3}n^{2/3}$, since this term dominates in the interesting range of $m = O(n^2)$ and $m = \Omega(\sqrt{n})$. We are now ready to derive our first sum-product bound.

**Theorem 2.2 (Elekes [2]).** *Every finite set $A$ of real numbers satisfies*

$$\max\{|A + A|, |AA|\} = \Omega\left(|A|^{5/4}\right).$$

*Proof.* Consider the planar point set

$$\mathcal{P} = \{(c, d) : \ c \in A + A \quad \text{and} \quad d \in AA\}.$$

We also consider the set of lines

$$\mathcal{L} = \{y = a(x - a') : \ a, a' \in A\}$$

(by $y = a(x - a')$ we refer to the line in $\mathbb{R}^2$ that is defined by this linear equation). Notice that $|\mathcal{L}| = |A|^2$ and $|\mathcal{P}| = |A + A| \cdot |AA|$. Figure 2 depicts such a point-line construction.

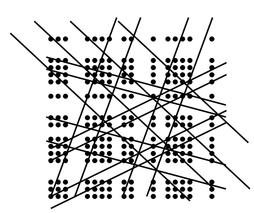

Figure 2: The point-line construction in Elekes' proof: A cartesian product of points and $|A|$ sets of $|A|$ parallel lines.

The proof is based on double counting the number of incidences $I(\mathcal{P}, \mathcal{L})$. A line that is defined by $y = a(x - a')$ contains every point of $\mathcal{P}$ of the form $(a' + b, ab)$ for every $b \in A$. That is, every line of $\mathcal{L}$ is incident to at least $|A|$ points of $\mathcal{P}$, and we have

$$I(\mathcal{P},\mathcal{L}) \geq |\mathcal{L}||A| = |A|^3. \tag{1}$$

On the other hand, by Theorem 2.1 we have

$$\begin{aligned} I(\mathcal{P},\mathcal{L}) &= O\left(|\mathcal{P}|^{2/3}|\mathcal{L}|^{2/3} + |\mathcal{P}| + |\mathcal{L}|\right) \\ &= O\left(|A+A|^{2/3}|AA|^{2/3}n^{4/3} + |A+A|\cdot|AA| + |A|^2\right). \end{aligned} \tag{2}$$

By combining (1) and (2), we obtain

$$|A|^3 = O\left(|A+A|^{2/3}|AA|^{2/3}n^{4/3} + |A+A|\cdot|AA| + |A|^2\right),$$

or

$$|A+A|\cdot|AA| = \Omega\left(|A|^{5/2}\right).$$

It is thus impossible for both $|A+A|$ and $|AA|$ to be asymptotically smaller than $|A|^{5/4}$, which completes the proof of the theorem. □

The following subsections present a couple of results that are related to Elekes' proof.

## 2.1 The case of small sum sets

By inspecting the proof of Theorem 2.2, we notice that if $|A+A| = k|A|$ (where $k$ may depend on $|A|$) then $|AA| = \Omega(|A|^{3/2}/k)$. In this section we present a bound by Elekes and Ruzsa [4] that is stronger when $k = O^*(|A|^{1/6})$. First, we rely on Theorem 2.1 to prove the following.

**Corollary 2.3.** *(a) Let $\mathcal{P}$ be a set of $m$ points in $\mathbb{R}^2$ and let $k$ be a positive integer. Then the number of lines in $\mathbb{R}^2$ that are incident to at least $k$ points of $\mathcal{P}$ is $O\left(\frac{m^2}{k^3} + \frac{m}{k}\right)$.*
*(b) Let $A$ be a set of $m$ real numbers. Then the number of collinear triples in $A \times A \subset \mathbb{R}^2$ (that is, triples of points that are on a common line) is $O(m^4 \lg m)$.*

*Proof.* We first prove part (a). As a trivial bound, we have $O(m^2)$ lines for every $k \geq 2$ since there are $O(m^2)$ lines that pass through at least two points of $\mathcal{P}$. This completes the proof when $k$ is a constant, so we may assume that $k$ is larger than the constant in the $O(\cdot)$-notation of Theorem 2.1.

