COUNTABLE BOREL QUASI-ORDERS

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ABSTRACT OF THE DISSERTATION

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In recent years, much work in descriptive set theory has been focused on the Borel complexity of naturally occurring classification problems, in particular, the study of countable Borel equivalence relations and their structure under the quasi-order of Borel reducibility. Following the approach of Louveau and Rosendal in [25] for the study of analytic equivalence relations, we study countable Borel quasi-orders.

We are largely concerned in this thesis with universal countable Borel quasi-orders, i.e. countable Borel quasi-orders above all other countable Borel quasi-orders with regard to Borel reducibility. We first establish that there is a universal countable Borel quasi-order, using a Feldman-Moore-type result for countable Borel quasi-orders and an argument similar to that of Dougherty, Jackson, and Kechris in [5]. We then establish that several countable Borel quasi-orders are universal. An important example is an embeddability relation on descriptive set theoretic trees. This is used in many of the other proofs of universality.

Our main result is Theorem 5.5.2, which states that embeddability of finitely generated groups is a universal countable Borel quasi-order, answering a question of Louveau and Rosendal in [25]. This immediately implies that biembeddability of finitely generated groups is a universal countable Borel equivalence relation. Although it may

have been possible to prove this only using results on countable Borel equivalence relations, the use of quasi-orders seems to be the most direct route to this result. The proof uses small cancellation theory. The same techniques are also used to show that embeddability of countable groups is a universal analytic quasi-order.

Finally, we discuss the structure of countable Borel quasi-orders under Borel reducibility, and we present some open problems.

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Contents

Chapter 1

Introduction to Countable Borel Equivalence Relations and Quasi-Orders

Mathematicians often work to classify the objects of their study up to some notion of equivalence, for example classifying finitely generated groups up to isomorphism, or unitary operators on infinite-dimensional Hilbert space up to unitary equivalence. Descriptive set theory gives us tools to compare the relative complexity of many naturally occurring classification problems throughout mathematics.

In this thesis, we are particularly interested in the complexity of classifying finitely generated groups up to biembeddability. In this case, it is more natural to study embeddability of finitely generated groups, which is a quasi-order and not an equivalence relation. Previously, in [25], Louveau and Rosendal studied the class of analytic quasi-orders to prove results about the analytic equivalence relations which arise by symmetrizing them. Following their approach, we introduce and study the class of countable Borel quasi-orders. Not only is this a natural way to study the complexity of the equivalence relations which arise from these quasi-orders, such as biembeddability of finitely generated groups, but also this class of relations is interesting in its own right.

The rest of this chapter introduces the concepts mentioned above in greater detail. After setting out some of the basic definitions used in descriptive set theory, the use of descriptive set theory to study classification problems and how they are related is explained in more detail. This is followed by a discussion of the countable Borel equivalence relations. Finally there is a section on quasi-orders and how they fit into the descriptive set-theoretic framework. The precise statements of the results of this thesis can be found in Chapter 2.

1.1 Descriptive set theory basics

Descriptive set theory provides a framework for the study of classification problems from throughout mathematics. The core of descriptive set theory is the study of "definable" subsets of Polish spaces. A *Polish space* is a separable topological space which can be equipped with a compatible complete metric. Examples of Polish spaces include $\mathbb{R}$ and $\mathbb{R}^n$ with their standard topologies, $[0,1]$ with the subspace topology, $\{0,1\}^{\mathbb{N}}$ (written $2^{\mathbb{N}}$) with the product topology, and the p-adics $\mathbb{Q}_p$ with their usual topology.

A subset of a Polish space X is *Borel* if it is a member of the smallest σ-algebra containing the open sets of X, and *analytic* if it is a continuous image of a Borel set. Note that every Borel set is analytic, although the converse is not true. These collections of sets will be sufficient for the purposes of this thesis.

We will also need the notion of a *standard Borel space*, a measurable space $(X, \mathcal{S})$ for which there is a Polish topology $\mathcal{T}$ on X that gives rise to $\mathcal{S}$ as its Borel σ-algebra. Obviously every Polish space equipped with the induced Borel σ-algebra is a standard Borel space. It can be shown that every Borel subset Y of a standard Borel space $(X, \mathcal{B})$ is a standard Borel space when equipped with the Borel sets

$$\mathcal{B} \restriction Y = \{A \cap Y \mid A \in \mathcal{B}\}.$$

Note that the Polish topology placed on Y is not necessarily the subspace topology which it inherits from X. For example, open intervals in $\mathbb{R}$ are standard Borel spaces, but are not complete in the subspace topology.

If $(X, \mathcal{B}_X), (Y, \mathcal{B}_Y)$ are standard Borel spaces, we call $f \colon X \to Y$ *Borel* if for every Borel set $U \subseteq Y$, $f^{-1}(U)$ is a Borel subset of X. This is equivalent to requiring that the graph of f is a Borel subset of $X \times Y$. It follows that if $f \colon X \to Y$ is Borel and a bijection then $f^{-1} \colon Y \to X$ is Borel, as $\operatorname{graph}(f^{-1})$ is a Borel subset of $Y \times X$. In this case f is said to be a *Borel isomorphism*. It is a remarkable theorem of Kuratowski that every uncountable standard Borel space is Borel isomorphic to $\mathbb{R}$. (See Theorem 15.6 in [22].) For more background on descriptive set theory, see [22] or [34].

1.2 Classification problems and equivalence relations

Many collections of mathematical objects may be viewed as forming a Polish space X, and then a classification problem corresponds to understanding an equivalence relation E on X. For example consider the problem of classifying the subsets of $\mathbb{N}$ up to recursive isomorphism, written $\equiv_1$. Recall that a function from $\mathbb{N}$ to $\mathbb{N}$ is recursive if it is computable by a Turing machine, and sets $A, B \subseteq \mathbb{N}$ are said to be recursively isomorphic if there is a recursive permutation $\pi\colon \mathbb{N} \to \mathbb{N}$ such that $\pi(A) = B$. The set $\mathcal{P}(\mathbb{N})$ may be identified with the Polish space $2^{\mathbb{N}}$ in the obvious way. Recursive isomorphism is then an equivalence relation on $2^{\mathbb{N}}$, which may be viewed as a subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. Note that $\equiv_1$ is $\cup_n \operatorname{graph}(f_n)$, where the $f_n\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ are the maps induced by the recursive permutations of $\mathbb{N}$. As each f_n is continuous, their graphs are closed, and so $\equiv_1$ is a Borel subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$. This is a general phenomenon; many of the equivalence relations corresponding to natural classification problems, when viewed as subsets of the appropriate product spaces, are either Borel or analytic. We call such equivalence relations Borel or analytic.

Classifying the objects in X up to the equivalence relation E amounts to finding a set of complete invariants I and a map $c\colon X \to I$ such that

$$xEx' \iff c(x) = c(x').$$

This is the same as finding an injective map $c'\colon X/E \to I$, where X/E is the quotient space corresponding to E. If this is all that we require, then classification is trivial but useless. Most interesting examples of Borel equivalence relations have $2^{\aleph_0}$ distinct equivalence classes. By the Axiom of Choice we may assign each equivalence class a unique real number, but this classification is not useful if we have no idea what the assignment looks like.

1.3 Borel reductions

In order to avoid such difficulties, we restrict ourselves to more explicit classification maps. Borel sets and maps can be thought of as being in some sense explicit, and avoiding the "pathologies" of the Axiom of Choice. For example, every Borel set of reals is Lebesgue measurable and has the property of Baire, and every Borel map is Baire measurable. So we require our classification maps to be Borel.

An ideal classification scheme for an analytic equivalence relation E on a space X would be a Borel map $c\colon X \to \mathbb{R}$, assigning each E-class a unique real number. If E admits such a classification, then E is said to be *smooth*. Finding such a classification scheme is not always possible, since there are non-smooth equivalence relations. For example, define E_0 on $2^{\mathbb{N}}$ by

$$x \, E_0 \, y \iff \exists N \in \mathbb{N} \ \ \forall n \geq N \ \ (x(n) = y(n)).$$

In other words, E_0 is eventual equality of infinite binary sequences. Suppose $f\colon 2^{\mathbb{N}} \to \mathbb{R}$ is a Borel map such that $x \, E_0 \, y \Rightarrow f(x) = f(y)$. Then basic category or measure arguments can be used to show that there must exist $\alpha, \beta \in 2^{\mathbb{N}}$ such that $\alpha \not{E}_0 \beta$ and $f(\alpha) = f(\beta)$, meaning f does not give a complete classification for E_0.

This leads one to consider different types of invariants. We can use objects identified up to some equivalence relation to classify other objects up to an equivalence relation. This gives us a natural way to compare the complexity of various equivalence relations.

Definition 1.3.1. *Suppose that E, F are Borel (or analytic) equivalence relations on the standard Borel spaces $(X, \mathcal{B}_X), (Y, \mathcal{B}_Y)$ respectively. Then we say E Borel reduces to F, written $E \leq_B F$, if there exists a Borel map $f\colon X \to Y$ such that for all $x, x' \in X$*

$$x \, E \, x' \iff f(x) \, F \, f(x').$$

We write $E <_B F$ if $E \leq_B F$ and $F \not\leq_B E$. We write $E \sim_B F$ if $E \leq_B F$ and $F \leq_B E$, and say that E and F are Borel bireducible.

A Borel reduction can be seen as an explicit assignment of complete invariants from the quotient space Y/F to the objects in X. This means that if we have a classification scheme for the objects in Y up to F-equivalence, we can use the Borel reduction to turn it into a classification scheme for the objects in X up to E-equivalence. In light of this, if $E \leq_B F$, then we consider F to be at least as complicated as E.

