Chapter 6: Third Moment Energy

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1 Introduction

In this chapter we will see how to obtain various results by relying on a generalization of the concept of energy. While such generalizations exist for any abelian group, we focus on the case of $\mathbb{R}$. In the current section we describe the main object that we will study, and in the following sections we will see applications for this object.

For a finite set $A \subset \mathbb{R}$ and a positive integer d, we define

$$E_d^+(A) = \sum_{x \in A-A} r_A^-(x)^d.$$

Notice that $E_1^+(A) = |A|^2$ and $E_2^+(A) = E^+(A)$. In this chapter we are specifically interested in $E_3^+(A)$, which is sometimes called the *third moment additive energy* of A.[1] Since $r_A^-(x) \le |A|$ for any $x \in A - A$, we have

$$E_3^+(A) = \sum_{x \in A-A} r_A^-(x)^3 \le |A| \sum_{x \in A-A} r_A^-(x)^2 = |A| \cdot E^+(A).$$

By the Cauchy-Schwarz inequality, we have

$$E_3^+(A) = \sum_{x \in A-A} r_A^-(x)^3 \ge \frac{\left(\sum_{x \in A-A} r_A^-(x)^{3/2} r_A^-(x)^{1/2}\right)^2}{\sum_{x \in A-A} r_A^-(x)} = \frac{E^+(A)^2}{|A|^2}.$$

Consider a set A with $E^+(A) = \delta |A|^3$. By the two above bounds we have that $\delta^2 |A|^4 \le E_3^+(A) \le \delta |A|^4$

For $s \in A-A$, we set $A_s = A \cap (A-s)$. We have that $|A_s| = \sum_{y \in \mathbb{R}} 1_A(y) 1_A(s+y) = r_A^-(s)$. The following lemma is taken from [3].

Lemma 1.1. *For any finite $A \subset \mathbb{R}$ we have $E_3^+(A) = \sum_{s \in \mathbb{R}} E^+(A, A_s)$.*

Proof. We first note that

$$\sum_{s \in \mathbb{R}} E^+(A, A_s) = \sum_{s \in \mathbb{R}} \sum_{t \in \mathbb{R}} r_A^-(t) r_{A_s}^-(t) = \sum_{t \in \mathbb{R}} r_A^-(t) \sum_{s \in \mathbb{R}} r_{A_s}^-(t). \tag{1}$$

[1] Another object that is sometimes called the third energy of A is the quantity $|\{(a,b,c,d,e,f) \in A : a+b+c = d+e+f\}|$. We do not study this quantity in the lecture notes, although it does appear in the third assignment.

We study the expression $\sum_{s\in\mathbb{R}} r^-_{A_s}(t)$ for a fixed t. First, we have

$$\begin{aligned} r^-_{A_s}(t) &= \sum_{x\in\mathbb{R}} 1_{A_s}(x)1_{A_s}(t+x) = \sum_{x\in\mathbb{R}} 1_A(x)1_{A-s}(x)1_A(t+x)1_{A-s}(t+x) \\ &= \sum_{x\in\mathbb{R}} 1_A(x)1_A(s+x)1_{A-t}(x)1_{A-t}(s+x) = \sum_{x\in\mathbb{R}} 1_{A_t}(x)1_{A_t}(s+x) = r^-_{A_t}(s). \end{aligned}$$

By recalling that $|A_t| = r^-_A(t)$, we obtain

$$\sum_{s\in\mathbb{R}} r^-_{A_s}(t) = \sum_{s\in\mathbb{R}} r^-_{A_t}(s) = |A_t|^2 = r^-_A(t)^2.$$

Combining this with (1) gives

$$\sum_{s\in\mathbb{R}} E^+(A, A_s) = \sum_{t\in\mathbb{R}} r^-_A(t) \sum_{s\in\mathbb{R}} r^-_{A_s}(t) = \sum_{t\in\mathbb{R}} r^-_A(t)^3 = E^+_3(A).$$

□

2 Convex sets

Consider a set $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}$ such that $a_1 < a_2 < \cdots < a_n$. We say that A is *convex* if for every $2 \le j \le n-1$ we have $a_{j+1} - a_j > a_j - a_{j-1}$. It intuitively seems that a convex set does not have a good additive structure (e.g., it cannot contain a large arithmetic progression). This leads to the problem: How small can the doubling of a convex set be? As a first bound, recall a result that was proved in Chapter 2 of these lecture notes.

Theorem 2.1. *Let A be a finite set of real numbers and let f be a strictly convex or strictly concave function. Then*

$$|A + A| \cdot |f(A) + f(A)| = \Omega\left(|A|^{5/2}\right).$$

Corollary 2.2. *Let $B \subset \mathbb{R}$ be a finite convex set. Then $|B + B| = \Omega(|B|^{3/2})$.*

Proof. Write $B = \{b_1, \ldots, b_n\}$ so that $b_1 > b_2 > \cdots > b_n$. Let $A = \{1, 2, \ldots, n\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function such that for every $j \in A$ we have $f(j) = b_j$ (such a function exists since B is convex). Applying Theorem 2.1 to A gives $|A + A| \cdot |f(A) + f(A)| = \Omega(n^{5/2})$. Since $|A + A| = \Theta(n)$ and $f(A) = B$, we get that $|B + B| = \Omega(n^{3/2})$. □

Schoen and Shkredov [2] proved that if A is convex then $|A + A| = \Omega^*(|A|^{14/9})$. We now prove a different result from the same paper.

