Chapter 4: Arithmetic Progressions in Dense Sets

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1 Introduction

In this chapter we discuss how dense can a set be without containing an arithmetic progression of some length. This discussion leads us to studying the Fourier transform. We begin by mentioning a well-known result, without proof.

Theorem 1.1 (Szemerédi's theorem). *Given $\varepsilon > 0$ and a positive integer k, there exists a sufficiently large n_0 that satisfies: For any $n \geq n_0$ and any subset $A \subset \{1, \ldots, n\}$ with $|A| \geq \varepsilon n$, there exists a k-term arithmetic progression in A.*

In this chapter we focus on the case of $k = 3$. As a first example, it is easy to notice that in any subset $A \subset \{1, \ldots, n\}$ with $|A| \geq \lceil 2n/3 \rceil$ there exists a 3-term arithmetic progression in A. Indeed, partition $\{1, \ldots, n\}$ into triples of consecutive numbers $(1, 2, 3)$, $(4, 5, 6)$, and so on (possibly with one or two elements without a triple at the end), and notice that a set without a 3-term arithmetic progression can have at most two elements out of each triple. In Section 5 we prove that every set $A \subset \{1, \ldots, n\}$ with $|A| \geq \frac{cn}{\lg \lg n}$ (for some constant c) contains a 3-term arithmetic progression. This result is known as *Roth's theorem* and is one of the two theorem for which Roth received his fields medal. Before that, in Section 4 we consider a somewhat simpler variant of the problem in $\mathbb{F}_3^n$.

After observing Theorem 1.1, one natural question that arises is: What happens when $|A|$ is of a smaller density? The following theorem shows that the theorem is false for a somewhat smaller density.

Theorem 1.2 (Behrend [3]). *For every sufficiently large n, there exists a set $A \subset \{1, \ldots, n\}$ with $|A| \geq n2^{-c\sqrt{\lg n}}$ that contains no 3-term arithmetic progression (for some small constant $c > 1$).*

We prove Theorem 1.2 in Section 2. It is not known whether this result is tight (for more details, see Section 5 below). The following is another main problem concerning dense sets that contain large arithmetic progressions.

Conjecture 1.3 (Erdős and Turan [6]). *Let $A \subset \mathbb{N}$ satisfy $\sum_{x \in A} \frac{1}{x} = \infty$. Then A contains arbitrarily long arithmetic progressions.*

Erdős offered 3000$ for proving or disproving this conjecture — one of the largest prizes that Erdős ever offered. It is known that the set $\mathbb{P}$ of prime numbers satisfies $\sum_{p \in \mathbb{P}} \frac{1}{p} = \infty$. Green and Tao [7] proved that $\mathbb{P}$ indeed contains infinitely long arithmetic progressions.

2 Behrend's Construction

In this section we prove Theorem 1.2. Before presenting the proof, we first introduce the concept of Freiman homomorphisms.

Let $\mathbb{F}$ be a field, let m, n, and t be positive integers, let $A \subset \mathbb{F}^n$, and consider a function $\tau : \mathbb{F}^n \to \mathbb{F}^m$. We say that τ is a *Freiman t-homomorphism* of A if $\tau(a_1)+\tau(a_2)+\cdots+\tau(a_t) = \tau(b_1)+\tau(b_2)+\cdots+\tau(b_t)$ (where $a_1,\ldots,a_k,b_1,\ldots,b_k \in A$) implies $a_1 + a_2 + \cdots + a_t = b_1 + b_2 + \cdots + b_t$.

For example, let $A \subset \mathbb{F}_5^n$ and let $\tau : \mathbb{F}_5^n \to \mathbb{Z}$ be defined as $\tau(a_1, a_2, \ldots, a_n) = \sum_{j=1}^n a_j 10^{j-1}$. That is, the j'th decimal digit of $\tau(a_1, a_2, \ldots, a_n)$ is determined only by a_j. In this case τ is a Freiman 2-homomorphism of A. Indeed, assume that $\tau(a) + \tau(b) = \tau(c) + \tau(d)$ for $a, b, c, d \in \mathbb{F}_5^n$, and notice that the j'th decimal digit of $\tau(a)+\tau(b)$ is a_j+b_j. Since this holds for every j, we obtain that $a+b = c+d$. Similarly, let $A \subset \mathbb{F}_4^n$ and let $\tau' : \mathbb{F}_4^n \to \mathbb{Z}$ be defined as $\tau'(a_1, a_2, \ldots, a_n) = \sum_{j=1}^n a_j 10^{j-1}$. Then τ' is a Freiman 3-homomorphism of A.

Proof of Theorem 1.2. We define a *hypersphere* in $\mathbb{R}^d$ as a the set of points that are at a given distance r from a given point $a = (a_1, a_2, \ldots, a_d)$. That is, a hypersphere is defined by an equation of the form $(x_1 - a_1)^2 + \cdots + (x_d - a_d)^2 = r^2$. A line in $\mathbb{R}^d$ is defined by a point $a \in \mathbb{R}^d$ and a direction $v \in \mathbb{R}^d \setminus \{0\}$, as $\{a + cv : c \in \mathbb{R}\}$. The proof is based on the observation that a line in $\mathbb{R}^d$ intersects a hypersphere in at most two points.[1]

[1] Asking for a point $(p_1, \ldots, p_d)$ to lie on a line and on a hypersphere corresponds to $d - 1$ independent linear equations (with variables $p_1, \ldots, p_d$) and one quadratic equation. Such a system has at most two solutions.

For positive integers m and d that will be set below, we consider an $m \times \cdots \times m$ section of the integer lattice $\mathcal{L} = \{(a_1, \ldots, a_d) \in \mathbb{Z}^d : 0 \le a_j \le m-1\}$. For a positive integer r we define the hypersphere $S_r = \{(a_1, \ldots, a_d) \in \mathbb{R}^d : a_1^2 + \cdots + a_d^2 = r\}$. Notice that S_r is a hypersphere of radius $\sqrt{r}$ that is centered at the origin. Our next observation is that every point of $\mathcal{L}$ is contained in one of the spheres $S_1, S_2, \ldots, S_{dm^2}$. Since $|\mathcal{L}| = m^d$, at least one of these hyperspheres contains at least m^{d-2}/d points of $\mathcal{L}$. We pick an arbitrary hypersphere with this property and denote it as S. We then define the finite point set $\mathcal{P} = S \cap \mathcal{L}$; notice that $|\mathcal{P}| \ge m^{d-2}/d$. Since $\mathcal{P}$ is fully contained in the hypersphere S, every line contains at most two points of $\mathcal{P}$.