One well-known example of these more general invariants is the classification of rank one torsion-free abelian groups due to Baer in [3] using sequences of elements from $\mathbb{N} \cup \{\infty\}$. Here two groups are isomorphic if and only if they are assigned sequences which eventually agree, and always agree where they equal ∞. It is easy to see this is an equivalence relation on such sequences.

Let $\mathcal{C}$ be a collection of equivalence relations. Then E is *universal* for $\mathcal{C}$ if $E \in \mathcal{C}$ and $F \leq_B E$ for every $F \in \mathcal{C}$. A universal equivalence relation for $\mathcal{C}$ can be thought of as being as complicated as possible among all equivalence relations in $\mathcal{C}$, lending it special significance. There may be several equivalence relations in $\mathcal{C}$ which are universal, but the definition easily implies that they are the same up to Borel bireducibility.

Thus far the only collections of equivalence relations that we have mentioned are the collections of all Borel equivalence relations and all analytic equivalence relations. It is a result of H. Friedman and L. Stanley in [9] that there is no universal Borel equivalence relation. In fact they prove something more:

Theorem 1.3.2. *Suppose E is a Borel equivalence relation on a Polish space X. Let $[x]_E$ denote the E-equivalence class of x and define E^+ on $X^{\mathbb{N}}$ by*

$$(x_n)\, E^+\, (y_n) \iff \{[x_n]_E\}_{n\in\mathbb{N}} = \{[y_n]_E\}_{n\in\mathbb{N}}.$$

Then E^+ is Borel and $E <_B E^+$.

On the other hand, there is a universal analytic equivalence relation. This is due to Becker and Kechris in section 3.5 of [4], and relies on the existence of what are known as universal analytic sets. There are several natural examples of universal analytic equivalence relations, including:

- Bi-embeddability of countable graphs. [25]
- Isometric bi-embeddability between Polish metric spaces. [25]
- Isomorphism of separable Banach spaces. [8]

1.4 Countable Borel equivalence relations

One class of equivalence relations of particular interest in this area is the class of countable Borel equivalence relations. Here a Borel equivalence relation is said to be *countable* if all of its equivalence classes are countable. Some simple examples are equality on $\mathbb{R}$, eventual equality of infinite binary sequences, recursive isomorphism of sets of natural numbers, and isomorphism of finitely generated groups. (We will define the space of finitely generated groups in section 3.2.)

Many countable Borel equivalence relations arise in practice as the orbit equivalence relations of Borel actions of countable groups. Let G be a countable group acting in a Borel way on a Polish space X. Then the orbit equivalence relation E_G^X is defined by

$$x E_G^X y \iff (\exists g \in G)\ \ g \cdot x = y.$$

It is easy to see that the first three equivalence relations listed in the previous paragraph can be realized as orbit equivalence relations of Borel actions of suitable countable groups. However, Turing equivalence $\equiv_T$ of sets of natural numbers is a countable Borel equivalence relation for which there is no obvious countable group G such that $\equiv_T$ coincides with $E_G^{2^{\mathbb{N}}}$. Nonetheless, we have the following result of Feldman and Moore [7]:

Theorem 1.4.1. *If E is a countable Borel equivalence relation on the Polish space X, then there is a countable group G which acts on X in a Borel way such that $E = E_G^X$.*

In [5], Dougherty, Jackson, and Kechris used the Feldman-Moore Theorem to show that there is a universal countable Borel equivalence relation. There are several natural examples of such an equivalence relation.

- The shift equivalence relation on subsets of $\mathbb{F}_2$, the free group on 2 generators, written E_∞. (Two subsets A, B of a group G are shift equivalent if there is some $g \in G$ such that $gA = B$.) [5]

- The conjugacy equivalence relation on subgroups of $\mathbb{F}_2$, or even on subgroups of groups embedding $\mathbb{F}_2$. [39], [11], [2]

- Isomorphism of finitely generated groups. [39]
- Arithmetic equivalence of subsets of $\mathbb{N}$. [27]

1.5 The structure of countable Borel equivalence relations under $\leq_B$

There has been a great deal of research concerning the structure of the countable Borel equivalence relations under $\leq_B$. Let $\Delta(X)$ denote the equality relation on X. If we let n denote a discrete n-element space, then n is Polish and $\Delta(n)$ is a countable Borel equivalence relation. Similarly $\Delta(\mathbb{N})$ is a countable Borel equivalence relation on $\mathbb{N}$. Clearly

$$\Delta(1) <_B \Delta(2) <_B \Delta(3) <_B \ldots <_B \Delta(\mathbb{N}).$$

By a result of Silver [32], if E is any Borel equivalence relation, either $E \leq_B \Delta(\mathbb{N})$ or $\Delta(\mathbb{R}) \leq_B E$. Again it is clear that $\Delta(\mathbb{N}) <_B \Delta(\mathbb{R})$. Thus $\Delta(\mathbb{R})$ is an immediate successor to $\Delta(\mathbb{N})$ under $\leq_B$.

Recall the definition of E_0 from section 1.3. Perhaps more surprising than Silver's theorem is the following result of Harrington, Kechris, and Louveau in [15]:

Theorem 1.5.1. *Suppose that E is a Borel equivalence relation. Then either*

$$E \leq_B \Delta(\mathbb{R}) \text{ or } E_0 \leq_B E.$$

We saw in section 1.3 that $\Delta(\mathbb{R}) <_B E_0$. Thus E_0 is an immediate successor to $\Delta(\mathbb{R})$ under $\leq_B$ among the Borel equivalence relations, and so also among the countable Borel equivalence relations. As previously mentioned, there are also universal countable Borel equivalence relations such as E_∞, the shift equivalence relation on subsets of $\mathbb{F}_2$. By an argument of Slaman and Steel [33], $E_0 <_B E_\infty$, and hence our picture so far is

$$\Delta(\mathbb{R}) <_B E_0 <_B E_\infty.$$

For some time, it was not known if the countable Borel equivalence relations were linearly ordered by $<_B$, or how many countable Borel equivalence relations there were. This question was settled in 2000 by Adams and Kechris in [1], where they show there are $2^{\aleph_0}$ countable Borel equivalence relations up to Borel bireducibility, and that the structure of the corresponding quasi-order is quite complicated.

Theorem 1.5.2. *There is a map $A \mapsto E_A$ assigning to each Borel subset $A \subseteq 2^{\mathbb{N}}$ an equivalence relation E_A such that $A_1 \subseteq A_2 \Leftrightarrow E_{A_1} \leq_B E_{A_2}$.*

The proof of this result relied on the Zimmer Cocycle Superrigidity Theorem [40], which implies that certain orbit equivalence relations remember information about the groups that created them. Since the Adams-Kechris result, many examples of countable Borel equivalence relations intermediate between E_0 and E_∞ have been found, often using superrigidity theorems. For example, let $\cong_n$ denote the isomorphism relation on torsion-free abelian groups of rank n. The result of Baer mentioned earlier shows that $\cong_1 \sim_B E_0$. Thomas showed in [35] that $\cong_n <_B \cong_{n+1}$ for all n, establishing that there was a natural strictly increasing countable chain of equivalence relations between E_0 and E_∞.

1.6 Special classes of countable Borel equivalence relations

A Borel equivalence relation F is *hyperfinite* if there is an increasing sequence $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$ of finite equivalence relations (i.e. equivalence relations with every class finite) such that $F = \cup_n F_n$. Clearly every hyperfinite Borel equivalence relation is countable. It is also easy to see that E_0 is hyperfinite. In fact, Dougherty, Jackson, and Kechris [19] showed that every non-smooth hyperfinite countable Borel equivalence relation is Borel bireducible with E_0. This is notable because there are several conditions equivalent to hyperfiniteness, but not obviously so. For example, in the early 80's Slaman and Steel [33] showed that E is hyperfinite if and only if E is induced by a Borel action of $\mathbb{Z}$. Since then, a great deal of work has been done to expand the collection of groups for which it is known that their Borel actions always induce hyperfinite equivalence relations. In particular, Gao and Jackson have shown in [13] that every Borel action of a countable abelian group induces a hyperfinite Borel equivalence relation. For example, define E_c on $\mathbb{R}^+$ by

$$x \, E_c \, y \iff x/y \in \mathbb{Q}^+.$$

This is induced by a Borel action of $\mathbb{Q}^+$ and hence must be hyperfinite by the Gao-Jackson result, although this fact is far from obvious.

Let X be a Polish space. A Borel graph on X is a graph with vertex set X whose edge set $\Gamma \subseteq X \times X$ is Borel. We identify the graph with its edge set. A countable Borel equivalence relation E on X is *treeable* if there is an acyclic Borel graph $\Gamma \subseteq X \times X$ for which the connected components of Γ are precisely the equivalence classes of E. Every hyperfinite equivalence relation is treeable, as one can use the fact that E arises from a Borel $\mathbb{Z}$-action to show that it is possible to associate a discrete linear order to each equivalence class in a Borel way, and together these form an acyclic Borel graph. Jackson, Kechris, and Louveau showed in [19] that there is a universal treeable equivalence relation $E_{\infty T}$, and that $E_0 <_B E_{\infty T} <_B E_\infty$. In particular there are non-hyperfinite treeable equivalence relations. Hjorth went on to show in [18] that there are $2^{\aleph_0}$ distinct treeable equivalence relations up to Borel bireducibility. In fact, Hjorth showed that there are $2^{\aleph_0}$ treeable equivalence relations which are pairwise incomparable with respect to Borel reducibility, although his proof did not produce any explicit examples of incomparable treeable equivalence relations. This remains an open problem.