Consider a larger value of $k$. Let $\mathcal{L}$ denote the set of lines that are incident to at least $k$ points of $\mathcal{P}$, and set $n_k = |\mathcal{L}|$. By definition, we have $I(\mathcal{P}, \mathcal{L}) \geq n_k k$. On the other hand, Theorem 2.1 implies $I(\mathcal{P}, \mathcal{L}) = O(m^{2/3}n_k^{2/3} + n_k + m)$. Combining these two bounds yields $n_k k = O(m^{2/3}n_k^{2/3} + n_k + m)$. Since $k$ is larger than the constant in the $O(\cdot)$-notation, the dominating term in the bound cannot be $n_k$, so $n_k k = O(m^{2/3}n_k^{2/3} + m)$. This immediately implies $n_k = O\left(\frac{m^2}{k^3} + \frac{m}{k}\right)$, as required.

We move to prove part (b) of the corollary. As before, for every $k \geq 3$ we denote by $n_k$ the number of lines that contain at least $k$ points of $A \times A$. Similarly, we denote by $n_{=k}$ the number of lines that contain *exactly* $k$ points of $A \times A$. Let $\mathcal{L}$ denote the set of lines that contain at least three points of $A \times A$.

We use a standard trick called *dyadic decomposition*, in which we partition $\mathcal{L}$ into subsets such that all of the lines in the same subset contain a similar number of points of $A \times A$. Specifically, for every $1 \leq k \leq \lg m$ we separately consider the lines that contain between $2^k$ and $2^{k+1} - 1$ points. By applying the bound of part (a), we get that the number of collinear triples in $A \times A$ is

$$\sum_{j=3}^{m} \binom{j}{3} n_{=j} < \sum_{k=1}^{\lg m} \binom{2^{k+1}}{3} n_{2^k} = \sum_{k=1}^{\lg m} \binom{2^{k+1}}{3} \cdot O\left(\frac{m^4}{2^{3k}} + \frac{m^2}{2^k}\right)$$
$$= \sum_{k=1}^{\lg m} O\left(m^4 + m^2 2^{2k}\right) = O\left(m^4 \lg m\right). \qquad \square$$

Given a finite set $A \subset \mathbb{R}$, the *multiplicative energy* of $A$ is

$$E^\times(A) = \left|\left\{(a_1, a_2, a_3, a_4) \in A^4 : a_1 a_2 = a_3 a_4\right\}\right|.$$

Notice that $|A|^2 \leq E^\times(A)$ since this is the number of solutions with $a_1 = a_3$ and $a_2 = a_4$. Similarly, $E^\times(A) \leq |A|^3$ holds since for any choice of $a_1, a_2$, and $a_3$ there is at most one valid choice for $a_4$.

For any $x \in AA$ we set $r_A^\times(x) = |\{(a_1, a_2) \in A^2 : a_1 a_2 = x\}|$. Since every pair of $A^2$ participates in exactly one $r_A^\times(x)$, we have $\sum_{x \in AA} r_A^\times(x) = |A|^2$. By the Cauchy-Schwarz inequality, we have

$$E^\times(A) = \sum_{x \in AA} r_A^\times(x)^2 \geq \frac{\left(\sum_x r_A^\times(x)\right)^2}{|AA|} = \frac{|A|^4}{|AA|}. \qquad (3)$$

Intuitively (and not accurately), the smaller $|AA|$ is, the larger $E^\times(A)$ is. Some examples to illustrate this:

- If $A$ is a geometric progression then $|AA| = 2|A| - 1$ and $E^{\times}(A) = \Theta(|A|^3)$.
- If $A$ is a random set, then we expect $|AA| = \Theta(|A|^2)$ and $E^{\times}(A) = \Theta(|A|^2)$.
- Let $A = HR$ with $H$ being a geometric progression of size $\Theta(|A|^{\alpha})$ and $R$ being a random set of size $\Theta(|A|^{1-\alpha})$ (for some $0 < \alpha < 1$). Then $|AA| \approx |A|^{2-\alpha}$ and $E^{\times}(A) \approx |A|^{2+\alpha}$.