We consider the projection $\tau : \mathbb{R}^d \to \mathbb{R}$ that is defined as

$$\tau(x_1, \ldots, x_d) = \sum_{j=1}^{d} x_j (2m)^{j-1}.$$

Let $\mathcal{P}' = \tau(\mathcal{P}) = \{\tau(p) : p \in \mathcal{P}\}$. Since every coordinates of every points of $\mathcal{P}$ is at most $m-1$, the map τ is Freiman 2-homomorphism. It is also a bijection between $\mathcal{P}$ and $\mathcal{P}'$, which implies $|\mathcal{P}'| = |\mathcal{P}| \ge m^{d-2}/d$.

Assume for contradiction that $\mathcal{P}'$ contains a 3-term arithmetic progression p', q', r'; that is, $p' + r' = 2q'$. Set $p = \tau^{-1}(p')$, $q = \tau^{-1}(q')$, and $r = \tau^{-1}(r')$. Since τ is a Freiman 2-homomorphism, we have $p + r = 2q$. This in turn implies that p, q, and r are collinear, contradicting the fact that no line contains three points of $\mathcal{P}$. Thus, the set $\mathcal{P}' \subset \mathbb{Z}$ contains no 3-term arithmetic progression. We set $A = P' \setminus \{0\}$. Since $\mathcal{P}' \subset \{0, 1, 2, \ldots, (2m)^d\}$, we set $n = (2m)^d$. To conclude the proof, it remains to derive a lower bound for $|\mathcal{P}'|$.

We set $d = \sqrt{\lg_2 n}$ and $m = 2^{d-1}$, which implies $n = 2^{d^2} = (2m)^d$. This in turn implies

$$|\mathcal{P}'| \ge \frac{m^{d-2}}{d} = \frac{n}{d 2^d m^2} \ge n \cdot 2^{-c\sqrt{\lg n}},$$

for some constant c and sufficiently large n. □

3 The Fourier transform

To prove more advanced results concerning density and arithmetic progressions, we require the Fourier transform. This section contains a basic introduction for this tool. For now, we only work over $\mathbb{F}_p^n$, where n is a positive integer and p is a prime.

We denote the set of p'th roots of unity in $\mathbb{C}$ as $S_p = \{e^{2k\pi i/p} : 0 \le k < p\}$. By the formula for a geometric sum, we have

$$\sum_{s\in S_p} s = \sum_{k=0}^{p-1} e^{2k\pi i/p} = \frac{1-e^{2\pi i}}{1-e^{2\pi i/p}} = 0. \tag{1}$$

The *characters* of $\mathbb{F}_p^n$ are the homomorphisms[2] from $\mathbb{F}_p^n$ to S_p. For any $\alpha \in \mathbb{F}_p^n$ we have the character

$$\chi_\alpha(x) = e^{2\pi i(x\cdot\alpha)/p},$$

where $x\cdot\alpha$ is the standard inner product in $\mathbb{F}_p^n$. The following claim presents several basic properties of the characters χ_α.

Claim 3.1. *(i) For any $\alpha, x, y \in \mathbb{F}_p^n$, we have $\chi_\alpha(x+y) = \chi_\alpha(x)\chi_\alpha(y)$.*
(ii) For any $\alpha, \beta, x \in \mathbb{F}_p^n$, we have $\chi_{\alpha+\beta}(x) = \chi_\alpha(x)\chi_\beta(x)$.
(iii) For any $\alpha \in \mathbb{F}_p^n \setminus \{0\}$, we have $\sum_{x\in\mathbb{F}_p^n} \chi_\alpha(x) = 0$.

Proof. For (i), notice that

$$\chi_\alpha(x+y) = e^{2\pi i((x+y)\cdot\alpha)/p} = e^{2\pi i(x\cdot\alpha)/p} e^{2\pi i(y\cdot\alpha)/p} = \chi_\alpha(x)\chi_\alpha(y).$$

For (ii), we have

$$\chi_{\alpha+\beta}(x) = e^{2\pi i(x\cdot(\alpha+\beta))/p} = e^{2\pi i(x\cdot\alpha)/p} e^{2\pi i(x\cdot\beta)/p} = \chi_\alpha(x)\chi_\beta(x).$$

For (iii), we assume without loss of generality that the n'th coordinate of α is non-zero. We partition the elements of $\mathbb{F}_p^n$ into p^{n-1} subset, each consisting of p elements. Specifically, for every $y \in \mathbb{F}_{p-1}^n$ we consider together the p elements of $\mathbb{F}_p^n$ that can be obtained by adding to y an n'th coordinate. That is, we consider the set $S_y = \{(y,0),(y,1),\ldots,(y,p-1)\} \subset \mathbb{F}_p^n$.

Given a specific $y \in \mathbb{F}_{p-1}^n$, we set $c = \alpha\cdot(y,0)$. For $\alpha \in \mathbb{F}_p^n$, let α_n be the n'th coordinate of α. We get that

$$\sum_{x\in S_y} \chi_\alpha(x) = \sum_{x\in S_y} e^{2\pi i(x\cdot\alpha)/p} = \sum_{k=0}^{p-1} e^{2\pi i(c+k\cdot\alpha_n)/p} = e^{2\pi ic}\sum_{k=0}^{p-1} e^{2\pi i(k\cdot\alpha_n)/p} = 0.$$

The last step holds by (1), since we sum up the p'th roots of unity. Indeed, since $\mathbb{F}_p$ is a group under multiplication, we have that $\{k\cdot\alpha_n : 0\le k\le p-1\} = \{0,1,2,\ldots,p-1\}$. Part (iii) of the claim is obtained by summing this up over every $y \in \mathbb{F}_{p-1}^n$. □

[2] Recall that a function $f : \mathbb{F}_p^n \to \mathbb{C}$ is a homomorphism if for every $a, b \in \mathbb{F}_p^n$ we have $f(a)f(b) = f(ab)$.

Consider a function $f : \mathbb{F}_p^n \to \mathbb{C}$. The *Fourier coefficient* of f with respect to $\alpha \in \mathbb{F}_p^n$ is defined as

$$\hat{f}(\alpha) = p^{-n} \sum_{x \in \mathbb{F}_p^n} f(x)\overline{\chi_\alpha(x)}.$$

These are called coefficients since we can use them to write any function $f : \mathbb{F}_p^n \to \mathbb{C}$ as a linear combination of the characters χ_α.