A common thread through the analysis of all of these types of equivalence relations is the importance of various properties of the corresponding group actions, and the proofs often involve tools and results from ergodic theory and related fields, such as the superrigidity results mentioned earlier. Most of these techniques only apply in the case of a free Borel action of a countable group G on a standard Borel space X. Recall that a group action is *free* if

$$\forall g \in G, \forall x \in X \ \ (g \cdot x = x \rightarrow g = e).$$

A countable Borel equivalence relation E on a Borel set X is *free* if it is induced by a free Borel action of a countable group on X. A Borel equivalence relation F is *essentially free* if there is a free countable Borel equivalence relation E such that $F \leq_B E$.

All of the treeable equivalence relations are essentially free. However, in [38], Thomas showed that the class of essentially free equivalence relations does not admit a universal element, and so in particular E_∞ is not essentially free. In fact, he proved

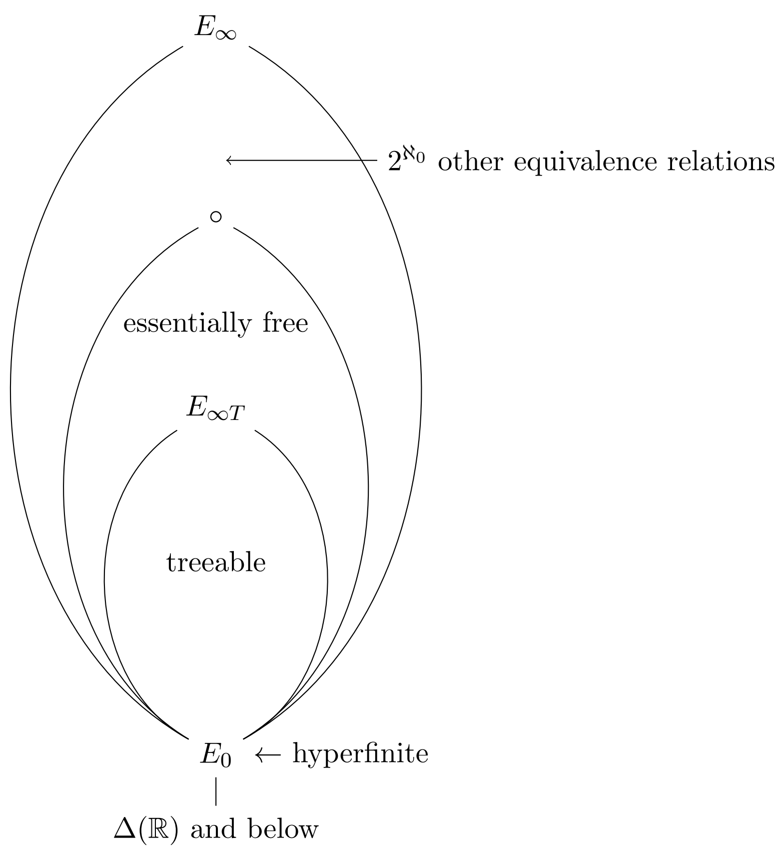

Figure 1.1: The countable Borel equivalence relations under $\leq_B$

there are $2^{\aleph_0}$ inequivalent essentially free non-treeable equivalence relations and $2^{\aleph_0}$ inequivalent non-essentially free equivalence relations. Thus the picture of the countable Borel equivalence relations looks like figure 1.1 (modeled after a figure in section 9.5 of [21]).

1.7 Quasi-orders

A quasi-order is a binary relation which is reflexive and transitive. Every quasi-order can be symmetrized to create an associated equivalence relation, i.e. if Q is a quasi-order, then there is an associated equivalence relation E_Q defined by

$$x \, E_Q \, y \iff x \, Q \, y \wedge y \, Q \, x.$$

We call E_Q the *symmetrization* of Q. Many naturally occurring equivalence relations arise most naturally by symmetrizing quasi-orders in this way.

For example, biembeddability of countable groups is most naturally seen as the symmetrization of the quasi-order of embeddability of countable groups. As another example, recall the notions of Turing reducibility $\leq_T$ and Turing equivalence $\equiv_T$ on $2^{\mathbb{N}}$. Here if $x, y \in 2^{\mathbb{N}}$, then $x \leq_T y$ if there is some Turing machine which can compute the digits of x, given an oracle for the digits of y. Then by definition $\equiv_T$ is $E_{\leq_T}$.

As with equivalence relations, many quasi-orders can be viewed as Borel or analytic subsets of standard Borel spaces, and the notion of Borel reducibility can also be defined for quasi-orders.

Definition 1.7.1. *Suppose that Q is a Borel (or analytic) quasi-order on a standard Borel space $(X, \mathcal{B}_X)$ and Q' is a Borel (or analytic) quasi-order on a standard Borel space $(Y, \mathcal{B}_Y)$. We say that Q is Borel reducible to Q', written $Q \leq_B Q'$, if there is a Borel function $f\colon X \to Y$ such that*

$$x \, Q \, y \iff f(x) \, Q' \, f(y).$$

We can once again think of $\leq_B$ as capturing a notion of relative complexity, so if Q and R are quasi-orders and $Q \leq_B R$, then we can think of R as being at least as complicated as Q. We can define Borel bireducibility and universality for a class of quasi-orders $\mathcal{C}$ as before. Quasi-orders which are universal for some class of quasi-orders often symmetrize to equivalence relations which are universal for a closely related class of equivalence relations. This is due to the following lemma.

Lemma 1.7.2. *Suppose F is an equivalence relation on X and Q is a quasi-order on Y. If $F \leq_B Q$, then $F \leq_B E_Q$.*

Proof. Suppose $f\colon X \to Y$ is a Borel reduction from F to Q. Then

$$\begin{aligned} f(x) \, Q \, f(y) &\iff x \, F \, y \\ &\iff y \, F \, x \\ &\iff f(y) \, Q \, f(x) \end{aligned}$$

Thus f is in fact a reduction of F to E_Q. $\square$

In [25], Louveau and Rosendal showed there is a universal analytic quasi-order $\leq_{max}$. By the last lemma, and the fact that every analytic equivalence relation is an analytic quasi-order, this means that $E_{\leq_{max}}$ is a universal analytic equivalence relation.

Louveau and Rosendal went on to present several natural examples of universal analytic quasi-orders, generally coming from embeddability notions. This explains why many of the universal analytic equivalence relations listed above are bi-embeddability relations. The connection between bi-embeddability relations and analytic equivalence relations was further explored by S. Friedman and L. Motto Ros in [10].

Louveau and Rosendal also looked at several other classes of quasi-orders, most notably K_σ quasi-orders; i.e. quasi-orders which are countable unions of compact sets. There is a universal K_σ quasi-order, which gives rise to a universal K_σ equivalence relation. Some examples include

- Embeddability of locally-finite combinatorial trees. [25]
- The quasi-order $\subseteq^{\mathbb{F}_2,t}_{\mathcal{P}(\mathbb{F}_2)}$ defined on subsets of $\mathbb{F}_2$ by

$$A \subseteq^{\mathbb{F}_2,t}_{\mathcal{P}(\mathbb{F}_2)} B \iff (\exists g \in \mathbb{F}_2)\ gA \subseteq B \quad [25].$$

- Surjectability of finitely-generated groups. [37]

In this thesis, we will largely be concerned with countable Borel quasi-orders. Here a quasi-order Q defined on a Polish space X is said to be *countable Borel* if Q is Borel when viewed as a subset of X^2 and for every $x \in X$, the set of predecessors of x, $\{y \mid yQx\}$, is countable. Clearly if Q is a countable Borel quasi-order, then the associated equivalence relation E_Q is also countable Borel. There are several natural examples of countable Borel quasi-orders. For example, if $A, B \subseteq \mathbb{N}$, then A is *1-reducible* to B, written $A \leq_1 B$, if there is a one-to-one recursive function $f\colon \mathbb{N} \to \mathbb{N}$ such that $f^{-1}(B) = A$. As there are only countably many recursive functions, 1-reducibility is a countable Borel quasi-order. So is Turing reducibility $\leq_T$ and many of the other standard computability-theoretic reducibilities. In addition, the embeddability relation on the space of finitely generated groups is a countable Borel quasi-order. We will see many other countable Borel quasi-orders throughout this thesis.

Chapter 2

Precise Statement of Results

We will begin by discussing the work of Louveau and Rosendal on analytic quasi-orders in more detail. They begin by proving that there is a universal analytic quasi-order, but the proof gives no real indication of what such a quasi-order might look like. They next find a universal analytic quasi-order with a simple combinatorial description. Finally, they use this to establish that several quasi-orders from various areas of mathematics are universal analytic quasi-orders.

Most of these universal analytic quasi-orders are given by an embeddability relation for some space of mathematical structures. Louveau and Rosendal noted that in many cases, if the embeddability relation is restricted to structures which are finitely generated or locally finite in some sense, then it becomes K_σ or compact. In many of these cases, the restricted quasi-orders are universal for the corresponding class of quasi-orders. This led Louveau and Rosendal to conjecture that embeddability of finitely generated groups, which we will write $\preccurlyeq_{em}$, was a universal K_σ quasi-order. (There is a formal definition of the Polish space of finitely generated groups and $\preccurlyeq_{em}$ in chapter 3.) Although the conjecture as stated is false, a slight modification of this conjecture is true, as we will prove later in the thesis.

To see that the stated conjecture is false, first observe that $\preccurlyeq_{em}$ is a countable quasi-order, since a finitely generated group has only countably many finitely generated subgroups. Thus biembeddability of finitely generated subgroups is a countable Borel equivalence relation. However, it is impossible for a countable Borel equivalence relation to be a universal K_σ equivalence relation. One reason for this is the following theorem of Kechris and Louveau in [23] regarding the equivalence relation E_1 on $(2^{\mathbb{N}})^{\mathbb{N}}$ of eventual equality of sequences of reals.

