

# Isomorphism of finitely generated solvable groups is weakly universal

Jay Williams<sup>1,\*</sup>

*Department of Mathematics  
California Institute of Technology  
Pasadena, CA 91125*

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## Abstract

We show that the isomorphism relation for finitely generated solvable groups of class 3 is a weakly universal countable Borel equivalence relation. This improves on previous results. The proof uses a modification of a construction of Neumann and Neumann. Elementary arguments show that isomorphism of finitely generated metabelian or nilpotent groups can not achieve this Borel complexity. In this sense the result is sharp, though it remains open whether the relation is in fact universal.

*Keywords:* Borel equivalence relations, solvable groups, weakly universal

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## 1. Introduction

Let  $E$  be an equivalence relation defined on a Polish space  $X$ . Then we call  $E$  *Borel* if  $E$  is Borel as a subset of  $X^2$ . We say  $E$  is *countable* if every equivalence class of  $E$  is countable. Given two equivalence relations  $E, F$  on Polish spaces  $X, Y$  respectively, we say that  $E$  *Borel reduces* to  $F$ , written  $E \leq_B F$ , if there is a Borel map  $f: X \rightarrow Y$  such that

$$x E y \Leftrightarrow f(x) F f(y)$$

Intuitively, if  $E \leq_B F$ , then the classification problem associated to  $F$  is at least as difficult as the classification problem associated to  $E$ . Thus the notion of a Borel reduction gives us a mathematical framework for comparing the complexity of assorted classification problems. In the particular case of countable Borel equivalence relations, the following definition is especially important.

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\*Corresponding Author: Jay Williams, Department of Mathematics, California Institute of Technology, Pasadena, CA 91125; Email, jaywill@caltech.edu; Phone, (626) 395-4338

*Email address:* jaywill@caltech.edu (Jay Williams)

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**Definition 1.1.** *Suppose that  $E$  is a countable Borel equivalence relation. Then  $E$  is universal if for every countable Borel equivalence relation  $F$ ,  $F \leq_B E$ .*

A universal countable Borel equivalence relation can then be thought of as being as complicated as possible. Thomas and Velickovic showed in [11] that isomorphism on the space of finitely generated groups  $\mathcal{G}$  is universal. (For a discussion of the topology on  $\mathcal{G}$ , see [2].) Their proof proceeds by constructing a family of groups using free products with amalgamation. Such groups are generally large from a group-theoretic standpoint; for example nearly all contain free non-abelian subgroups. It is natural to ask whether this complexity may be achieved using “small” groups, or other more natural classes of groups. Although thus far we have not been able to prove that isomorphism of any other class of finitely generated groups is universal, we have been able to prove strong lower bounds. In order to describe them we must first record one other definition.

**Definition 1.2.** *Suppose that  $E$  is a countable Borel equivalence relation on  $Y$ . Then  $E$  is weakly universal if for every countable Borel equivalence relation  $F$  on  $X$  there is a countable-to-1 Borel map  $f: X \rightarrow Y$  such that*

$$x F y \Rightarrow f(x) E f(y).$$

*Such a map is called a countable-to-1 Borel homomorphism from  $F$  to  $E$ .*

As of this writing, it is open whether or not every weakly universal equivalence relation is in fact universal. Conjecturally the answer is no, but the same conjecture implies that weakly universal equivalence relations are still much more complex than non-weakly-universal equivalence relations (see [9] for details). In any event, we expect that weakly universal equivalence relations are very complex. Recently, in [12], the author and Thomas established the following.

**Theorem 1.3** (Thomas-Williams [12]). *Isomorphism of Kazhdan groups is weakly universal.*

The class of Kazhdan groups contains many groups which are large or complex from a group-theoretic perspective; for example there are SQ-universal Kazhdan groups. One might ask if the groups themselves must be complex if isomorphism restricted to those groups is to be universal. As there are only countably many finitely generated abelian, nilpotent, polycyclic, and metabelian groups, isomorphism for these groups is simple with respect to Borel reducibility, and so we must look to more complicated groups. The main result of this paper is the following.