Claim 3.2. *For every $x \in \mathbb{F}_p^n$, we have $f(x) = \sum_{\alpha \in \mathbb{F}_p^n} \hat{f}(\alpha)\chi_\alpha(x)$.*

Proof. By the definition of $\hat{f}(\alpha)$ we have

$$\sum_{\alpha \in \mathbb{F}_p^n} \hat{f}(\alpha)\chi_\alpha(x) = \sum_{\alpha \in \mathbb{F}_p^n} \chi_\alpha(x) p^{-n} \sum_{y \in \mathbb{F}_p^n} f(y)\overline{\chi_\alpha(y)} = p^{-n} \sum_{y \in \mathbb{F}_p^n} f(y) \sum_{\alpha \in \mathbb{F}_p^n} e^{2\pi i((x-y)\cdot\alpha)/p}.$$

By part (iii) of Claim 3.1, for every $x \neq y$ we have $\sum_{\alpha \in \mathbb{F}_p^n} e^{2\pi i((x-y)\cdot\alpha)/p} = 0$. This implies

$$\sum_{\alpha \in \mathbb{F}_p^n} \hat{f}(\alpha)\chi_\alpha(x) = p^{-n} f(x) \sum_{\alpha \in \mathbb{F}_p^n} 1 = f(x).$$

□

The function $\hat{f}$ is called the *Fourier transform* of f, and the formula $f(x) = \sum_{\alpha \in \mathbb{F}_p^n} \hat{f}(\alpha)\chi_\alpha(x)$ is called the *Fourier inversion* of f. As a first example, consider the function $f : \mathbb{F}_2^n \to \mathbb{C}$ defined as $f(x) = e^{\pi i x \cdot x}$. Notice that $f(x) = 1$ if x consists of an even number of 1's, and otherwise $f(x) = e^{\pi i} = -1$. That is, $f(x) = (-1)^{|x|}$ (where $|x|$ is the sum of the coordinates of x). For every $\alpha \in \mathbb{F}_2^n$, we have

$$\hat{f}(\alpha) = 2^{-n} \sum_{x \in \mathbb{F}_2^n} f(x)\overline{\chi_\alpha(x)} = 2^{-n} \sum_{x \in \mathbb{F}_2^n} e^{\pi i x \cdot x} e^{-\pi i x \cdot \alpha} = 2^{-n} \sum_{x \in \mathbb{F}_2^n} (-1)^{|x|} e^{-\pi i x \cdot \alpha}.$$

Assume that the j'th coordinate of α is zero. If $x, x' \in \mathbb{F}_2^n$ differ only in their j'th coordinate, then $(-1)^{|x|}e^{-\pi i x \cdot \alpha} + (-1)^{|x'|}e^{-\pi i x' \cdot \alpha} = 0$. That is, by pairing up every element $x \in \mathbb{F}_2^n$ with another element that differs from x only in the j'th coordinate, we obtain that $\sum_{x \in \mathbb{F}_2^n} (-1)^{|x|}e^{-\pi i x \cdot \alpha} = 0$. This implies that $\hat{f}(\alpha) = 0$ for any α that contains at least one zero coordinate. Let 1_n denote the all 1's element of $\mathbb{F}_2^n$. By Claim 3.2 we obtain

$$\begin{aligned} f(x) &= \sum_{\alpha \in \mathbb{F}_2^n} \hat{f}(\alpha)\chi_\alpha(x) = \chi_{1_n}(x) \cdot \hat{f}(1_n) = e^{\pi i x \cdot 1_n} \cdot 2^{-n} \sum_{x \in \mathbb{F}_2^n} f(x)\overline{\chi_{1_n}(x)} \\ &= (-1)^{|x|} \cdot 2^{-n} \sum_{x \in \mathbb{F}_2^n} (-1)^{|x|} \cdot (-1)^{|x|} = (-1)^{|x|}. \end{aligned}$$

That is, the expression in Claim 3.2 indeed gives the correct function. The following claim presents several basic properties of the Fourier transform.

Claim 3.3. *(i) For any $f, g : \mathbb{F}_p^n \to \mathbb{C}$ we have $\widehat{f+g} = \hat{f} + \hat{g}$.*
(ii) For any $f : \mathbb{F}_p^n \to \mathbb{C}$ and $c \in \mathbb{C}$ we have $\widehat{cf} = c\hat{f}$.
(iii) For $f : \mathbb{F}_p^n \to \mathbb{C}$ and $y \in \mathbb{F}_p^n$ we set $g(x) = f(x-y)$. Under this notation $\hat{g}(\alpha) = e^{-2\pi i(\alpha \cdot y)/p} \cdot \hat{f}(\alpha)$.
(iv) **(Parseval's theorem).** *For any $f : \mathbb{F}_p^n \to \mathbb{C}$, we have*

$$\sum_{\alpha \in \mathbb{F}_p^n} |\hat{f}(\alpha)|^2 = p^{-n} \sum_{x \in \mathbb{F}_p^n} |f(x)|^2.$$

Proof. For (i), notice that

$$\begin{aligned}\widehat{f+g}(\alpha) &= p^{-n} \sum_{x \in \mathbb{F}_p^n} (f+g)(x)\overline{\chi_\alpha(x)} \\ &= p^{-n} \sum_{x \in \mathbb{F}_p^n} f(x)\overline{\chi_\alpha(x)} + p^{-n} \sum_{x \in \mathbb{F}_p^n} g(x)\overline{\chi_\alpha(x)} = \hat{f}(\alpha) + \hat{g}(\alpha).\end{aligned}$$

Similarly, for (ii) we have

$$\widehat{cf} = p^{-n} \sum_{x \in \mathbb{F}_p^n} (cf)(x)\overline{\chi_\alpha(x)} = cp^{-n} \sum_{x \in \mathbb{F}_p^n} f(x)\overline{\chi_\alpha(x)} = c\hat{f}(\alpha).$$

To obtain (iii) we set $z = x - y$. Claim 3.1 implies

$$\begin{aligned}\hat{g}(\alpha) &= p^{-n} \sum_{x \in \mathbb{F}_p^n} f(x-y)\overline{\chi_\alpha(x)} = p^{-n} \sum_{z \in \mathbb{F}_p^n} f(z)\overline{\chi_\alpha(z+y)} \\ &= p^{-n} \sum_{z \in \mathbb{F}_p^n} f(z)\overline{\chi_\alpha(z)\chi_\alpha(y)} = e^{-2\pi i(\alpha \cdot y)/p} \cdot \hat{f}(\alpha).\end{aligned}$$