By (3), a small value of $|AA|$ always implies a large value of $E^{\times}(A)$. However, the other direction does not always hold. For example, take $A = P \cup R$ where $P$ is a geometric progression of size $|A|/2$ and $R$ is a random set of size $|A|/2$. In this case we have $|AA| = \Theta(|A|^2)$ and $E^{\times}(A) = \Theta(|A|^3)$.

We are now ready to prove the bound of Elekes and Ruzsa.

**Theorem 2.4.** *Let $A \subset \mathbb{R}$ satisfy $|A + A| = k|A|$. Then $|AA| = \Omega\left(\frac{|A|^2}{k^4 \lg(|A|)}\right)$.*

One nice consequence of Theorem 2.4: If $|A + A| = O(|A|)$ then $|AA| = \Omega^*(|A|^2)$.

*Proof of Theorem 2.4.* We set $X = A \cup (A + A)$ and consider the product $\mathcal{P} = X \times X \subset \mathbb{R}^2$. Let $T$ denote the number of collinear triples in $\mathcal{P}$. The proof is based on double counting $T$. By part (b) of Corollary 2.3, we have

$$T = O\left(|P|^2 \lg |P|\right) = O\left(|A + A|^4 \lg |A + A|\right) = O\left(k^4 |A|^4 \lg |A|\right).$$

By (3), there exist $\Omega\left(\frac{|A|^4}{|AA|}\right)$ quadruples $(a_1, a_2, a_3, a_3) \in A^4$ with $a_1 a_2 = a_3 a_4$. For each such quadruple and $(b, c) \in A^2$, we have that

$$(b, c), \quad (b + a_1, c + a_3), \quad (b + a_4, c + a_2)$$

is a collinear triple of points of $\mathcal{P}$. Indeed, the line through the first two points has slope $\frac{a_3}{a_1}$, and the line through the first and third points is of slope $\frac{a_2}{a_4}$. Since $a_1 a_2 = a_3 a_4$, the two lines are identical. Thus, we have that $T = \Omega\left(\frac{|A|^6}{|AA|}\right)$.

Combining the two bounds for $T$ yields the assertion of the theorem. □

## 2.2 Sums and strictly convex curves

In this section we consider a variant of the sum-product problem which involves a strictly convex or strictly concave function. For this variant we need to introduce another well-known incidence bound.

**Theorem 2.5 (Pach and Sharir [9]).** *Let $\mathcal{P}$ be a set of $m$ points and let $\Gamma$ be a set of $n$ curves, both in $\mathbb{R}^2$. If every two curves of $\Gamma$ intersect in at most $t$ points and no $s$ points of $\mathcal{P}$ are contained in the intersection of $t$ curves of $\Gamma$ (for some constants $s$ and $t$), then*

$$I(\mathcal{P},\Gamma) = O_{s,t}\left(m^{\frac{s}{2s-1}} n^{\frac{2s-2}{2s-1}} + m + n\right).$$

To emphasize the strength of Theorem 2.5, we consider some common types of curves:

- If $\Gamma$ is a set of lines, since two lines intersect in at most one point, we can set $s = t = 2$. That is, Theorem 2.5 generalizes the Szemerédi-Trotter theorem (Theorem 2.1).
- If $\Gamma$ is a set of circles of radius 1, then we can set $s = 2$ and $t = 3$. That is, we again obtain the bound $I(\mathcal{P},\Gamma) = O\left(m^{2/3}n^{2/3} + m + n\right)$.
- If $\Gamma$ is a set of arbitrary circles, then we can set $s = 3$ and $t = 2$. In this case we obtain the bound $I(\mathcal{P},\Gamma) = O\left(m^{3/5}n^{4/5} + m + n\right)$.