Theorem 2.0.3 (Kechris-Louveau). *Suppose that G is a Polish group with a Borel action on the standard Borel space X. Then $E_1 \not\leq_B E_G^X$.*

In particular, along with the Feldman-Moore Theorem this implies that E_1 does not Borel reduce to any countable Borel equivalence relation. As E_1 is a K_σ equivalence relation, it follows that a countable Borel equivalence relation cannot be a universal K_σ equivalence relation. Thus it is not possible for $\preccurlyeq_{em}$ to be a universal K_σ quasi-order. We will prove instead that $\preccurlyeq_{em}$ is a universal countable Borel quasi-order. Following Louveau and Rosendal, we will first show that there exists a universal countable Borel quasi-order, and then later deduce that $\preccurlyeq_{em}$ is also universal.

Unless otherwise noted, all of the results mentioned in this chapter are due to the author, and their proofs will be presented in chapter 5.

2.1 Finding a universal countable Borel quasi-order

Before we can show that $\preccurlyeq_{em}$ is a universal countable Borel quasi-order, we first show that such a quasi-order exists. There is no analog of a universal analytic set to take advantage of in this case, and so we must use a different approach. As mentioned in section 1.5, the existence of a universal countable Borel equivalence relation relies on the theorem of Feldman and Moore, and so we begin by establishing a similar result.

Theorem 2.1.1. *If $\preccurlyeq$ is a countable Borel quasi-order on the standard Borel space X, then there is a monoid M which acts on X in a Borel way such that*

$$x \preccurlyeq y \iff (\exists m \in M)\ x = m \cdot y.$$

Although to the author's knowledge, this result is not in the literature, it is a straightforward application of the well-known Lusin-Novikov theorem, and should perhaps be considered as folklore. It is proven in section 5.1. Thus every countable Borel quasi-order comes from the Borel action of a countable monoid.

Definition 2.1.2. *For every standard Borel space X and countable monoid M, the corresponding canonical Borel action of M on X^M is defined by $(m \cdot f)(s) = f(sm)$*

for $m, s \in M$ *and* $f \in X^M$*. We denote the corresponding quasi-order by* $\preccurlyeq_M^X$*, i.e. for* $f, g \in X^M$*,*

$$f \preccurlyeq_M^X g \iff (\exists m \in M)\ f = m \cdot g.$$

To see that this is an action, let $m, n \in M$ and $f \in X^M$. Then

$$\begin{aligned}(m \cdot (n \cdot f))(s) &= (n \cdot f)(sm) \\ &= f(smn) \\ &= (mn \cdot f)(s)\end{aligned}$$

as desired.

Definition 2.1.3 (The quasi-order $\preccurlyeq_\omega$)**.** *Let* M_ω *be the free monoid on countably many generators. Then define* $\preccurlyeq_\omega$ *to be* $\preccurlyeq_{M_\omega}^{2^{\mathbb{N}}}$*.*

Suppose that M is a countable monoid acting on a standard Borel space X. This defines a quasi-order $\preccurlyeq$ by

$$x \preccurlyeq y \iff x = m \cdot y.$$

Every countable monoid M is a homomorphic image of M_ω. Suppose that $f\colon M_\omega \to M$ is such a homomorphism. Then we can define an action of M_ω on X by

$$m \cdot x = f(m) \cdot x.$$

This defines the same quasi-order as the M-action, so every countable Borel quasi-order actually comes from an action of M_ω. Once this is established, it is straightforward to prove the next theorem.

Theorem 2.1.4. $\preccurlyeq_\omega$ *is a universal countable Borel quasi-order.*

Unfortunately $\preccurlyeq_\omega$ is difficult to work with, and so we follow the approach of [5] to obtain a simpler universal countable Borel quasi-order.

Theorem 2.1.5. *Let* M_2 *denote the free monoid on 2 generators, and* $2 = \{0, 1\}$*. Then* $\preccurlyeq_{M_2}^2$ *is a universal countable Borel quasi-order.*

Note that if M is in fact a group, then $\preccurlyeq_M^X$ is the equivalence relation E_M^X. Indeed, $E_{\mathbb{F}_2}^2$ is the same as E_∞, the universal countable Borel equivalence relation mentioned earlier. In the case of a group G, E_G^2 can be identified with the orbit equivalence relation of the shift action of G on $\mathcal{P}(G)$. For monoids which are not groups, we must be more careful. Let $m \in M$ and $f \in 2^M$. In general, if $g = m \cdot f$, then the values of g only depend on the values of f on some subset of its domain. For this reason, $\preccurlyeq_{M_2}^2$ is not quite as simple as it first appears. This leads us to consider alternative quasi-orders.

In descriptive set theory, a *tree* on X is a set $T \subseteq X^{<\mathbb{N}}$ such that if $t \in T$, and s is an initial segment of t, then $s \in T$. We write $s \subset t$ if s is an initial segment of t, and we write $s^\frown t$ for the concatenation of s and t. If X is a countable discrete space, then $Tr(X)$, the space of trees on X, is a closed subspace of $2^{X^{<\mathbb{N}}}$, and so is a Polish space. One can identify M_2 with the complete binary tree $2^{<\mathbb{N}}$, and doing so leads us to the following quasi-order.

Definition 2.1.6. *Given a countable discrete space X, the quasi-order $\preccurlyeq_X^{tree}$ on the space of trees on X, $Tr(X)$, is given by $T \preccurlyeq_X^{tree} T'$ if there exists $u \in X^{<\mathbb{N}}$ such that $T = T'_u$, where $T'_u = \{v \in X^{<\mathbb{N}} \mid u^\frown v \in T'\}$.*

In section 5.2 we will prove the following:

Theorem 2.1.7. *$\preccurlyeq_2^{tree}$ is a universal countable Borel quasi-order.*

With this quasi-order, we have finally arrived at a universal countable Borel quasi-order which is easy to describe and work with. This will make it easier to prove that other countable Borel quasi-orders are universal.

2.2 Some group-theoretic universal countable Borel quasi-orders

Consider E_∞, the universal countable Borel equivalence relation induced by the shift action of $\mathbb{F}_2$ on $\mathcal{P}(\mathbb{F}_2)$. We have seen that this is the same as the quasi-order $\preccurlyeq_{\mathbb{F}_2}^2$, and so our universal quasi-order $\preccurlyeq_{M_2}^2$ is a natural modification of E_∞, simply moving from a group action to the analogous monoid action.

At this point, it is natural to turn our attention to other quasi-orders which can be seen as modifications of E_∞ and hope to arrive at a universal quasi-order as we did before. The most obvious generalization is to look at the quasi-order $\subseteq^{\mathbb{F}_2,t}_{\mathcal{P}(\mathbb{F}_2)}$ on $\mathcal{P}(\mathbb{F}_2)$ defined by

$$A \subseteq^{\mathbb{F}_2,t}_{\mathcal{P}(\mathbb{F}_2)} B \iff (\exists g \in \mathbb{F}_2)\ gA \subseteq B.$$

Replacing the $\subseteq$ symbol on the right-hand side of the definition with the $=$ symbol gives E_∞. Unfortunately the above quasi-order is clearly not countable, and in fact has been shown to be a universal K_σ quasi-order (in [25]). As explained earlier, a universal K_σ quasi-order is more complex than any countable Borel quasi-order; and so we must generalize in another way, which leads us to the following quasi-order.

Definition 2.2.1. *If G is a countable group, then $\preccurlyeq^G_t$ is the countable Borel quasi-order on $\mathcal{P}(G)$ defined by*

$$A \preccurlyeq^G_t B \iff (\exists g_1, \ldots, g_n \in G)\ A = g_1 B \cap \ldots \cap g_n B.$$

In section 5.3 we will show

Theorem 2.2.2. *$\preccurlyeq^{\mathbb{F}_2}_t$ is a universal countable Borel quasi-order.*

Given a countable group G, the space $\mathrm{Sg}(G)$ of subgroups of G is a standard Borel space, on which G acts by conjugation. The orbit equivalence relation of this action, which we will write $E_c(G)$, is Borel. In [39], Thomas and Velickovic showed that $E_c(\mathbb{F}_2)$ is a universal countable Borel equivalence relation, essentially by coding E_∞ into $E_c(\mathbb{F}_2)$. Recall that a subgroup H of a group G is said to be malnormal if $gHg^{-1} \cap H = \{1\}$ for all $g \in G \setminus H$. If H is a malnormal subgroup of G then it is easily seen that $E_c(H) \leq_B E_c(G)$. It follows from the Thomas and Velickovic result that if G is a countable nonabelian free group, then $E_c(G)$ is a universal countable Borel equivalence relation.

As we have the quasi-order $\preccurlyeq^{\mathbb{F}_2}_t$ which is analogous to E_∞, we might expect to be able to use a similar coding to find a new universal countable Borel quasi-order. Rather than attempt to adapt the coding of Thomas and Velickovic, we will use a simpler coding of Gao [11], which he used to prove the following result.

Theorem 2.2.3 (Gao)**.** *If $G = K * H$, where K has a nonabelian free subgroup and H is nontrivial cyclic, then $E_c(G)$ is a universal countable Borel equivalence relation.*

We consider the following countable Borel quasi-order:

Definition 2.2.4. *Let G be a countable group. Then $\preccurlyeq_c^G$ is the countable Borel quasi-order on* $\mathrm{Sg}(G)$ *defined by*

$$A \preccurlyeq_c^G B \iff (\exists g_1, \ldots, g_n \in G)\ A = g_1 B g_1^{-1} \cap \ldots \cap g_n B g_n^{-1}.$$

In section 5.3 we will use a straightforward adaptation of the argument in [11] to prove:

Theorem 2.2.5. *Suppose that K is a countable group containing a nonabelian free subgroup and that H is a nontrivial cyclic group. Then $\preccurlyeq_c^{K*H}$ is a universal countable Borel quasi-order.*

2.3 Group embeddability

One can view $\preccurlyeq_2^{tree}$ as a type of embeddability relation, while the other universal quasi-orders that we have seen so far do not relate as clearly to embedding notions. Thus to show that $\preccurlyeq_{em}$ is a universal countable Borel quasi-order, it is most natural to attempt to show that $\preccurlyeq_2^{tree} \leq_B \preccurlyeq_{em}$.