For (iv), we recall that $|a|^2 = a\overline{a}$ for any $a \in \mathbb{C}$. By combining this property with the definition of $\hat{f}$ and then with part (iii) of Claim 3.1, we obtain

$$\begin{aligned}\sum_{\alpha \in \mathbb{F}_p^n} |\hat{f}(\alpha)|^2 = \sum_{\alpha \in \mathbb{F}_p^n} \hat{f}(\alpha)\overline{\hat{f}(\alpha)} &= \sum_{\alpha \in \mathbb{F}_p^n} \left(p^{-n} \sum_{x \in \mathbb{F}_p^n} f(x)\overline{\chi_\alpha(x)} \right) \left(p^{-n} \sum_{y \in \mathbb{F}_p^n} \overline{f(y)}\chi_\alpha(y) \right) \\ = p^{-2n} \sum_{x,y \in \mathbb{F}_p^n} f(x)\overline{f(y)} \sum_{\alpha \in \mathbb{F}_p^n} e^{2\pi(\alpha \cdot (y-x))/p} &= p^{-n} \sum_{x \in \mathbb{F}_p^n} f(x)\overline{f(x)} = p^{-n} \sum_{x \in \mathbb{F}_p^n} |f(x)|^2.\end{aligned}$$

□

For more details about the Fourier transform, see for example [11].

4 Meshulam's theorem

We are now ready for our first use of the Fourier transform. Consider a set $A \subset \mathbb{F}_3^n$ for some large integer n. In this case, we say that A contains a 3-term arithmetic progression if there exist $a, b \in \mathbb{F}_3^n$ such that $b \neq 0$ and

$$\{a, a+b, a+2b\} \subset A.$$

Such a triple of points can also be thought of as a line that is defined by $n-1$ independent linear equations in $x_1, \ldots, x_n$.

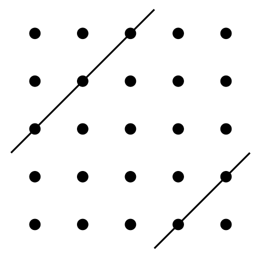

Figure 1: The "line" that is defined by $y = x + 2$ in $\mathbb{F}_5^2$.

This may be a good place to point out that lines behave somewhat differently in $\mathbb{F}_p^n$. For example, consider $\mathbb{F}_5^2$ as a 5×5 lattice in the plane. Figure 1 depicts the "line" that is defined by $y = x + 2$ in this plane. In $\mathbb{F}_3^n$ a linear equation defines a "hyperplane" that contains a third of the points of $\mathbb{F}_3^n$ and may consist of many distinct "connected components".

How large can a set $A \subset \mathbb{F}_3^n$ be without containing a 3-term arithmetic progression? A straightforward example of such a set is $A = \{0,1\}^n$. This set obviously contains no 3-term progression and is of size 2^n. Edel [5] derived sets of size 2.2174^n with no 3-term progression, which is currently the largest known lower bound. We now present an upper bound proof by Meshulam [8].

Theorem 4.1. *There exists a constant $c > 0$ such that any set $A \subset \mathbb{F}_3^n$ with $|A| \geq c \cdot 3^n/n$ contains a 3-term arithmetic progression.*

Proof. We prove the theorem by induction on n. For the induction basis, the claim holds for small n by taking c to be a sufficiently large constant.

For the induction step we assume for contradiction that there exists a set $A \subset \mathbb{F}_3^n$ such that $|A| = c \cdot 3^n/n$ and A contains no 3-term arithmetic progression. Recall that the indicator function $1_A(x)$ equals 1 if $x \in A$ and is otherwise zero. The proof is based on studying the Fourier coefficients of 1_A. First, we observe that

$$\widehat{1_A}(0) = 3^{-n} \sum_{a \in \mathbb{F}_3^n} 1_A(a)\overline{\chi_0(a)} = 3^{-n} \sum_{a \in \mathbb{F}_3^n} 1_A(a) = \frac{|A|}{3^n} = \frac{c}{n}. \tag{2}$$

Finding another large coefficient. Set $\delta = \max_{\alpha \in \mathbb{F}_3^n \setminus \{0\}} \left| \widehat{1_A}(\alpha) \right|$. We now show that δ cannot be too small. By the definition of a Fourier coefficient, we have

$$\sum_{\alpha \in \mathbb{F}_3^n} \widehat{1_A}(\alpha)^3 = \sum_{\alpha \in \mathbb{F}_3^n} \left(3^{-n} \sum_{x \in \mathbb{F}_3^n} 1_A(x)\overline{\chi_\alpha(x)} \right)^3 = 3^{-3n} \sum_{\alpha \in \mathbb{F}_3^n} \sum_{x,y,z \in A} e^{-2\pi i((x+y+z)\cdot\alpha)/3}. \tag{3}$$

By part (iii) of Claim 3.1, when $x+y+z \neq 0$ we have $\sum_{\alpha \in \mathbb{F}_3^n} e^{2\pi i(\alpha\cdot(x+y+z))/3} = 0$. That is, it suffices to sum over $x, y, z \in A$ with $x+y+z = 0$. Notice that $x+y+z = 0$ is equivalent to $x + z = 2y$. This holds either when $x = y = z$ or when x, y, z is a 3-term arithmetic progression. Since we assume that A contains no such progression, the only solution to $x + y + z = 0$ is $x = y = z$. Thus, (3) becomes

$$\sum_{\alpha \in \mathbb{F}_3^n} \widehat{1_A}(\alpha)^3 = 3^{-3n} \sum_{\alpha \in \mathbb{F}_3^n} \sum_{x \in A} 1 = 3^{-2n}|A| = \frac{c}{n3^n}.$$

By combining this with (2), we obtain

$$\frac{c}{n3^n} - \frac{c^3}{n^3} = \sum_{\substack{\alpha \in \mathbb{F}_3^n \\ \alpha \neq 0}} \widehat{1_A}(\alpha)^3.$$

By taking the absolute value of both sides and applying the triangle inequality, we obtain

$$\frac{c^3}{n^3} - \frac{c}{n3^n} = \left| \sum_{\substack{\alpha \in \mathbb{F}_3^n \\ \alpha \neq 0}} \widehat{1_A}(\alpha)^3 \right| \leq \sum_{\substack{\alpha \in \mathbb{F}_3^n \\ \alpha \neq 0}} \left| \widehat{1_A}(\alpha) \right|^3.$$

By recalling the definition of δ and applying Parseval's Theorem (Claim 3.3(iv)), we get

$$\frac{c^3}{n^3} - \frac{c}{n3^n} \leq \delta \sum_{\substack{\alpha \in \mathbb{F}_3^n \\ \alpha \neq 0}} \left| \widehat{1_A}(\alpha) \right|^2 \leq \frac{\delta}{3^n} \sum_{\alpha \in \mathbb{F}_3^n} |1_A(\alpha)|^2 = \frac{\delta}{3^n}|A| = \frac{\delta c}{n}.$$

That is, we obtain $\delta \geq \frac{c^2}{n^2} - \frac{1}{3^n}$.