Theorem 2.3. *Let A be a convex set. Then $|A - A| = \Omega^*(|A|^{8/5})$.*

To prove this theorem, we require the following lemma.

Lemma 2.4. *Let $A \subset \mathbb{R}$ be a convex set and let $B \subset \mathbb{R}$. Then for every $t \geq 1$, the number of elements $x \in A - B$ for which $r^-_{A,B}(x) \geq t$ is $O(|A||B|^2/t^3)$.*

Proof. Write $A = \{a_1, a_2, \ldots, a_{|A|}\}$ so that $a_1 < a_2 < \cdots < a_{|A|}$. Since A is a convex set, there exists a strictly convex function $f : \mathbb{R} \to \mathbb{R}$ such that for every $j \in \{1, 2, \ldots, |A|\}$ we have $f(j) = a_j$. For $\alpha, \beta \in \mathbb{R}$ we define the curve

$$\gamma_{\alpha,\beta} = \{(x, f(x)) : x \in \mathbb{R}\} + (\alpha, \beta).$$

We consider the set of curves

$$\Gamma = \{\gamma_{\alpha,\beta} : \alpha \in \{1, 2, \ldots, |A|\} \text{ and } \beta \in -B\}.$$

Consider $x \in A - B$ such that $r^-_{A,B}(x) = t_x$, and set $\mathcal{P}_x = \{(p, x) \in \mathbb{R}^2 : p \in \{2, 3, \ldots, 2|A|\}\}$. For any representation $x = a_j - b_j$ (with $a_j \in A$ and $b_j \in B$) and $\alpha \in \{1, 2, \ldots, |A|\}$ the curve $\gamma_{\alpha,-b_j}$ is incident to the point $(j + \alpha, x) \in \mathcal{P}_x$. That is, there are at least $|A| \cdot t_x$ incidences between the points of $\mathcal{P}_x$ and the curves of Γ. Since every point of $\mathcal{P}_x$ can be incident to at most t_x curves of Γ, at least $2|A|/3$ points of $\mathcal{P}_x$ are incident to at least $t_x/4$ curves of Γ.

Set $m = |\{x \in A - B : r^-_{A,B}(x) \geq t\}|$. Notice that $|\Gamma| = |A||B|$. By the above, there are at least $2m|A|/3$ points in $\mathbb{R}^2$ that are incident to at least $t/4$ curves of Γ. We denote by $\mathcal{P}$ a set of exactly $2m|A|/3$ of these points. Note that $I(\mathcal{P}, \Gamma) = \Omega(m|A|t)$. Recall that two translations of a strictly convex curve intersect in at most one point. Thus, by Theorem 2.5 of Chapter 2 (the incidence theorem of Pach and Sharir) we have

$$I(\mathcal{P}, \Gamma) = O(|\mathcal{P}|^{2/3}|\Gamma|^{2/3} + |\mathcal{P}| + |\Gamma|) = O(m^{2/3}|A|^{4/3}|B|^{2/3} + |A||B| + m|A|).$$

By combining the two above bounds for $I(\mathcal{P}, \Gamma)$ we obtain

$$m = O\left(m^{2/3}|A|^{1/3}|B|^{2/3}t^{-1} + |B|t^{-1} + mt^{-1}\right). \tag{2}$$

The claim of the lemma trivially holds for small t, so we may assume that t is larger than the constant in the $O(\cdot)$-notation in (2). Thus, the third term in (2) cannot dominate the bound. The other two terms imply $m = O(|A||B|^2/t^3 + |B|/t)$. We may assume that $t \leq \min\{|A|, |B|\}$ (since no difference in $A - B$ can has a larger number of representations), so $|B|/t = O(|A||B|^2/t^3)$. That is, $m = O(|A||B|^2/t^3)$. □

Corollary 2.5. *Let $A \subset \mathbb{R}$ be a convex set. Then $E^+_3(A) = O(|A|^3 \lg |A|)$.*

Proof. Set $N_j = |\{x \in A - A : r^-_A(x) \geq 2^j\}|$. By Lemma 2.4 we have $N_j = O(|A|^3/2^{3j})$. This implies

$$\begin{aligned} E^+_3(A) &= \sum_{x \in A-A} r^-_A(x)^3 = \sum_{j=0}^{\lg|A|} \sum_{2^j \leq r^-_A(x) < 2^{j+1}} r^-_A(x)^3 \\ &< \sum_{j=0}^{\lg|A|} N_j 2^{3(j+1)} = \sum_{j=0}^{\lg|A|} O(|A|^3) = O(|A|^3 \lg |A|). \end{aligned}$$

□

Proof of Theorem 2.3. We begin by double counting $E_3^+(A)$. Corollary 2.5 provides an upper bound for $E_3^+(A)$. To obtain a lower bound, we recall that Lemma 1.1 states that $E_3^+(A) = \sum_{s\in\mathbb{R}} E^+(A, A_s)$. Set $D = A - A$, $|D| = k|A|$, and

$$D^+ = \left\{ d \in D : \, |A_d| \geq \frac{|A|^2}{2|D|} \right\}.$$

Notice that

$$\sum_{d \notin D^+} |A_d| \leq |D| \cdot \frac{|A|^2}{2|D|} = |A|^2/2,$$

which implies that $\sum_{d\in D^+} |A_d| \geq |A|^2/2$.