Given a tree $T \in Tr(2)$, our general strategy is to create a finitely generated group G_T with subgroups corresponding to the trees T_w for $w \in 2^{<\mathbb{N}}$. We will start with two generators and then add other relations to this group according to the nodes present in T. The idea is to add relations which restrict the possible embeddings between these "tree-groups". We will use the results of small cancellation theory, which is discussed in chapter 4, in order to choose appropriate relations.

Before we prove anything about $\preccurlyeq_{em}$, which will be the focus of section 5.5, we will first discuss embeddability of countable groups, which we write as $\sqsubseteq_{Gp}$. By removing the restriction that the groups we work with should be finitely generated, we are allowed more freedom with regards to our construction, and the ideas we use when working with $\preccurlyeq_{em}$ can be seen more clearly. At the same time, removing this restriction means that $\sqsubseteq_{Gp}$ is an analytic quasi-order, rather than a countable Borel quasi-order.

In [25], Louveau and Rosendal showed that embeddability of countable graphs is a universal analytic quasi-order. In section 5.4 we will use small cancellation techniques to create countable groups whose relations encode countable graphs in such a way that embeddability of countable graphs reduces to $\sqsubseteq_{Gp}$, establishing the following theorem.

Theorem 2.3.1. *$\sqsubseteq_{Gp}$ is a universal analytic quasi-order.*

Emboldened by our success, we will use the same ideas to show that $\preccurlyeq_2^{tree} \leq_B \preccurlyeq_{em}$. The combinatorial details of the construction will be considerably more involved since we can only work with finitely many generators. Thus we will prove

Theorem 2.3.2. *$\preccurlyeq_{em}$ is a universal countable Borel quasi-order.*

2.4 The structure of countable Borel quasi-orders under $\leq_B$

As every countable Borel equivalence relation is a quasi-order, we know the structure of the countable Borel quasi-orders is quite complicated, thanks to the result of Adams and Kechris. However, this is not a satisfying answer to the question, since it completely ignores the asymmetric countable Borel quasi-orders.

Given a countable Borel equivalence relation E on a standard Borel space X with a Borel linear order $\leq$, define $E(\leq)$ to be $E \cap \leq$. Note that since any two uncountable Polish spaces are Borel isomorphic, we can put a Borel linear order on any Polish space X, for example by using a Borel isomorphism between $\mathbb{R}$ and X. Clearly $E(\leq)$ is an asymmetric countable Borel quasi-order (unless E is just equality on X, since $(\Delta(X))(\leq)$ is simply $(\Delta(X))$.). Note that

$$E(\leq) \leq_B F(\leq) \implies E \leq_B F$$

and so if $E(\leq)$ and $F(\leq)$ are Borel bireducible, then so are E and F. Thus by the result of Adams and Kechris we have the following theorem:

Theorem 2.4.1. *There are $2^{\aleph_0}$ quasi-orders of the form $E(\leq)$ up to Borel bireducibility.*

Every $E(\leq)$ symmetrizes to equality, i.e. the equivalence relation $E_{E(\leq)}$ is equality. Consequently, it would be nice to find examples of countable Borel quasi-orders which are not universal and do not symmetrize to a smooth countable Borel equivalence relation. A modest goal would be to find a quasi-order which symmetrized to a nonsmooth hyperfinite Borel equivalence relation.

The shift action of $\mathbb{Z}$ on $\mathcal{P}(\mathbb{Z})$ induces an equivalence relation, written $E_{\mathbb{Z}}$, which is Borel bireducible with E_0, since every Borel $\mathbb{Z}$-action induces a hyperfinite equivalence relation. Recall our definition of $\preccurlyeq_t^G$ in section 2.2, and let $E_t^{\mathbb{Z}}$ denote the symmetrization of $\preccurlyeq_t^{\mathbb{Z}}$. In section 5.6 we will prove

Theorem 2.4.2. $E_t^{\mathbb{Z}} = E_{\mathbb{Z}}$

Thus $\preccurlyeq_t^{\mathbb{Z}}$ is a countable Borel quasi-order which is not universal and does not symmetrize to a smooth countable Borel equivalence relation. Let E_G be the equivalence relation induced by the shift action of G on $\mathcal{P}(G)$. It is not true in general that $E_t^G = E_G$, and we will provide an explicit example of a group where the two equivalence relations differ.

2.5 Organization of this thesis

In chapter 3, we will discuss the space of finitely generated groups and the space of countable groups. We will show that the first is a Polish space and that the second is a standard Borel space. We will also show that embeddability of finitely generated groups is Borel, while embeddability of countable groups is analytic.

In chapter 4, we will discuss the basic results of small cancellation theory, and give a few examples of its use in proving results related to group embeddings.

In chapter 5 we will present proofs of all of the results mentioned in this chapter, as well as related lemmas, corollaries, etc. We will also discuss some of the open problems in this area.

Chapter 3

Spaces of groups

There are two spaces of groups which appear in this thesis: the space of countable groups and the space of finitely generated groups. Although the finitely generated groups form a Borel subspace of the space of countable groups, for technical reasons we will not be working with them as a subspace. Instead we will use a construction due to Grigorchuk in [14]. We will also show that $\preccurlyeq_{em}$ is a countable Borel quasi-order, and that $\sqsubseteq_{Gp}$ is an analytic quasi-order. We will start with the space of countable groups, which is simpler to describe.

3.1 The space of countable groups

When dealing with countable groups, we may always assume that the underlying set is $\mathbb{N}$. Let G be such a countable group. Then the multiplication table of G is the set

$$\circ_G = \{(k, l, m) \in \mathbb{N}^3 \mid k \circ l = m\}.$$

Thus each countable group may be identified with a subset of $\mathbb{N}^3$. The collection of subsets of $\mathbb{N}^3$ which correspond to multiplication tables is

$$Gp = \{x \in \mathcal{P}(\mathbb{N}^3) \mid \text{ the group axioms hold for } x\}$$

and this is easily seen to be a Borel subset of $\mathcal{P}(\mathbb{N}^3)$. Thus Gp is a standard Borel space.

3.1.1 Embeddability of countable groups is analytic

Let $I(\mathbb{N})$ denote the set of injections from $\mathbb{N}$ to $\mathbb{N}$. Recall that $\mathbb{N}^{\mathbb{N}}$ has the product topology that comes from giving each copy of $\mathbb{N}$ the discrete topology. In this topology,

$I(\mathbb{N})$ is a closed subset of $\mathbb{N}^{\mathbb{N}}$ and so is a Polish space when given the subspace topology. If $x \in \mathcal{P}(\mathbb{N}^3)$, $f \cdot x$ is the function defined by

$$f \cdot x(a, b, c) = x(f(a), f(b), f(c)).$$

Let $\sqsubseteq_{Gp}$ denote the quasi-order of embeddability of countable groups, i.e. for $G, H \in Gp$,

$$\begin{aligned} G \sqsubseteq_{Gp} H &\iff G \text{ embeds in } H \\ &\iff \exists f \in I(\mathbb{N}) \ \ (G = f \cdot H) \end{aligned}$$

Let $\Gamma \subset I(\mathbb{N}) \times Gp \times Gp$ be defined by

$$(f, x, y) \in \Gamma \iff x = f \cdot y.$$

We will show that Γ is Borel. If $x \neq f \cdot y$, then there is some (a, b, c) such that

$$y(f(a), f(b), f(c)) \neq x(a, b, c).$$

Suppose for example $y(f(a), f(b), f(c)) = 1$ and $x(a, b, c) = 0$. Then define

$$V = \{f\} \times \{x \mid x(a, b, c) = 0\} \times \{x \mid x(f(a), f(b), f(c)) = 1\}.$$

If we give $I(\mathbb{N})$ the subspace topology inherited from $\mathbb{N}^{\mathbb{N}}$ and Gp the subspace topology inherited from $2^{\mathbb{N}^3}$, then V is open. It follows that in this topology the complement of Γ is open and so Γ is closed. The Borel sets of Gp as a standard Borel space coincide with the Borel sets of Gp given the subspace topology, so Γ is a Borel subset of the standard Borel space $I(\mathbb{N}) \times Gp \times Gp$. Finally, $\sqsubseteq_{Gp}$ is the projection of Γ onto its second two coordinates, and so it is analytic.

3.2 The space of finitely generated groups

There are two equivalent ways to define the space of finitely generated groups.

A *marked group* (G, S) is a group G along with an ordered list of generators $S = (s_1, \ldots, s_n)$. If S contains n elements, then we say that (G, S) is marked by n elements. The list of generators need not be canonical or minimal in any sense, and may include repetitions or even the identity of G. Two marked groups $(G, (s_1, \ldots, s_n))$ and $(G', (s'_1, \ldots, s'_n))$ are identified if the map sending $s_1 \mapsto s'_1$, $s_2 \mapsto s'_2$, etc. extends to an isomorphism. We call such an isomorphism a marked isomorphism.