Studying the large coefficient. Since $\delta \geq \frac{c^2}{n^2} - \frac{1}{3^n}$, there exists $\alpha \in \mathbb{F}_3^n \setminus \{0\}$ with $|\widehat{1_A}(\alpha)| \geq \frac{c^2}{n^2} - \frac{1}{3^n}$. We now study what such a large Fourier coefficient implies about A.

A *hyperplane* in $\mathbb{F}_3^n$ is the set of points that are defined by a linear equation in $x_1, \ldots, x_n$. Notice that every hyperplane contains exactly 3^{n-1} points of $\mathbb{F}_3^n$. For $j \in \{0, 1, 2\}$ (and α as set above), we denote by $H_{\alpha,j}$ the hyperplane that is defined by $\alpha \cdot (x_1, \ldots, x_n) = j$. Notice that $H_{\alpha,0}, H_{\alpha,1}$, and $H_{\alpha,2}$ are three "parallel" hyperplanes that together cover $\mathbb{F}_3^n$. We can think of these hyperplanes as being orthogonal to the vector α.

For $0 \leq j \leq 2$, we set $k_j = |A \cap H_{\alpha,j}|/3^{n-1} - c/n$. Notice that the first term in this difference is the density of A in $H_{\alpha,j}$ and the second term is the density of A in $\mathbb{F}_3^n$. We have

$$\begin{aligned}
\widehat{1_A}(\alpha) &= 3^{-n} \sum_{x \in \mathbb{F}_3^n} 1_A(x) e^{-2\pi i (x \cdot \alpha)/3} \\
&= 3^{-n} \left(\sum_{x \in h_{\alpha,0}} 1_A(x) + \sum_{x \in h_{\alpha,1}} 1_A(x) e^{-2\pi i/3} + \sum_{x \in h_{\alpha,2}} 1_A(x) e^{-4\pi i/3} \right) \\
&= 3^{-1} \left((k_0 + c/n) + (k_1 + c/n) e^{-2\pi i/3} + (k_2 + c/n) e^{-4\pi i/3} \right).
\end{aligned}$$

Since the sum of the 3'rd roots of unity is zero (that is, $1 + e^{-2\pi i/3} + e^{-4\pi i/3} = 0$), we get

$$\frac{c^2}{n^2} - \frac{1}{3^n} \leq \left| \widehat{1_A}(\alpha) \right| = \left| 3^{-1} \left(k_0 + k_1 e^{-2\pi i/3} + k_2 e^{-4\pi i/3} \right) \right|.$$

By the triangle inequality, we have

$$\frac{c^2}{n^2} - \frac{1}{3^n} \leq \frac{|k_0| + |k_1| + |k_2|}{3}.$$

That is, there exists $j \in \{0, 1, 2\}$ such that $\frac{c^2}{n^2} - \frac{1}{3^n} \leq |k_j|$. Notice that $k_0 + k_1 + k_2 = |A|/3^{n-1} - 3c/n = 0$. Thus, if $\frac{c^2}{n^2} - \frac{1}{3^n} \leq -k_j$ then there exists $j' \in \{0, 1, 2\} \setminus \{j\}$ with $k_{j'} \geq \frac{c^2}{2n^2} - \frac{1}{2 \cdot 3^n}$. Either way, we have $\max_{j \in \{0,1,2\}} k_j \geq \frac{c^2}{2n^2} - \frac{1}{2 \cdot 3^n}$.

To recap, we showed that the existence of a large Fourier coefficient $\widehat{1_A}(\alpha)$ (where $\alpha \neq 0$) implies that A is not well-distributed along the hyperplanes $H_{\alpha,j}$. That is, there exists a hyperplane $H = H_{\alpha,j}$ for some $j \in \{0, 1, 2\}$ such that $|A \cap H| \geq 3^{n-1}(\frac{c}{n} + \frac{c^2}{2n^2} - \frac{1}{2 \cdot 3^n})$.

Concluding the proof. Consider the set $A' = A \cap H$. We consider H as $\mathbb{F}_3^{n-1}$ and A' as a subset of $\mathbb{F}_3^{n-1}$. By the above, for sufficiently large c we have

$$|A'| = 3^{n-1}\left(\frac{c}{n} + \frac{c^2}{2n^2} - \frac{1}{2\cdot 3^n}\right) > 3^{n-1}\frac{c}{n-1}.$$

By the induction hypothesis, this means that A' contains a 3-term arithmetic progression. Since A' is a subset of A, we obtain a contradiction to A not containing such a progression. This contradiction completes the induction step, and proves the assertion of the theorem. $\square$

In the proof of Theorem 4.1 we showed that a large Fourier coefficient $\widehat{1_A}(\alpha)$ (where $\alpha \neq 0$) implies that A is not well-distributed along the hyperplanes that are orthogonal to α. It is not difficult to show that the complement statement also holds — no large Fourier coefficients (excluding $\widehat{1_A}(0)$) implies that the point set A is well-distributed. That is, for any direction α the points of A are well-distributed among the three hyperplanes that are orthogonal to α.

Bateman and Katz [2] improved the bound of Theorem 4.1 to $c \cdot 3^n/n^{1+\varepsilon}$, for some small $\varepsilon > 0$. They did this by showing that there are many large coefficient, and studying what this implies. This still leaves a significantly large bound between the current best lower and upper bounds for this problem.

5 Roth's theorem

Theorem 1.2 stated that there exist somewhat dense sets that do not contain a 3-term arithmetic progression. Roth [9] proved the complement claim — any sufficiently dense set must contain a 3-term arithmetic progression.