**Theorem 1.4.** *Isomorphism of finitely generated solvable groups of class 3 is weakly universal.*

**Corollary 1.5.** *Isomorphism of finitely generated amenable groups is weakly universal.*

Thus isomorphism of finitely generated groups is weakly universal on a class of small groups, and in fact for the smallest solvability class possible. Further, this improves the previous best known lower bound on the complexity of isomorphism of finitely generated amenable groups, which was established by Thomas in [10], based on the work of Giordano-Putnam-Skau [4], Bezuglyi-Medynets [1], Matui [7], and Juschenko-Monod [5] on topological full groups. The proof of Theorem 1.4 is based on a construction due to Neumann and Neumann in [8] and uses nothing more complicated than wreath products. There are other constructions which show that solvable groups can be quite complicated from an algorithmic point of view, such as the one from [6], so the result is not entirely unexpected. It does not finish the story, however.

*Conjecture.* Isomorphism of finitely generated solvable groups of class 3 is a universal countable Borel equivalence relation.

More generally one might expect that, in the absence of cardinality issues, isomorphism for any algebraically interesting class of finitely generated groups should be of high complexity. In this vein we ask the following question.

*Question.* What is the Borel complexity of isomorphism of finitely generated simple groups?

Thomas' construction in [10] establishes a non-trivial lower bound. However, the lower bound is very weak, so there is much room for improvement.

## 2. Background

For the proof of the main theorem we need the following definition, which arises quite naturally in the study of SQ-universality.

**Definition 2.1.** *A subgroup  $H$  of a group  $G$  is a CEP-subgroup if for every  $N \trianglelefteq H$ , there is some  $M \trianglelefteq G$  such that  $N = M \cap H$ . We will write  $H \leq_{CEP} G$ .*

This is called an E-subgroup in [8]. It is an immediate consequence of the definition that if  $H$  is a CEP-subgroup of  $G$  and  $R \subseteq H$ , then  $\langle R^H \rangle = \langle R^G \rangle \cap H$ . We will note some further properties of CEP-subgroups momentarily.

Recall the definition of the (unrestricted) wreath product of two groups  $G$  and  $H$ ,  $G \text{ Wr } H$ . First let  $B = G^H$ , then let  $G \text{ Wr } H = B \rtimes H$ , where  $H$  acts on  $B$  by the shift action. So if  $f, f' \in B, h, h' \in H$ , we have

$$(hf)(h'f') = hh'f^{h'}f'$$

Let  $G_0 \leq G \text{ Wr } H$  denote the group of constant functions from  $H$  to  $G$ . Given  $h \in H$ , let  $G_h = \{f \in B \mid \forall h' \neq h, f(h') = e\}$  be a coordinate subgroup of  $G \text{ Wr } H$ . Proofs of the following results can be found in [8].

**Lemma 2.2.** *[Neumann and Neumann [8]]*

1. *If  $H \leq_{CEP} K \leq_{CEP} L$ , then  $H \leq_{CEP} L$ .*
2. *If  $H \leq_{CEP} G$  and  $H \leq G' \leq G$ , then  $H \leq_{CEP} G'$ .*

3. For any groups  $G, H$  and any  $h \in H$ ,  $G_0 \leq_{CEP} G \text{Wr} H$  and  $G_h \leq_{CEP} G \text{Wr} H$ .

Finally, let  $E_\infty$  denote the equivalence relation on  $\mathcal{P}(\mathbb{Z}^2)$  arising from the action of  $SL(2, \mathbb{Z})$  on  $\mathbb{Z}^2$ . By identifying  $\mathcal{P}(\mathbb{Z}^2)$  with  $2^{\mathbb{Z}^2}$  given the product topology, we see that this is a Borel equivalence relation on a Polish space. The following result is due to Gao.

**Theorem 2.3** (Gao [3]).  $E_\infty$  is a universal countable Borel equivalence relation.

### 3. Proof of main theorem

The proof of the main theorem uses a slight modification of the construction from [8]. Let  $F$  be the free abelian group whose generators are  $\{g_{i,j} \mid (i,j) \in \mathbb{Z}^2\}$ . We will prove the following:

**Lemma 3.1.** *There is a finitely generated solvable group  $H$  of class 3 for which*

1.  $F \leq_{CEP} H$ .
2. For every  $M \in SL(2, \mathbb{Z})$ , there is an automorphism  $\theta_M: H \rightarrow H$  such that  $\theta(g_{i,j}) = g_{M(i,j)}$  for all  $(i,j) \in \mathbb{Z}^2$ .

For now we take this lemma for granted.