Definition 3.2.1. *For each n, the set of marked groups $\mathcal{G}_n$ is the set of groups marked by n elements identified up to marked isomorphism.*

Let $\mathbb{F}_n$ denote the free group on n generators $\{x_1, \ldots, x_n\}$. For every $(G, S) \in \mathcal{G}_n$, there is an epimorphism $\theta_{G,S} \colon \mathbb{F}_n \to G$ sending $x_1 \mapsto s_1$, $x_2 \mapsto s_2$, etc. Note that $(G, S), (G', S') \in \mathcal{G}_n$ are the same up to marked isomorphism if and only if $\ker_{G,S}(\theta) = \ker_{G',S'}(\theta')$. Thus we may identify $\mathcal{G}_n$ with the space $\mathcal{N}_n$ of normal subgroups of $\mathbb{F}_n$. It is easy to see that $\mathcal{N}_n$ is a closed subset of $2^{\mathbb{F}_n}$, and therefore is a compact Polish space.

Clearly $\mathcal{G}_n \hookrightarrow \mathcal{G}_{n+1}$ via the map

$$(G, (s_1, \ldots, s_n)) \mapsto (G, (s_1, \ldots, s_n, 1))$$

or equivalently the corresponding normal subgroup $N \in \mathcal{N}_n$ is mapped to the normal closure of $N \cup \{x_{n+1}\}$ in $\mathbb{F}_{n+1}$. Under this map, $\mathcal{G}_n$ maps to a clopen subset of $\mathcal{G}_{n+1}$. Thus it makes sense to define $\mathcal{G} = \cup_n \mathcal{G}_n$, and we call $\mathcal{G}$ the space of marked groups. As each $\mathcal{G}_n$ is compact, $\mathcal{G}$ is locally compact. Furthermore, as the union of Hausdorff, second countable spaces, it is also Hausdorff and second countable. Thus by [22, Theorem 5.3], $\mathcal{G}$ is a Polish space.

Let $\mathbb{F}_\infty$ be the free group on countably many generators $\{x_1, x_2, \ldots\}$. Our previous discussion shows that $\mathcal{G}_n$ may be identified with

$$\mathcal{N}_n = \{N \trianglelefteq \mathbb{F}_\infty \mid N \text{ contains } x_m \text{ for all } m > n\}.$$

Then we may identify $\mathcal{G}$ with $\mathcal{N} = \cup_n \mathcal{N}_n$; i.e. $\mathcal{N}$ is the set of normal subgroups $N \trianglelefteq \mathbb{F}_\infty$ such that N contains all but finitely many of the x_n.

3.2.1 Embeddability of finitely generated groups is Borel

If $G, H \in \mathcal{G}$, then we write $G \preccurlyeq_{em} H$ if and only if there is a group embedding from G into H. To prove this is Borel, it is easier to work with $\mathcal{N}$, and we will do this in what follows. In this space, if $A, B \in \mathcal{N}$, then we write $A \preccurlyeq_{em} B$ if and only if there is a group embedding from $\mathbb{F}_\infty/A$ into $\mathbb{F}_\infty/B$, or in other words if $\mathbb{F}_\infty/A$ is isomorphic to a subgroup of $\mathbb{F}_\infty/B$. Write $\preccurlyeq_{em}^n$ for $\preccurlyeq_{em} \restriction \mathcal{N}_n$. Note that if A, B are normal subgroups of $\mathbb{F}_n = \langle x_1, \ldots, x_n \rangle$, then $A \preccurlyeq_{em}^n B$ iff there exist $g_1, \ldots, g_n \in \mathbb{F}_n$ such that for any word

$w(x_1, \ldots, x_n)$ in the generators of $\mathbb{F}_n$, $w(x_1, \ldots, x_n) \in A \leftrightarrow w(g_1, \ldots, g_n) \in B$, where $w(g_1, \ldots, g_n)$ indicates the element of $\mathbb{F}_n$ that comes from replacing x_i in $w(x_1, \ldots, x_n)$ with g_i. Let $B_g = \{N \in \mathcal{N} \mid g \in N\}$, which is Borel in $\mathcal{N}_n$. Then

$$\preccurlyeq_{em}^n = \bigcup_{g_1, \ldots, g_n \in F_n} \bigcap_{w(\underline{x}) \in \mathbb{F}_n} (B_{w(\underline{x})} \times B_{w(\underline{g})}) \cup (B^c_{w(\underline{x})} \times B^c_{w(\underline{g})}).$$

This is a Borel set, and hence the union $\preccurlyeq_{em}$ of the $\preccurlyeq_{em}^n$ is Borel. To see that it is countable, simply note that any finitely generated group only has countably many finitely generated subgroups, and that there are only countably many ways of marking each subgroup.

Chapter 4

Small cancellation theory

Small cancellation theory has its roots in work of Dehn on the word problem for the fundamental groups of closed orientable surfaces. Recall that the word problem for a group presentation $G = \langle S \mid R \rangle$ is the problem of determining whether a word w in the generators S represents the identity. Dehn proved that the following algorithm would work to solve the word problem for fundamental groups of closed of closed orientable surfaces, which all have finite presentations with a single defining relation r. Suppose w is a word in the generators $\{s_1, s_2, \ldots, s_n\}$ for such a fundamental group G.

1. Remove all subwords of w of the form $s_i s_i^{-1}$ and $s_i^{-1} s_i$.

2. Suppose w contains more than half of some cyclic permutation r^* of r or r^{-1}, i.e. w contains a subword r_1 such that $r^* = t r_1 v$ and $|r_1| > \frac{1}{2}|r|$. Then replace r_1 in w with $t^{-1} v^{-1}$. This does not change the element represented by w, since r^* is a relation for G. Note that this replacement strictly decreases the length of w.

3. Return to step 1 until the operations in steps 1 and 2 cannot be performed.

4. If $w = 1$ then it represents the identity, otherwise it does not.

Dehn's algorithm shows that the only words in G that represent the identity are the obvious ones. For a general group presentation, one could attempt to use the same algorithm (modulo the obvious changes to deal with multiple relations). However, it will not always work. For example, any group with unsolvable word problem must have words which represent the identity which Dehn's algorithm does not reduce to 1.

Dehn's proof that his algorithm worked in the restricted case that he was looking at relied on the fact that if s is a cyclic permutation of r or r^{-1}, there is very little overlap between r and s, i.e. there is very little cancellation in the product rs^{-1}. Small cancellation theory tells us that given a presentation with defining relations of a particular form and a similar restriction on the overlaps between them, Dehn's algorithm may be used to solve the word problem for the group.

4.1 The small cancellation hypotheses and basic consequences

Let $G = \langle x_1, x_2, \ldots \mid r_1, r_2 \ldots \rangle$ be a group presentation. We will refer to the set of defining relations collectively as R. Recall that a word $w = y_1 \ldots y_n$ is cyclically reduced if w is freely reduced and y_1 is not the inverse of y_n. This is equivalent to saying that every cyclic permutation of w is freely reduced. We say the set of words R is *symmetrized* if every element of R is cyclically reduced and whenever $r \in R$, all cyclic permutations of r and r^{-1} are in R.

Suppose R is a cyclically reduced set of words on $\{x_1, x_2, \ldots\}$. Let $R' \supseteq R$ be the set obtained by closing R under inverses and cyclic permutations, so that R' is symmetrized. We call R' the *symmetrization* of R. Then

$$\langle x_1, x_2, \ldots \mid R \rangle \cong \langle x_1, x_2, \ldots \mid R' \rangle$$

since the normal closure of R must contain all of the words that we added to R to create R'. Thus restricting ourselves to symmetrized sets of relations does not change the set of groups we can consider.

In this thesis, we are largely concerned with the following small cancellation property, which quantifies the extent to which relators in particular group presentations overlap.

Definition 4.1.1. *A symmetrized set R in a free group F is said to satisfy the $C'(\lambda)$ small cancellation condition if for every pair of distinct $r_1, r_2 \in R$, if we can write $r_1 = bc_1$ and $r_2 = bc_2$, then $|b| < \lambda \min\{|r_1|, |r_2|\}$.*

The significance of this property can be seen in the following theorems (Theorem V.4.4 and Theorem V.10.1 in [26])

Theorem 4.1.2. *Let F be a free group. Let R be a symmetrized subset of F and N its normal closure. If R satisfies $C'(\lambda)$ for some $\lambda \leq 1/6$, then every non-trivial element $w \in N$ contains a subword s of some $r \in R$ with $|s| > (1 - 3\lambda)|r| \geq \frac{1}{2}|r|$.*

It follows immediately that if $G = \langle X \mid R \rangle$ is a group presentation satisfying the $C'(1/6)$ condition, then Dehn's algorithm succeeds in solving the word problem for G. This theorem also leads to other nice results on small cancellation groups.

Theorem 4.1.3. *Suppose that $G = \langle x_1, x_2, \ldots \mid R \rangle$ is such that R is a symmetrized subset of $\langle x_1, x_2, \ldots \rangle$ satisfying the $C'(1/6)$ small cancellation condition. If w represents a word of finite order in G, then there is some $r \in R$ of the form $r = v^n$ such that w is conjugate to a power of v.*

In other words, the only words in G which represent torsion elements are the obvious ones.

4.2 Small cancellation and embeddability results

Many theorems regarding group embeddings rely on small cancellation techniques in their proofs. This section offers a small sample. The following results will not be used later in this thesis, but instead are presented here to give an idea of how small cancellation is used in proofs about group embeddings.

Definition 4.2.1. *A countable group K is SQ-universal if every countable group can be embedded in a quotient group of K.*

The celebrated result of Higman, Neumann, and Neumann [17] that every countable group can be embedded in a two generator group may be interpreted as saying that $\mathbb{F}_2$ is SQ-universal. Small cancellation theory can be used to show a much wider class of groups is SQ-universal.

Theorem 4.2.2 (Schupp [31])**.** *Let P be any non-trivial free product $P = X * Y$, with the single exception of $P = C_2 * C_2$. Then P is SQ-universal. In fact, every countable group may be embedded in a simple quotient of P.*

The proof of this uses small cancellation theory on free products, which we have not specifically defined but which is similar to what we have discussed.