Recall that $|A_s| = r_A^-(s)$. For any $s \in D^+$, the Cauchy-Schwarz inequality implies

$$E^+(A, A_s) = \sum_{x\in A-A_s} r_{A,A_s}^-(x)^2 \geq \frac{(\sum_{x\in A-A_s} r_{A,A_s}^-(x))^2}{|A - A_s|} = \frac{|A|^2|A_s|^2}{|A - A_s|}.$$

This in turn implies $|A| \cdot |A_s| \leq E^+(A, A_s)^{1/2}|A - A_s|^{1/2}$. By summing this up for every $s \in D^+$ and applying the Cauchy-Schwarz inequality once more, we obtain

$$\begin{aligned}\sum_{s\in D^+} |A| \cdot |A_s| &\leq \sum_{s\in D^+} E^+(A, A_s)^{1/2}|A - A_s|^{1/2} \\ &\leq \left(\sum_{s\in D^+} E^+(A, A_s)\right)^{1/2} \left(\sum_{s\in D^+} |A - A_s|\right)^{1/2}.\end{aligned}$$

Combining this with Lemma 1.1 and with $\sum_{s\in D^+} |A_s| \geq |A|^2/2$ gives

$$\sum_{s\in D^+} |A - A_s| \geq \frac{|A|^6}{4E_3^+(A)}. \tag{3}$$

Next, we notice that $A - A_s = A - (A \cap (A - s)) = D \cap (D + s)$. That is, for every element of $A - A_s$ there exist $d, d' \in D$ such that $d = d' + s$, which in turn implies $r_D^-(s) \geq |A - A_s|$. By combining this with (3) we get

$$\frac{|A|^6}{4E_3^+(A)} \leq \sum_{s\in D^+} |A - A_s| \leq \sum_{s\in D^+} r_D^-(s).$$

This in turn implies

$$\begin{aligned}E^+(A, D) &= \sum_{s\in D} r_A^-(s) r_D^-(s) \geq \sum_{s\in D^+} r_A^-(s) r_D^-(s) \\ &\geq \frac{|A|^2}{2|D|} \sum_{s\in D^+} r_D^-(s) \geq \frac{|A|^8}{8|D|E_3^+(A)} = \frac{|A|^7}{8kE_3^+(A)}.\end{aligned}$$

By Corollary 2.5 we have $E_3^+(A) = O(|A|^3 \lg |A|)$. Combining this with the above gives $E^+(A,D) = \Omega^*\left(|A|^4/k\right)$. Let N_j denote the number of elements $x \in A - D$ for which $r^-_{A,D}(x) \geq 2^j$. Notice that

$$E^+(A,D) = \sum_{x \in A-D} r^-_{A,D}(x)^2 = \sum_{j=0}^{\lg |A|} \sum_{2^j \leq r^-_A(x) < 2^{j+1}} r^-_A(x)^2 < \sum_{j=0}^{\lg |A|} N_j 2^{2j+2}.$$

By dyadic pigeonholing, there exists j such that $N_j 2^{2j+2} = \Omega^*(E^+(A,D)) = \Omega^*(|A|^4/k)$. We have the trivial bound $N_j \leq |A||D|/2^j = |A|^2 k/2^j$. Combining these two bounds gives $2^j = \Omega^*(|A|^2/k^2)$. Lemma 2.4 implies $N_j = O(|A||D|^2/2^{3j}) = O(|A|^3 k^2/2^{3j})$. Thus, we have

$$|A|^4/k = O^*(N_j 2^{2j}) = O^*\left(\frac{|A|^3 k^2}{2^{3j}} \cdot 2^{2j}\right) = O^*(|A|k^4).$$

This implies $k = \Omega^*(|A|^{5/3})$, which completes the proof. □

3 Another BSG bound

Our next application of the third energy is to derive a variant of Schoen's theorem from Chapter 3 (this variant is from the same paper of Scheon [1]).

To appear???

References

[1] T. Schoen, New bounds in Balog-Szemerdi-Gowers theorem, *Combinatorica*, to appear.

[2] T. Schoen and I. D. Shkredov, On sumsets of convex sets, *Combinatorics, Probability and Computing*, **20** (2011), 793–798.

[3] T. Schoen and I. D. Shkredov, Additive properties of multiplicative subgroups of $\mathbb{F}_p$, *The Quarterly Journal of Mathematics* **63** (2012), 713–722.