*Proof of Theorem 1.4.* For  $A \subseteq \mathbb{Z}^2$ , define  $N_A \trianglelefteq F$  to be the subgroup generated by  $\{g_{i,j} \mid (i,j) \in A\}$ . By part i) of lemma 3.1 and the definition of CEP-subgroup, the map  $A \mapsto H/\langle N_A^H \rangle$  is one-to-one. Every group in its range is solvable of class 3. Suppose that  $A, B \subseteq \mathbb{Z}^2$  and  $A E_\infty B$  as witnessed by  $M \in SL(2, \mathbb{Z})$ . Then let  $\theta_M$  be as in part ii) of lemma 3.1. Clearly  $\theta_M(N_A) = N_B$ , and so

$$\begin{aligned} H/\langle N_A^H \rangle &\cong \theta_M(H)/\theta_M(\langle N_A^H \rangle) \\ &\cong H/\langle N_B^H \rangle \end{aligned}$$

Thus  $A E_\infty B$  implies  $H/\langle N_A^H \rangle \cong H/\langle N_B^H \rangle$ , and so there is a countable-to-1 Borel homomorphism from  $E_\infty$  to isomorphism of finitely generated solvable groups of class 3. As  $E_\infty$  is universal, this establishes the result.  $\square$

*Proof of Lemma 3.1.* We begin by constructing  $H$ . Let  $F^*$  be an isomorphic copy of  $F$  with generators  $\{a_{m,n} \mid (m,n) \in \mathbb{Z}^2\}$ . Let  $Z = \langle z \rangle$  be an infinite cyclic group and form the unrestricted wreath product  $P = F^* \text{Wr} Z$ . For each generator  $a_{m,n}$  of  $F^*$ , we define the function  $f_{m,n} \in (F^*)^Z$  by  $f_{m,n}(z^i) = a_{m,n}^{-i}$  for all  $i \in \mathbb{Z}$ . A simple calculation shows that  $[f_{m,n}, z] \in (F^*)^Z$  is the constant  $a_{m,n}$  function, which we will call  $k_{m,n}$ .

Next, let  $B = \mathbb{Z}^3$ . Let  $b_{m,n} = (1, m, n)$ . Note that  $b_{m_1, n_1} + b_{m_2, n_2} \neq b_{m_3, n_3}$  for any  $m_i, n_j \in \mathbb{Z}$ , and also  $b_{m_1, n_1} + b_{m_2, n_2} \neq (0, 0, 0)$ . Let  $Q = P \text{Wr} B$ . Let  $q \in P^B$  be the function defined by

$$q(i, j, k) = \begin{cases} z & \text{if } (i, j, k) = (0, 0, 0) \\ f_{m,n} & \text{if } (i, j, k) = -b_{m,n} \\ e_P & \text{otherwise.} \end{cases}$$

*Claim.*  $F$  is a subgroup of  $H = \langle q, B \rangle \leq Q$ .

*Proof of claim.* Let  $(m, n) \in \mathbb{Z}^2$  be given. Define  $g_{m,n} = [q^{b_{m,n}}, q] \in P^B \cap H$ . Note that  $q^{b_{m,n}}(i, j, k) = q(i-1, j-m, k-n)$ , and so

$$g_{m,n}(0, 0, 0) = [f_{m,n}, z] = k_{m,n}.$$

Meanwhile, for  $(i, j, k) \neq (0, 0, 0)$  we have either  $q^{b_{m,n}}(i, j, k) = e_P$  or  $q(i, j, k) = e_P$  by our choice of  $b_{m,n}$ , and so  $g_{m,n}(i, j, k) = e_P$ . Thus the group generated by the  $g_{m,n}$  is isomorphic to  $F^*$  and in particular is free abelian on those generators.  $\square$

Note that  $F^* \cong P_0$ , and by Lemma 2.2,  $P_0 \leq_{CEP} P$ . The isomorphism from  $P$  to  $Q_{(0,0,0)}$  takes  $P_0$  to  $F$ , so  $F \leq_{CEP} Q_{(0,0,0)}$ . Again using Lemma 2.2,  $Q_{(0,0,0)} \leq_{CEP} Q$ , so  $F \leq_{CEP} Q$ , and thus  $F \leq_{CEP} H$ .