A group G is *hopfian* if every surjection from G onto G is an automorphism. A group G is *co-hopfian* if every injection from G into G is an automorphism. For example, $\mathbb{Z}$ is hopfian, since every subgroup of $\mathbb{Z}$ has finite index, and so any surjection from $\mathbb{Z}$ onto $\mathbb{Z}$ must have trivial kernel. However, $\mathbb{Z}$ is not co-hopfian, since for example $\mathbb{Z} \cong 2\mathbb{Z}$, which is a proper subgroup of $\mathbb{Z}$. An example of a hopfian and co-hopfian group is $\mathbb{Q}$. Since every non-zero element of $\mathbb{Q}$ is a root of a power of every other, it follows that every injection of $\mathbb{Q}$ into $\mathbb{Q}$ must also be onto, and every surjection of $\mathbb{Q}$ onto $\mathbb{Q}$ must have trivial kernel.

These two properties imply that the collection of endomorphisms of G is limited in some sense. Small cancellation theory gives us the tools to enforce such limitations on the endomorphisms of a group.

Theorem 4.2.3 (Miller and Schupp [29])**.** *Any countable group H can be embedded in a two-generator hopfian group G. If there is some $m \in \mathbb{N}$ such that H has no elements of order m, then G can be chosen to be co-hopfian as well.*

Here is a sketch of the proof. Let $H = \langle h_1, h_2, \ldots \rangle$ be a countable group and define $F = H * C_5 * C_7$. Write the generators of C_5 and C_7 as x and y respectively. Let

$$r_0 = xyxy^2(xy)^2xy^2 \ldots (xy)^{80}xy^2$$

and for $i = 1, 2, \ldots$ let

$$r_i = h_i^{-1} \prod_{j=80i+1}^{80(i+2)} ((xy)^j xy^2).$$

It is easy to see that the symmetrized set R generated by the r_i satisfies the $C'(1/10)$ condition. Let N be the normal closure of R in F and $G = F/N$.

It is a result of small cancellation theory for free products that H, C_5, and C_7 each embed into G. The r_i ensure that each h_i is equal to some word on x and y, so G is actually a quotient of $C_5 * C_7$.

Let $\psi\colon G \to G$ be a surjection. Then ψ is determined by its values on x and y, and $\psi(x)$ and $\psi(y)$ generate G. This implies that each is nontrivial, since otherwise G would be cyclic of order 5 or 7. By the generalization of Theorem 4.1.3 to free products, $\psi(x)$ must be in some conjugate of C_5 or H in F. Similarly $\psi(y)$ must have order 7 and be in some conjugate of C_7 or H in F.

From here, one need only show that in order for the r_i to map to relations for G, it must be that ψ is an inner automorphism. Hence by following ψ by an inner automorphism, we may assume that $\psi(x) \in C_5$ or $\psi(x) \in H$. Since ψ is onto, there is some word α in x and y such that $\psi(\alpha) = y$. Using the appropriate generalization of Theorem 4.1.2, one finds that $\psi(y) \in C_7$, since if $\psi(y) \notin C_7$ then it would not be possible for $\psi(\alpha)$ to contain enough of a relator so that $y^{-1}\psi(\alpha) \in N$. Arguing similarly for x, one finds that $\psi(x) \in C_5$. At this point we have $\psi(x) = x^\delta$ and $\psi(y) = y^\gamma$ with neither equal to 1. By looking at $\psi(r_0)$ and then again using the appropriate generalization of Theorem 4.1.2, one finds that $\psi(x) = x$ and $\psi(y) = y$.

Now suppose that H has no elements of order 5. (More generally, if H has no elements of order m, then we can do this same proof with $F = H * C_m * C_q$, where q is some prime greater than m.) Then if θ is an injection from G into G, we examine $\theta(x)$ and $\theta(y)$. Because H has no elements of order 5, the appropriate generalization of 4.1.3 implies that, up to an inner automorphism, $\theta(x) \in C_5$. From here, the proof is similar to that in the previous paragraph.

The above proof heavily uses the fact that torsion elements of small cancellation groups are of a very particular form. This is what gives control over the surjections and injections from G to itself. We will encounter this idea of using torsion elements to control embeddings again in the next chapter.

Chapter 5

Proofs of results

5.1 A universal countable Borel quasi-order

We start by proving an analogue of the Feldman-Moore Theorem [7] for countable Borel quasi-orders. This is a straightforward application of the following theorem of Lusin and Novikov (see Theorem 18.10 in Kechris [22]):

Theorem 5.1.1 (Lusin-Novikov)**.** *Let X, Y be standard Borel spaces and let $P \subseteq X \times Y$ be Borel. If every section $P_x = \{y \mid (x, y) \in P\}$ is countable, then P can be written as $\cup_n P_n$, where each P_n is the graph of a partial Borel function.*

Theorem 5.1.2. *If $\preccurlyeq$ is a countable Borel quasi-order on the Polish space X, there is a monoid M which acts on X in a Borel way such that*

$$x \preccurlyeq y \iff (\exists m \in M)\; x = m \cdot y.$$

Proof. First, note that by definition for all $y \in X$, $\preccurlyeq_y = \{x \mid x \preccurlyeq y\}$ is countable, which implies the set $\preccurlyeq \subseteq X \times X$ has countable sections with respect to its second coordinate. By the Lusin-Novikov theorem, $\succcurlyeq = \cup_n f_n$, where each $f_n \colon E_n \to X$ is a Borel function, with $E_n \subseteq X$ Borel.

We can extend these to total functions on X by letting $f_n(y) = y$ for $y \in X \setminus E_n$. These functions are still Borel, and their union is still equal to $\succcurlyeq$ by reflexivity. We may also add the identity function to our collection without changing the union, again by reflexivity. With all this in place, the f_n generate a monoid M under composition, and M acts on X by $m \cdot x = m(x)$. If $x \preccurlyeq y$ then there exists $m \in M$ such that $x = m \cdot y = m(y)$, and the transitivity of $\preccurlyeq$ ensures that for all $m \in M$ and $x \in X$, $m \cdot x \preccurlyeq x$. $\square$

We wish to use this result to show that there is a universal countable Borel quasi-order. Our approach closely follows the proof of Dougherty, Jackson, and Kechris in [5] that there is a universal countable Borel equivalence relation. Recall the definition of $\preccurlyeq_\omega$ (Definition 2.1.3).

Theorem 5.1.3. *$\preccurlyeq_\omega$ is a universal countable Borel quasi-order.*

Proof. Let $\preccurlyeq$ be a countable Borel quasi-order on a Polish space X. By Theorem 5.1.2, there is a countable monoid M such that $\preccurlyeq$ is the quasi-order induced by a Borel action of M on X. Let $f\colon M_\omega \to M$ be a surjective homomorphism. Then we can define an action of M_ω on X by

$$m \cdot x = f(m) \cdot x.$$

This action is Borel and also induces $\preccurlyeq$, and so without loss of generality we may assume that $M = M_\omega$.

Let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of Borel sets in X which separates points. Then we define $\phi\colon X \to (2^{\mathbb{N}})^{M_\omega}$ by $x \mapsto \phi_x$, with

$$\phi_x(s)(i) = 1 \iff s \cdot x \in U_i.$$

We will show this is Borel. Let V be a basic open set in $(2^{\mathbb{N}})^{M_\omega}$. Then there is some $u \in 2^{<\mathbb{N}}$ and $(v, s) \in (\mathbb{N} \times M_\omega)^{<\mathbb{N}}$ such that $|u| = |(v, s)|$ and

$$V = \bigcap_{0\le k<|u|} \{f \in (2^{\mathbb{N}})^{M_\omega} \mid f(s_k)(v_k) = u_k\}.$$

Let A_k equal U_{v_k} if $u_k = 1$, the complement of U_{v_k} otherwise. Let $S_k\colon X \to X$ be the Borel map $S_k(x) = s_k \cdot x$ and define $B_k = (S_k)^{-1}(A_k)$. Then $\phi^{-1}(V) = \cap_{0\le k<|u|} B_k$ is a Borel set, and so ϕ is a Borel map.

Since the U_i separate points, we see that ϕ is injective. Furthermore, if $t \in M_\omega$, then $t \cdot \phi_x = \phi_{t\cdot x}$. To see this, let $s \in M_\omega$, and $i \in \mathbb{N}$. Then

$$\begin{aligned}
\phi_{t\cdot x}(s)(i) = 1 &\iff s \cdot t \cdot x \in U_i \\
&\iff \phi_x(st)(i) = 1 \\
&\iff t \cdot \phi_x(s)(i) = 1
\end{aligned}$$

Now suppose that $x \preccurlyeq y$. Then there exists $m \in M_\omega$ such that $x = m \cdot y$. It follows that $\phi_x = \phi_{m \cdot y} = m \cdot \phi_y$, and so $\phi_x \preccurlyeq_\omega \phi_y$. The same reasoning works in reverse, and hence $\phi_x \preccurlyeq_\omega \phi_y$ implies that $x \preccurlyeq y$. Thus ϕ is a Borel reduction. $\square$

Thus there exists a universal countable Borel quasi-order. Next we wish to find universal countable Borel quasi-orders which are easier to work with. We proceed by a series of easily proven lemmas which are the analogues of propositions 1.4-1.8 in Dougherty, Jackson, and Kechris [5]. The proofs are virtually the same, although we must take care to ensure that they also work for monoid actions (as opposed to group actions). We will use several quasi-orders of the form $\preccurlyeq_M^X$ (see Definition 2.1.2).

Lemma 5.1.4. *If M, N are monoids and M is a homomorphic image of N, then* $\preccurlyeq_M^X \leq_B \preccurlyeq_N^X$.