We rely on Theorem 2.5 to derive the following result of Elekes, Nathanson, and Ruzsa [3]. Given a set $A \subset \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$, we write $f(A) = \{f(a) : a \in A\}$.

**Theorem 2.6.** *Let $A$ be a finite set of real numbers and let $f$ be a strictly convex or strictly concave function. Then*

$$|A + A| \cdot |f(A) + f(A)| = \Omega\left(|A|^{5/2}\right).$$

The theorem states that it is impossible for a set to have a small sum set and to still have a small sum set after applying to it a strictly convex/concave function.

*Proof of Theorem 2.6.* We consider the planar point set

$$\mathcal{P} = (A + A) \times (f(A) + f(A)),$$

and the set of planar curves

$$\Gamma = \{y = f(x - b) + c : \ b \in A \text{ and } c \in f(A)\}.$$

Notice that $\Gamma$ is a set of $|A|^2$ translations of the curve $y = f(x)$, which is either strictly convex or strictly concave. Any two such translations intersect in at most one point. Thus, we may apply Theorem 2.5 with $s = 2$ and $t = 2$, to obtain

$$I(\mathcal{P},\Gamma) = O\left(|\mathcal{P}|^{2/3}|\Gamma|^{2/3} + |\mathcal{P}| + \Gamma\right) = O\left(|\mathcal{P}|^{2/3}|A|^{4/3} + |\mathcal{P}| + |A|^2\right).$$

On the other hand, a curve of $\Gamma$ that is defined by $y = f(x-b)+c$ contains the point $(a+b, f(a)+c) \in \mathcal{P}$ for every $a \in A$. Thus, we have

$$I(\mathcal{P}, \Gamma) \geq |A| \cdot |\Gamma| = |A|^3.$$

Combining the two above bounds for $I(\mathcal{P}, \Gamma)$ implies

$$|A|^3 = O\left(|\mathcal{P}|^{2/3}|A|^{4/3} + |\mathcal{P}| + |A|^2\right),$$

or $|\mathcal{P}| = \Omega\left(|A|^{5/2}\right)$. This completes the proof of the theorem, since $|\mathcal{P}| = |A+A| \cdot |f(A)+f(A)|$. □

# 3 Solymosi's bound

In this section we derive the second sum-product bound of Solymosi, which introduced a novel and elegant geometric approach for the sum-product problem.

**Theorem 3.1 (Solymosi [11]).** *Every finite set $A$ of positive real numbers satisfies* $\max\{|A+A|, |AA|\} \geq \frac{|A|^{4/3}}{2\lg^{1/3}|A|}$.

While Theorem 3.1 has some restrictions on the set $A$, it immediately implies a similar bound for every set $A$.

**Corollary 3.2.** *Every finite set $A \subset \mathbb{R}$ satisfies* $\max\{|A+A|, |AA|\} = \Omega\left(\frac{|A|^{4/3}}{\lg^{1/3}|A|}\right)$.

*Proof.* Let $A' = A \setminus \{0\}$. If at least half of the elements of $A'$ are positive, let $A''$ be the set of positive elements of $A'$. Otherwise, let $A''$ be the set of absolute values of the negative elements of $A'$. Either way, $A''$ is a set of at least $\frac{|A|-1}{2}$ positive real numbers. The corollary is immediately obtained by applying Theorem 3.1 on $A''$ and noting that $\max\{|A+A|, |AA|\} \geq \max\{|A''+A''|, |A''A''|\}$. □