Theorem 5.1 (Roth's theorem). *There exists a constant $c > 0$ such that the following holds for every positive integer n. Any set $A \subset \{1, 2, \dots, n\}$ with $|A| \geq cn/\lg\lg n$ contains a 3-term arithmetic progression.*

Notice that there is a gap of between the density $e^{-c\sqrt{\lg n}}$ in Theorem 1.2 and the density $\frac{1}{\lg\lg n}$ in Theorem 5.1. Over the years, Theorem 5.1 has been improved to smaller densities. The current best bound is by Bloom [4], stating that sets of density at least $c\frac{(\lg\lg n)^4}{\lg n}$ must contain a 3-term arithmetic progression.

To prove Theorem 5.1, we will rely on the following two number theoretic results (the first is taken from [1]; for the second, see for example [10, Section II.1])

Theorem 5.2. *For any sufficiently large integer n, the interval $\{\lfloor n - n^{0.525} \rfloor, \dots, n\}$ contains a prime number.*

Theorem 5.3 (Dirichlet's approximation theorem). *For every $\gamma \in \mathbb{R}$ and positive integer N, there exist integers p and $1 \leq q \leq N$ such that $|\gamma - \frac{p}{q}| \leq \frac{1}{N \cdot q}$.*

We will also rely on the following claim, whose technical proof can be found below in Appendix A.

Claim 5.4. *Let $I \subset \mathbb{R}$ be a continues interval of length $\beta > 0$, and let $f : I \to \mathbb{R}$ be a function that satisfies $|f(x)| \leq \gamma$ for any $x \in I$. Then for any $x_1, \dots, x_m \in I$ we have*

$$\left| \sum_{j=1}^{m} f(x_j) e^{-2\pi i x_j} \right| \leq \left| \sum_{j=1}^{m} f(x_j) \right| + 2\pi\beta\gamma m.$$

Proof of Theorem 5.1. We first claim that it suffices to prove the theorem for the case where n is prime. Indeed, by Theorem 5.2 for every n there exists a prime of size smaller than $2n$. That is, proving the theorem for every prime n when $|A| \geq \frac{cn}{\lg \lg n}$ implies the theorem for every positive integer n when $|A| \geq \frac{2cn}{\lg \lg n}$.

We imitate the proof of Theorem 4.1 (this is opposite of what actually happened — Meshulam adapted Roth's proof). We prove the theorem by contradiction, assuming that there exists an n and $A \subset \{1, 2, \dots, n\}$ such that $|A| = \frac{cn}{\lg \lg n}$ and A contains no 3-term arithmetic progression (for a sufficiently large constant c). Let n be the smallest prime for which such a set A exists. In various parts of our analysis, we may assume that n is sufficiently large by taking c to be sufficiently large (by taking c to be large we get $\frac{cn}{\lg \lg n} > n$ for small values of n, preventing the existence of a set A of this size).

We decrease every element of A by one, so that $A \subset \{0, 1, \dots, n-1\}$. To use the Fourier transform as in the proof of Theorem 4.1, we will work over the finite field $\mathbb{F}_n$ (this is why we asked for n to be prime). Although A might not contain 3-term arithmetic progressions, it might contain such progressions in $\mathbb{F}_n$; for example, $\{1, 8, 10\}$ is not an arithmetic progression over $\mathbb{R}$ but is a progression over $\mathbb{F}_{11}$. To address this issue, we notice the following property. In any 3-term arithmetic progression in $\mathbb{R}$, the first and third elements have the same parity. On the other hand, if a 3-term progression was created in A due to working in $\mathbb{F}_n$, then the first and third elements in this progression have opposite parities.

If at least $|A|/2$ elements of A are even, we denote by B the set of even elements of A. Otherwise, we denote by B the set of odd elements of A. As before, we denote the indicator functions of A and B as 1_A and 1_B, respectively. We observe that

$$\begin{aligned}\widehat{1_A}(0) &= n^{-1}\sum_{a\in\mathbb{F}_n} 1_A(a)\overline{\chi_0(a)} = n^{-1}\sum_{a\in\mathbb{F}_n} 1_A(a) = \frac{|A|}{n}.\\ \widehat{1_B}(0) &= n^{-1}\sum_{a\in\mathbb{F}_n} 1_B(a)\overline{\chi_0(a)} = n^{-1}\sum_{a\in\mathbb{F}_n} 1_B(a) = \frac{|B|}{n}.\end{aligned} \tag{4}$$

Finding another large coefficient. Set $\delta = \max_{\alpha\in\mathbb{F}_n\setminus\{0\}}\left|\widehat{1_A}(\alpha)\right|$. We now show that δ cannot be too small. By the definition of a Fourier coefficient, we have

$$\begin{aligned}\sum_{\alpha\in\mathbb{F}_n}\widehat{1_B}(\alpha)^2\widehat{1_A}(-2\alpha) &= \sum_{\alpha\in\mathbb{F}_n}\left(n^{-1}\sum_{x\in\mathbb{F}_n}1_B(x)\overline{\chi_\alpha(x)}\right)^2\left(n^{-1}\sum_{y\in\mathbb{F}_n}1_A(y)\overline{\chi_{-2\alpha}(y)}\right)\\ &= n^{-3}\sum_{\alpha\in\mathbb{F}_n}\sum_{x,z\in B}\sum_{y\in A} e^{-2\pi i((x+z-2y)\cdot\alpha)/n}.\end{aligned} \tag{5}$$

By part (iii) of Claim 3.1, when $x+z-2y\neq 0$ we have $\sum_{\alpha\in\mathbb{F}_n} e^{2\pi i(\alpha\cdot(x+z-2y))/n}=0$. That is, it suffices to sum over x,y,z with $x+z=2y$. This holds either when $x=y=z$ or when x,y,z is a 3-term arithmetic progression. Since A contains no such progressions in $\mathbb{R}$ and since x and z have the same parity, the only solution to $x+z=2y$ is $x=y=z$. Thus, (5) becomes

$$\sum_{\alpha\in\mathbb{F}_n}\widehat{1_B}(\alpha)^2\widehat{1_A}(-2\alpha) = n^{-3}\sum_{\alpha\in\mathbb{F}_n}\sum_{x\in B}1 = n^{-2}|B|.$$

By combining this with (4), we obtain

$$\frac{|B|}{n^2} - \frac{|A||B|^2}{n^3} = \sum_{\substack{\alpha\in\mathbb{F}_n\\ \alpha\neq 0}}\widehat{1_A}(\alpha)^2\widehat{1_A}(-2\alpha).$$