Now, suppose that  $M \in SL(2, \mathbb{Z})$ . Then we may define the automorphism  $\Phi_M: F \rightarrow F$  by  $\Phi_M(g_{m,n}) = g_{M(m,n)}$ . We similarly define  $\phi_M: F^* \rightarrow F^*$ . We wish to extend  $\Phi_M$  to an automorphism  $\Theta_M: H \rightarrow H$ .

We will start by defining an automorphism on  $Q$  starting from  $\phi_M$ , which will agree with  $\Phi_M$ . Define the mapping  $\phi^+: (F^*)^{\mathbb{Z}} \rightarrow (F^*)^{\mathbb{Z}}$  by  $\phi^+(f)(z^n) = \phi_M(f(z^n))$ , then extend  $\phi^+$  to  $P$  by letting  $\phi^+(z^n f) = z^n \phi^+(f)$ . One can easily check this is an automorphism of  $P$ .

Now we extend  $\phi^+$  to an automorphism  $\phi^*$  of  $Q$  in the same fashion. First we extend  $\phi^+$  to  $P^B$  as before, so for  $g \in P^B$ ,  $y \in B$ , we have  $\phi^*(g)(y) = \phi^+(g(y))$ . Then we extend this to an automorphism of  $Q$  as before, so for  $\beta \in B$ ,  $g \in P^B$  we have  $\phi^*(\beta g) = \beta \phi^*(g)$ . Note that

$$\begin{aligned} \phi^*(g_{m,n})(0, 0, 0) &= \phi^+(f_{m,n}) \\ &= f_{M(m,n)} \\ &= g_{M(m,n)}(0, 0, 0), \end{aligned}$$

and since  $g_{m,n}(i, j, k) = e_P$  for all other coordinates, this shows that  $\phi^*$  extends  $\Phi_M$ .

Given  $M \in SL(2, \mathbb{Z})$ , we also define the automorphism  $\alpha_M: B \rightarrow B$  to be given by the matrix  $\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}$ . Note that  $\alpha_M(b_{m,n}) = b_{M(m,n)}$ . We extend this to an automorphism  $\alpha^*$  of  $Q$  in two steps. First, for  $f \in P^B$ , let

$$\alpha^*(f)(i, j, k) = f(\alpha_M^{-1}(i, j, k)).$$

Then, for  $\beta f \in Q$ , let  $\alpha^*(\beta f) = \alpha_M(\beta) \alpha^*(f)$ . Note that  $\alpha^*(g_{m,n}) = g_{m,n}$ .

Now let  $\theta_M = \phi^* \circ \alpha^*$ . Then  $\Theta_M$  extends  $\Phi_M$ . We wish to show that  $\Theta_M$  is an automorphism of  $H$ . First, we check that  $\Theta_M(B) = B$ . By definition,

$$\begin{aligned} \Theta_M(i, j, k) &= \phi^*(\alpha^*((i, j, k))) \\ &= \phi^*(M(i, j, k)) \\ &= M(i, j, k). \end{aligned}$$

Next, we will show that  $\Theta_M(q) = q$ , finishing the proof. Let  $y \in B$ . Then

$$\begin{aligned}\Theta_M(q)(y) &= \phi^*(\alpha^*(q)(y)) \\ &= \phi^*(q)(\alpha_M^{-1}(y)) \\ &= \phi^+(q(\alpha_M^{-1}(y)))\end{aligned}$$

So for  $y = (0, 0, 0)$  we get

$$\begin{aligned}\Theta_M(q)(y) &= \phi^+(q(\alpha_M^{-1}(y))) \\ &= \phi^+(q(y)) \\ &= \phi^+(z) \\ &= z,\end{aligned}$$

and for  $y = -b_{m,n}$  we get

$$\begin{aligned}\Theta_M(q)(y) &= \phi^+(q(\alpha_M^{-1}(-b_{m,n}))) \\ &= \phi^+(q(-b_{M^{-1}(m,n)})) \\ &= \phi^+(f_{M^{-1}(m,n)}) \\ &= f_{m,n}.\end{aligned}$$

Finally, for any other  $y$ , we know that  $M^{-1}(y)$  is not one of the previous values, so

$$\begin{aligned}\Theta_M(q)(y) &= \phi^+(q(\alpha_M^{-1}(y))) \\ &= \phi^+(e_P) \\ &= e_P.\end{aligned}$$

□

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