Before proving Theorem 3.1, we point out another property of multiplicative energy. Similarly to $r_A^{\times}(x)$, for any $\lambda \in A/A$ we set $r_A^{/}(\lambda) = |\{(a_1, a_2) \in A^2 : a_1/a_2 = \lambda\}|$. As with $r_A^{\times}(x)$, we have $\sum_{\lambda \in A/A} r_A^{/}(\lambda) = |A|^2$ since every pair $(a,b) \in A^2$ contributes to exactly one of the $r_A^{/}(\lambda)$. By inspecting the definition of multiplicative energy that was given in Section 2.1, we notice that

$$E^{\times}(A) = \left|\left\{(a_1, a_2, a_3, a_4) \in A^4 : (a_1, a_3) = (\lambda a_4, \lambda a_2) \text{ with } \lambda \in A/A\right\}\right|.$$

Thus, as in (3) we have

$$E^\times(A) = \sum_{\lambda \in A/A} r'_A(\lambda)^2 \geq \frac{\left(\sum_{\lambda\in A/A} r'_A(\lambda)\right)^2}{|A/A|} = \frac{|A|^4}{|A/A|}. \tag{4}$$

*Proof of Theorem 3.1.* The proof is based on double counting $E^\times(A)$. We use the lower bound for $E^\times(A)$ that was given in (3), so it remains to derive an upper bound.

We perform a dyadic decomposition on the sum in (4):

$$E^\times(A) = \sum_{j=0}^{\lg|A|-1} \sum_{\substack{\lambda\in A/A \\ 2^j \leq r'_A(\lambda) < 2^{j+1}}} r'_A(\lambda)^2.$$

This implies that there exists $0 \leq j < \lg|A|$ with

$$\sum_{\substack{\lambda\in A/A \\ 2^j \leq r'_A(\lambda) < 2^{j+1}}} r'_A(\lambda)^2 \geq \frac{E^\times(A)}{\lg|A|}. \tag{5}$$

We set $\Lambda = \{\lambda \in A/A : 2^j \leq r'_A(\lambda) < 2^{j+1}\}$ and write $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \cdots\}$ so that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Notice that $|\Lambda| > \frac{E^\times(A)}{2^{2j+2}\lg|A|}$. This concludes the pruning step of the proof, in which we found a subset of the elements of $A/A$ that have approximately the same number of divisors and correspond to a large part of $E^\times(A)$.

We consider the planar point set $\mathcal{P} = A \times A \subset \mathbb{R}^2$, and double count $|\mathcal{P}+\mathcal{P}|$. Since $\mathcal{P}+\mathcal{P} = (A+A)\times(A+A)$, we immediately obtain the bound $|\mathcal{P}+\mathcal{P}| = |A+A|^2$. For $1 \leq i \leq |\Lambda|$, let $\ell_i$ denote the line in $\mathbb{R}^2$ that is defined by $y = \lambda_i x$. That is, we have a set of $|\Lambda|$ lines that are incident to the origin, each containing between $2^j$ and $2^{j+1}-1$ points of $\mathcal{P}$; an example is depicted in Figure 3(a).

Consider two lines $\ell_k$ and $\ell_{k+1}$ (with consecutive slopes in $\Gamma$), and notice that every point of $(\mathcal{P}\cap\ell_k)+(\mathcal{P}\cap\ell_{k+1})$ lies in the wedge between $\ell_k$ and $\ell_{k+1}$; an example is depicted in Figure 3(b). Thus, for every $k \neq k'$ we have

$$\big((\mathcal{P}\cap\ell_k)+(\mathcal{P}\cap\ell_{k+1})\big) \bigcap \big((\mathcal{P}\cap\ell_{k'})+(\mathcal{P}\cap\ell_{k'+1})\big) = \emptyset.$$

If $a_1, a_2 \in \ell_k$ and $a_3, a_4 \in \ell_{k+1}$, then $a_1+a_3 \neq a_2+a_4$ (since $(a, a\cdot\lambda_k)+(b, b\cdot\lambda_{k+1}) = (p_x, p_y)$ has a unique solution). This implies

$$|(\mathcal{P}\cap\ell_k)+(\mathcal{P}\cap\ell_{k+1})| = |\mathcal{P}\cap\ell_k|\cdot|\mathcal{P}\cap\ell_{k+1}|.$$