For sufficiently large n, we have $\frac{|B|}{n^2} < \frac{|A||B|^2}{n^3}$ (for any $c\geq 1$). Thus, taking the absolute value of both sides and applying the triangle inequality gives

$$\frac{|A||B|^2}{n^3} - \frac{|B|}{n^2} = \left|\sum_{\substack{\alpha\in\mathbb{F}_n\\ \alpha\neq 0}}\widehat{1_B}(\alpha)^2\widehat{1_A}(-2\alpha)\right| \leq \sum_{\substack{\alpha\in\mathbb{F}_n\\ \alpha\neq 0}}\left|\widehat{1_B}(\alpha)\right|^2\left|\widehat{1_A}(-2\alpha)\right|.$$

By recalling the definition of δ and applying Parseval's theorem (part (iv) of Claim 3.3), we get

$$\frac{|A||B|^2}{n^3} - \frac{|B|}{n^2} \leq \delta \sum_{\substack{\alpha \in \mathbb{F}_n \\ \alpha \neq 0}} \left|\widehat{1_B}(\alpha)\right|^2 \leq \frac{\delta}{n} \sum_{\alpha \in \mathbb{F}_n} |1_B(\alpha)|^2 = \frac{\delta}{n}|B|.$$

Since $|B| \geq |A|/2$, we have $\delta \geq \frac{|A|^2}{2n^2} - \frac{1}{n}$.

Using the large coefficient. Since $\delta > \frac{|A|^2}{2n^2} - \frac{1}{n}$, there exists $\alpha \in \mathbb{F}_n \setminus \{0\}$ with $|\widehat{1_A}(\alpha)| \geq \frac{|A|^2}{2n^2} - \frac{1}{n}$. We now study what such a large Fourier coefficient implies about A. Notice that the density of A in $\{0, 1, 2, 3, \ldots, n-1\}$ is $c/\lg\lg n$. By part (iii) of Claim 3.1, for sufficiently large n we obtain

$$\begin{aligned} \left|\sum_{x \in \mathbb{F}_n} \left(1_A(x) - \frac{c}{\lg\lg n}\right) e^{-2\pi i \alpha x/n}\right| &= \left|\sum_{x \in \mathbb{F}_n} 1_A(x) e^{-2\pi i \alpha x/n}\right| = n \cdot \left|\widehat{1_A}(\alpha)\right| \\ &\geq \frac{|A|^2}{2n} - 1 > \frac{c^2 n}{3(\lg\lg n)^2}. \end{aligned} \tag{6}$$

By applying Theorem 5.3 with $\gamma = \alpha/n$ and $N = \sqrt{n}$, we obtain integers r and $1 \leq q \leq \sqrt{n}$ such that

$$\left|\frac{\alpha}{n} - \frac{r}{q}\right| \leq \frac{1}{\sqrt{n} \cdot q}. \tag{7}$$

We partition the set $\{0, 1, 2, 3, \ldots, n-1\}$ into the q arithmetic progressions $S_a = \{a + qb : 0 \leq b \leq \lfloor (n-a)/q \rfloor\}$, where $a \in \{0, 1, 2, \ldots, q-1\}$. Notice that for any such a we have $n/q - 1 \leq |S_a| \leq n/q$. For a k that we will set below, we further subdivide each S_a into k arithmetic progressions $S_{a,j}$ (where $0 \leq j \leq k-1$), each consisting of either $\lceil |S_a|/k \rceil$ or $\lfloor |S_a|/k \rfloor$ consecutive elements of S_a. This process creates kq arithmetic progressions, each with difference q and of size between $n/qk - 2$ and $n/qk + 1$.

The reason for partitioning $\{0, 1, 2, \ldots, n-1\}$ into the above arithmetic progressions is that every element $x \in S_{a,j}$ (for a fixed progression $S_{a,j}$) gives the expression $e^{-2\pi i(\alpha \cdot x)/n}$ almost the same value. Indeed, by (7) we have $e^{-2\pi i \alpha q/n} = e^{-2\pi i (r + \varepsilon q)} = e^{-2\pi i \varepsilon q}$ for some $|\varepsilon| \leq \frac{1}{\sqrt{n} \cdot q}$. By combining (6) and the triangle inequality, we have

$$\frac{c^2 n}{3(\lg\lg n)^2} < \left|\sum_{x \in \mathbb{F}_n} \left(1_A(x) - \frac{c}{\lg\lg n}\right) e^{-2\pi i \alpha x/n}\right| \leq \sum_{S_{a,j}} \left|\sum_{x \in S_{a,j}} \left(1_A(x) - \frac{c}{\lg\lg n}\right) e^{-2\pi i \alpha x/n}\right|$$

Consider the value of $\left|\sum_{x\in S_{a,j}}\left(1_A(x)-\frac{c}{\lg\lg n}\right)e^{-2\pi i\alpha x/n}\right|$ for a specific progression $S_{a,j}$. We write the elements of $S_{a,j}$ as $a'+bq$ (where $0\le b\le n/qk$), and recall the definition of ε above. For every such x we have

$$e^{-2\pi i\alpha x/n}=e^{-2\pi i\alpha(a'+bq)/n}=e^{-2\pi i\alpha a'/n}e^{-2\pi i(r+\varepsilon qb)}=e^{-2\pi i\alpha a'/n}e^{-2\pi i\varepsilon qb}.$$

As we take different elements $x\in S_{a,j}$, the above expression changes only in $e^{-2\pi i\varepsilon qb}$. We can thus apply Claim 5.4 to the function $f(y)=1_A(a'+y/\varepsilon)-\frac{c}{\lg\lg n}$, where $y=\varepsilon bq$ and $0\le b\le n/qk$. That is, we may take $\beta=\sqrt{n}/qk\ge\varepsilon q(n/qk)$, $\gamma=c$, and $m=2n/qk$, to obtain

$$\begin{aligned}\frac{c^2n}{3(\lg\lg n)^2}&<\sum_{S_{a,j}}\left(\left|\sum_{x\in S_{a,j}}\left(1_A(x)-\frac{c}{\lg\lg n}\right)\right|+\frac{14cn^{3/2}}{q^2k^2}\right)\\&\le\sum_{S_{a,j}}\left|\sum_{x\in S_{a,j}}\left(1_A(x)-\frac{c}{\lg\lg n}\right)\right|+\frac{14cn^{3/2}}{qk}.\end{aligned}$$