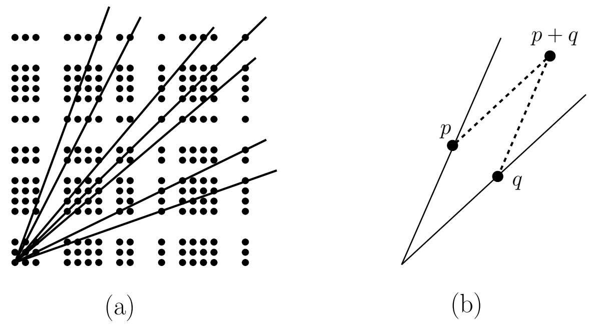


Figure 3: (a) A lattice $\mathcal{P}$ and lines with slopes in $\Lambda$. (b) Every point of $(\mathcal{P} \cap \ell_i) + (\mathcal{P} \cap \ell_{i+1})$ lies in the wedge between $\ell_i$ and $\ell_{i+1}$.

By combining the two observations above we obtain

$$|\mathcal{P}+\mathcal{P}| > \sum_{k=1}^{|\Lambda|-1} \left|(\mathcal{P}\cap \ell_k) + (\mathcal{P}\cap \ell_{k+1})\right| = \sum_{k=1}^{|\Lambda|-1} |\mathcal{P}\cap \ell_k| \cdot |\mathcal{P}\cap \ell_{k+1}| \geq (|\Lambda|-1)2^{2j}.$$

By rephrasing $|\Lambda| > \frac{E^{\times}(A)}{2^{2j+2}\lg|A|}$ as $1 > \frac{E^{\times}(A)}{|\Lambda|2^{2j+2}\lg|A|}$, we get

$$|\mathcal{P}+\mathcal{P}| > (|\Lambda|-1)2^{2j}\frac{E^{\times}(A)}{|\Lambda|2^{2j+2}\lg|A|} > \frac{E^{\times}(A)}{8\lg|A|}.$$

By combining this with the aforementioned bound $|\mathcal{P}+\mathcal{P}| = |A+A|^2$, we get

$$E^{\times}(A) < 8|A+A|^2 \lg|A|.$$

Finally, combining this with (3) implies

$$|A+A|^2|AA| > \frac{|A|^4}{8\lg|A|}. \tag{6}$$

By inspecting (6), we notice that if $|AA| < \frac{|A|^{4/3}}{2\lg^{1/3}|A|}$ then $|A+A| > \frac{|A|^{4/3}}{2\lg^{1/3}|A|}$. A symmetric argument applies for the case of $|A+A| \leq \frac{|A|^{4/3}}{2\lg^{1/3}|A|}$, and completes the proof. □

Notice that the above proof connects the sum set and product set of $A$ by considering the multiplicative energy of $A$. The connection between $|AA|$ and $E^{\times}(A)$ is straightforward. The connection between $|A+A|$ and $E^{\times}(A)$ is based on a geometric argument concerning the lattice $(A+A)\times(A+A)$ and lines with slopes in $|A/A|$.

Theorem 3.1 is obtained by deriving (6). Unfortunately, this equation is asymptotically tight up to polylogarithmic factors. By taking $A$ to be an arithmetic progression we have $|A+A| = \Theta(|A|)$ and $|AA| = \Theta(|A|^2 \lg^\beta |A|)$ with $\beta \approx -0.086$ (this result is attributed to Erdős in [4]).

The following subsection presents another application of Solymosi's technique. In this case the technique gives a tight bound.

## 3.1 The size of $\frac{A+A}{A+A}$

We now study the more involved expression $\frac{A+A}{A+A}$, where $A$ consists of positive real numbers (as before, we can generalize the result to any set of real numbers at the cost of losing a constant factor). Specifically, we are interested in how small can $\frac{A+A}{A+A}$ be with respect to $|A|$.