By setting $k=84\sqrt{n}(\lg\lg n)^2/cq$, for sufficiently large n we have

$$\frac{c^2n}{3(\lg\lg n)^2}<\sum_{S_{a,j}}\left|\sum_{x\in S_{a,j}}\left(1_A(x)-\frac{c}{\lg\lg n}\right)\right|+\frac{c^2n}{6(\lg\lg n)^2},$$

or

$$\frac{c^2n}{6(\lg\lg n)^2}<\sum_{S_{a,j}}\left|\sum_{x\in S_{a,j}}\left(1_A(x)-\frac{c}{\lg\lg n}\right)\right| \tag{8}$$

On the other hand, notice that

$$\sum_{S_{a,j}}\sum_{x\in S_{a,j}}\left(1_A(x)-\frac{c}{\lg\lg n}\right)=\sum_{x\in\mathbb{F}_n}\left(1_A(x)-\frac{c}{\lg\lg n}\right)=\frac{cn}{\lg\lg n}-\frac{cn}{\lg\lg n}=0. \tag{9}$$

Recall that we have kq arithmetic progressions $S_{a,j}$. For both (8) and (9) to hold, there must exist $S_{a,j}$ such that

$$\sum_{x\in S_{a,j}}\left(1_A(x)-\frac{c}{\lg\lg n}\right)>\frac{c^2n}{12(\lg\lg n)^2}\cdot\frac{1}{kq}. \tag{10}$$

To recap, we showed that the existence of a large Fourier coefficient $\widehat{1_A}(\alpha)$ (where $\alpha \neq 0$) implies that A is not well-distributed along the progressions $S_{a,j}$.

Concluding the proof. We are done with the Fourier analysis part of the proof and return to work over $\mathbb{R}$. Let $S_{a,j}$ be the set satisfying (10) and let $D = S_{a,j} \cap A$. Since $|A \cap S_{a,j}| = |D \cap S_{a,j}|$, we get

$$|D| - \frac{c|S_{a,j}|}{\lg\lg n} = \sum_{x \in S_{a,j}} \left(1_D(x) - \frac{c}{\lg\lg n} \right) > \frac{c^2 n}{12(\lg\lg n)^2} \cdot \frac{1}{kq}. \tag{11}$$

We translate and dilate $S_{a,j}$ so that $S_{a,j}$ becomes the set $\{1, 2, 3, \ldots, |S_{a,j}|\}$. Denote by D' be the set that is obtained by applying the same translation and dilation to D. Let n' denote the smallest prime that satisfies $n' \geq |S_{a,j}|$. By Theorem 5.2, $n' \leq |S_{a,j}| + |S_{a,j}|^{0.53}$, which in turn implies $n' \leq |S_{a,j}| + (n')^{0.53}$. Notice that

$$\frac{2n}{kq} \geq n' \geq \frac{n}{2kq} \geq \frac{c\sqrt{n}}{168(\lg\lg n)^2} \quad \text{and} \quad (n')^{0.53} < \frac{n'}{\lg\lg n}.$$

Notice also that the inequality $\frac{10}{(\lg\lg n)^2} + \frac{1}{\lg\lg n} > \frac{1}{\lg\lg n^{1/3}}$ holds for any $n > 31$. By combining the above observations and taking n and c to be sufficiently large, we get that (11) gives

$$\begin{aligned} |D'| = |D| > \frac{c^2 n}{12(\lg\lg n)^2} \cdot \frac{1}{kq} + \frac{c|S_{a,j}|}{\lg\lg n} &\geq \frac{11cn'}{(\lg\lg n)^2} + \frac{c\,(n' - (n')^{0.53})}{\lg\lg n} \\ &> \frac{10cn'}{(\lg\lg n)^2} + \frac{cn'}{\lg\lg n} > \frac{cn'}{\lg\lg n^{1/3}} > \frac{cn'}{\lg\lg n'}. \end{aligned}$$

By the minimality of n, since D' is a subset of $\{1, 2, \ldots, n'\}$ and $|D'| > \frac{cn'}{\lg\lg n'}$, there is a 3-term arithmetic progression in D'. Since A contains a translated and dilated copy of D', A also contains such a progression. This contradiction completes the proof. ☐

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A Claim 5.4

In this appendix we prove Claim 5.4. We first repeat the statement of the claim.

Claim 5.4. *Let $I \subset \mathbb{R}$ be a continues interval of length $\beta > 0$, and let $f : I \to \mathbb{R}$ be a function that satisfies $|f(x)| \leq \gamma$ for any $x \in I$. For $x_1, \ldots, x_m \in I$ we have*

$$\left|\sum_{j=1}^{m} f(x_j)e^{-2\pi i x_j}\right| \leq \left|\sum_{j=1}^{m} f(x_j)\right| + 2\pi\beta\gamma m.$$

Proof. By the triangle inequality, we have

$$\begin{aligned}\left|\sum_{j=1}^{m} f(x_j)e^{-2\pi i x_j}\right| &= \left|f(x_1) + \sum_{j=2}^{m} f(x_j)\frac{e^{-2\pi i x_j}}{e^{-2\pi i x_1}}\right| \\ &= \left|f(x_1) + \sum_{j=2}^{m}\left(f(x_j) - f(x_j) + f(x_j)\frac{e^{-2\pi i x_j}}{e^{-2\pi i x_1}}\right)\right| \\ &\leq \left|\sum_{j=1}^{m} f(x_j)\right| + \sum_{j=2}^{m}\left|f(x_j)\frac{e^{-2\pi i x_j}}{e^{-2\pi i x_1}} - f(x_j)\right| \\ &= \left|\sum_{j=1}^{m} f(x_j)\right| + \sum_{j=2}^{m} f(x_j)\left|e^{-2\pi i(x_j - x_1)} - 1\right|. \qquad (12)\end{aligned}$$

It is known that $|1 - e^{-2\pi i x}| \leq 2\pi\|x\|$, where $\|x\|$ is the distance between x and the closest integer (e.g., see [13, Section 4.4]). Combining this with (12) gives

$$\left|\sum_{j=1}^{m} f(x_j)e^{-2\pi i x_j}\right| \leq \left|\sum_{j=1}^{m} f(x_j)\right| + \sum_{j=2}^{m} f(x_j)2\pi\beta \leq \left|\sum_{j=1}^{m} f(x_j)\right| + 2\pi\beta\gamma m.$$

$\square$