As a first example, consider the case $A = \{1,2,3\}$ and notice that

$$\frac{A+A}{A+A} = \left\{1,2,3,\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{1}{3},\frac{2}{3},\frac{4}{3},\frac{5}{3},\frac{3}{4},\frac{5}{4},\frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{6}{5},\frac{5}{6}\right\}.$$

In this case we have $\left|\frac{A+A}{A+A}\right| = 17 = 2|A|^2 - 1$. Similarly, there exist arbitrary large values of $n$ such that $A = \{1,2,3,\cdots,n\}$ satisfies $\left|\frac{A+A}{A+A}\right| < \frac{24}{\pi^2}|A|^2$. The following theorem shows that the above bound is tight.

**Theorem 3.3 (Balog and Roche-Newton [1]).** *Let $A$ be a finite set of positive real numbers. Then $\left|\frac{A+A}{A+A}\right| \geq 2|A|^2 - 1$.*

*Proof.* We set $\mathcal{P} = A \times A$ and let $\mathcal{L}$ be the set of lines that are incident to the origin and to at least one point of $\mathcal{P}$. We write $\mathcal{L} = \{\ell_1, \ell_2, \ldots, \ell_{|A/A|}\}$ so that the lines are ordered by increasing slopes. That is, if $m_j$ is the slope of $\ell_j$ then $m_1 < m_2 < \cdots < m_{|A/A|}$. Since every point of $\mathcal{P}$ is contained in exactly one line of $\mathcal{L}$, we have

$$\sum_{j=1}^{|A/A|} |\ell_j \cap P| = |A|^2. \tag{7}$$

Similarly, consider the point set $\mathcal{P}' = (A+A) \times (A+A)$ and let $\mathcal{L}'$ denote the set of lines that are incident to the origin and to at least one point of $\mathcal{P}'$. Notice that $|\mathcal{L}'| = \left|\frac{A+A}{A+A}\right|$, so it suffices to prove that $|\mathcal{L}'| \geq 2|A|^2 - 1$. Given a point $p = (p_x, p_y) \in \mathbb{R}^2 \setminus \{0\}$, we set $R(p) = p_y/p_x$. That is, $R(p)$ is the slope of the line that is incident to the origin and to $p$. To complete the proof, we will show that the points of $\mathcal{P}'$ determine at least $2|A|^2 - 1$ such slopes.

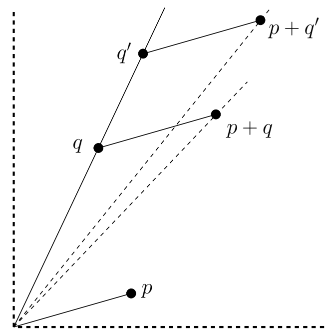


Figure 4: If $p \in \ell_j$, $q, q' \in \ell_{j+1}$, and $q_x < q'_x$, then $R(p+q) < R(p+q')$.

For some $1 \leq j < |A/A|$ consider the lines $\ell_j$ and $\ell_{j+1}$. As in the proof of Theorem 3.1, we notice that for $p \in \ell_j$ and $q \in \ell_{j+1}$ the point $p+q$ lies in the wedge that is bound by $\ell_j$ and $\ell_{j+1}$. If $q'$ is a point on $\ell_{j+1}$ with an $x$-coordinate larger than that of $q$, then $R(p+q) < R(p+q')$. We do not prove this straightforward property, which is illustrated in Figure 4. Similarly, if $p'$ is a point on $\ell_j$ with an $x$-coordinate larger than that of $p$, then $R(p+q) > R(p'+q)$.

Let $n_j = |\ell_j \cap \mathcal{P}|$ and $n_{j+1} = |\ell_{j+1} \cap \mathcal{P}|$. We denote the points of $\ell_j \cap \mathcal{P}$ as $p_1, \ldots, p_{n_j}$ and the points of $\ell_{j+1} \cap \mathcal{P}$ as $q_1, \ldots, q_{n_{j+1}}$, both ordered by increasing $x$-coordinates. Notice that

$$R(p_1 + q_{n_{j+1}}) > R(p_1 + q_{n_{j+1}-1}) > \cdots > R(p_1 + q_1) > R(p_2 + q_1) > \cdots > R(p_{n_j} + q_1).$$

Since $\mathcal{P}' = \mathcal{P} + \mathcal{P}$, the above implies that there are at least $n_j + n_{j+1} - 1$ distinct slopes of the form $R(p)$ with $p \in \mathcal{P}'$ and $m_j < R(p) < m_{j+1}$. By summing this over every $1 \leq j < |A/A|$ and recalling (7), we obtain

$$\left|\frac{A+A}{A+A}\right| \geq \sum_{j=1}^{|A/A|-1} (n_j + n_{j+1} - 1) = 2\left(\sum_{j=1}^{|A/A|} n_j\right) - n_1 - n_{|A/A|} - (|A/A| - 1)$$
$$= 2|A|^2 - n_1 - n_{|A/A|} - (|A/A| - 1).$$

Our next observation is that $\ell_1$ contains a single point of $\mathcal{P}$: the point whose $y$-coordinate is the minimum element of $A$ and whose $x$-coordinate is the maximum element of $A$. That is, $n_1 = 1$. By a symmetric argument we get that $n_{|A/A|} = 1$. We thus have $\left|\frac{A+A}{A+A}\right| \geq 2|A|^2 - |A/A| - 1$.

Finally, notice that for every $1 \leq j \leq |A/A|$ there exists a point $p \in \mathcal{P}'$ with $R(p) = m_j$. Indeed, for an arbitrary $q \in \mathcal{P} \cap \ell_j$ we can take $p = q + q$. Thus, there are at least $|A/A|$ elements of $\frac{A+A}{A+A}$ that we did not count above. This implies $\left|\frac{A+A}{A+A}\right| \geq 2|A|^2 - 1$, as asserted. $\square$

# References


[1] A. Balog and O. Roche-Newton, New sum-product estimates for real and complex numbers, *Discrete Comput. Geom.* **53** (2015), 825–846.

[2] G. Elekes, On the number of sums and products, *Acta Arith.* **81** (1997), 365–367.

[3] G. Elekes, M. B. Nathanson, and I. Z. Ruzsa, Convexity and sumsets, *Journal of Number Theory* **83** (2000), 194–201.

[4] G. Elekes and I. Z. Ruzsa, Few sums, many products, *Studia Scientiarum Mathematicarum Hungarica* **40** (2003), 301–308.

[5] P. Erdős and E. Szemerédi, On sums and products of integers, *Studies in pure mathematics, To the memory of Paul Turn,* 213–218, Birkhäuser, Basel-Boston, Mass., 1983.

[6] K. Ford, Sums and products from a finite set of real numbers, *The Ramanujan Journal* **2** (1998), 59–66.

[7] S. Konyagin and I. D. Shkredov, New results on sum-products in $\mathbb{R}$, arXiv:1602.03473.

[8] M. Nathanson, On sums and products of integers, *Proceedings of the American Mathematical Society* **125** (1997), 9–16.

[9] J. Pach and M. Sharir, Repeated angles in the plane and related problems, *Journal of Combinatorial Theory A* **59** (1992), 12–22.

[10] J. Solymosi, On the number of sums and products, *Bulletin of the London Mathematical Society* **37** (2005), 491–494.

[11] J. Solymosi, Bounding multiplicative energy by the sumset, *Advances in mathematics* **222** (2009), 402–408.

[12] E. Szemerédi and W. T. Trotter, Extremal problems in discrete geometry, *Combinatorica* **3** (1983), 381–392.