Proof. Let $\pi \colon N \to M$ be a surjective homomorphism. Then we define $f \colon X^M \to X^N$ by $f(p) = p^*$, where $p^*(h) = p(\pi(h))$. Then if $n \in N$,

$$\begin{aligned}(n \cdot p^*)(h) &= p^*(hn) \\ &= p(\pi(hn)) \\ &= p(\pi(h)\pi(n)) \\ &= \pi(n) \cdot p(\pi(h)) \\ &= (\pi(n) \cdot p)^*(h)\end{aligned}$$

So $n \cdot p^* = (\pi(n) \cdot p)^*$.

Now, if $p \preccurlyeq_M^X q$, then there is some $m \in M$ such that $p = m \cdot q$. Since π is surjective, there is some $n \in N$ such that $m = \pi(n)$. Thus $p = \pi(n) \cdot q$, and so $p^* = n \cdot q^*$ by the above, whence $p^* \preccurlyeq_N^X q^*$.

Next suppose that $p^* \preccurlyeq_N^X q^*$, so that $p^* = n \cdot q^* = (\pi(n) \cdot q)^*$ for some $n \in N$. Then for all $m \in M$, there exists $s \in N$ such that $m = \pi(s)$, and hence

$$\begin{aligned} p(m) &= p(\pi(s)) \\ &= p^*(s) \\ &= (\pi(n) \cdot q)^*(s) \\ &= (\pi(n) \cdot q)(\pi(s)) \\ &= (\pi(n) \cdot q)(m) \end{aligned}$$

So $p = \pi(n) \cdot q$, and f is a Borel reduction. $\square$

Lemma 5.1.5. *For any countable monoid M, $\preccurlyeq_M^{2^{\mathbb{Z}-\{0\}}} \leq_B \preccurlyeq_{M\times\mathbb{Z}}^3$.*

Proof. Define $f\colon (2^{\mathbb{Z}-\{0\}})^M \to 3^{M\times\mathbb{Z}}$ by $p \mapsto p^*$, where

$$p^*(s,n) = \begin{cases} p(s)(n) & \text{if } n \neq 0 \\ 2 & \text{if } n = 0 \end{cases}.$$

Suppose that $p = g \cdot q$. Then for $n \neq 0$,

$$\begin{aligned} ((g,0) \cdot q^*)(s,n) &= q^*(sg,n) \\ &= q(sg)(n) \\ &= (g \cdot q)(s)(n) \\ &= p(s)(n) \\ &= p^*(s,n) \end{aligned}$$

If $n = 0$, then $((g,0) \cdot q^*)(s,n) = q^*(sg,n) = 2 = p^*(s,n)$.

Now, if $(g,n) \cdot q^* = p^*$, then there are two cases. If $n = 0$, then everything is as above and $p = g \cdot q$. Suppose $n \neq 0$. Then we have $((g,n) \cdot q^*)(s,-n) = q^*(sg,0) = 2$, but $p^*(s,-n)$ is 0 or 1, a contradiction. Thus f is a Borel reduction. $\square$

Lemma 5.1.6. *For any countable monoid M, $\preccurlyeq_M^3 \leq_B \preccurlyeq_{M\times\mathbb{Z}_2}^2$.*

Proof. Define the map $p \in 3^M \mapsto p^* \in 2^{M \times \mathbb{Z}_2}$ by

$$p^*(m,i) = \begin{cases} 0 & \text{if } p(m) = 0 \\ 0 & \text{if } p(m) = 1, i = 0 \\ 1 & \text{if } p(m) = 1, i = 1 \\ 1 & \text{if } p(m) = 2 \end{cases}.$$

Suppose that $p = g \cdot q$. Then

$$((g,0) \cdot q^*)(s,i) = q^*(sg,i) = \begin{cases} 0 & \text{if } q(sg) = p(s) = 0 \\ 0 & \text{if } q(sg) = p(s) = 1, i = 0 \\ 1 & \text{if } q(sg) = p(s) = 1, i = 1 \\ 1 & \text{if } q(sg) = p(s) = 2 \end{cases},$$

and we see that this equals $p^*(s,i)$.

Now, suppose that we have $(g,i) \cdot q^* = p^*$. We wish to show that $g \cdot q = p$. We will break it down case-by-case.

Case $i = 0$**:** Suppose that $s \in M$ satisfies $p^*(s,0) = 0$. Then $((g,0) \cdot q^*)(s,0) = 0$ as well, and $p(s) = 0$ or 1 and $q(sg) = 0$ or 1. If also $p^*(s,1) = 0 = (g,0) \cdot q^*(s,1)$, then $p(s) = 0 = q(sg)$. If instead $p^*(s,1) = 1$, then $p(s) = 1 = q(sg)$. If $p^*(s,0) = 1 = (g,0) \cdot q^*(s,0)$, then $p(s) = 2 = q(sg)$. Thus $p = g \cdot q$.

Case $i = 1$**:** Suppose instead that $(g,1) \cdot q^* = p^*$. We look at the case that $p(s) = 1$ for some $s \in M$. Then $p^*(s,0) = 0 = q^*(sg,1)$ and $p^*(s,1) = 1 = q^*(sg,0)$. But then $q^*(sg,1) = 0$ can only happen if $q(sg) = 0$, and $q^*(sg,0) = 1$ can only happen if $q(sg) = 2$, so we have a contradiction. Thus $p(s) = 0$ or 2 for all $s \in M$.

Now suppose that $q(sg) = 1$ for some $s \in M$. Then $((g,1) \cdot q^*)(s,0) = 1$ and $(g,1) \cdot q^*(s,1) = 0$. This implies $p^*(s,0) = 1$, which can only happen if $p(s) = 2$, and $p^*(s,1) = 0$, which can only happen if $p(s) = 0$, another contradiction. So we find that $q(sg) = 0$ or 2 for all $s \in M$.

By the above, for any $s \in M$, if $p^*(s,0) = 1 = q^*(sg,1)$, then $p(s) = 2 = q(sg)$. If $p^*(s,1) = 0 = q^*(sg,0)$, then $p(s) = 0 = q(sg)$. Thus if either of these two cases occurs, we are guaranteed to have $p(s) = q(sg)$. Suppose that $p^*(s,0) = 0$ and $p^*(s,1) = 1$, the only remaining case. The first equality implies $p(s) = 0$, and the second implies $p(s) = 2$, a contradiction. Thus $p = g \cdot q$, and f is indeed a Borel reduction. $\square$

Lemma 5.1.7. *Let M_2 denote the free monoid on 2 generators. Then*

$$\preccurlyeq^2_{M_\omega} \leq_B \preccurlyeq^2_{M_2} .$$

Proof. We start by embedding M_ω into M_2 in order to view it as a submonoid of M_2. Let $M_\omega = \langle x_1, x_2, \ldots \rangle$ and $M_2 = \langle a, b \rangle$. We define our embedding by $e \mapsto e$ and $x_n \mapsto ab^n$ for all $n \in \mathbb{N}^+$.

Next we note that if $h \in M_2$, then we can canonically write h as a product $h = h'g$, with $g \in M_\omega$ and $h' \in M_2 \setminus M_\omega$, possibly with $g = e$ or $h' = e$, by finding the longest word in M_ω at the end of h. Define $L \colon M_2 \to \mathbb{N}$ by

$$L(h) = \text{ the length of } h'$$

where $h = h'g$ is the canonical form of h. This function has the desirable property that multiplying an element $h \in M_2$ on the right by an element $g \in M_\omega$ does not change the given length, i.e. $L(h) = L(hg)$.

Define $f \colon 2^{M_\omega} \to 2^{M_2}$ by $p \mapsto p^*$ where

$$p^*(h) = \begin{cases} p(h) & \text{if } L(h) = 0 \\ 1 & \text{if } L(h) = 1 \\ 0 & \text{if } L(h) > 1 \end{cases} .$$

Suppose that $p \preccurlyeq^2_{M_\omega} q$. Then $\exists g \in M_\omega$ such that $p = g \cdot q$. So if $h \in M_\omega$,

$$\begin{aligned} (g \cdot q^*)(h) &= q^*(hg) \\ &= q(hg) \\ &= (g \cdot q)(h) \\ &= p(h) \\ &= p^*(h) \end{aligned}$$

If $h \in M_2 \setminus M_\omega$, then since $L(h) = L(hg)$, we find

$$\begin{aligned}(g \cdot q^*)(h) &= q^*(hg) \\ &= p^*(h)\end{aligned}$$

So $p^* \preccurlyeq^X_{M_2} q^*$.

Now suppose that $p^* \preccurlyeq^2_{M_2} q^*$. Then there exists $g \in M_2$ such that $p^* = g \cdot q^*$. Clearly if $g \in M_\omega$, then $p = g \cdot q$. If instead $g \in M_2 \setminus M_\omega$ with $L(g) = n$, $n \geq 1$, then we should have that $p^*(b) = (g \cdot q^*)(b) = q^*(bg)$. But $p^*(b) = 1$, while $L(bg) > 1$, and so $q^*(bg) = 0$. Thus this case cannot happen, and hence f is a Borel reduction. $\square$

Theorem 5.1.8. $\preccurlyeq_\omega \leq_B \preccurlyeq^2_{M_2}$. *It follows that* $\preccurlyeq^2_{M_2}$ *is universal.*

Proof. Using the preceding lemmas, we find that

$$\begin{aligned}\preccurlyeq^{2^{\mathbb{N}}}_{M_\omega} &\leq_B \preccurlyeq^{2^{\mathbb{Z}-\{0\}}}_{M_\omega} \\ &\leq_B \preccurlyeq^3_{M_\omega \times \mathbb{Z}} \text{ by Prop. 5.1.5} \\ &\leq_B \preccurlyeq^2_{M_\omega \times \mathbb{Z} \times \mathbb{Z}_2} \text{ by Prop. 5.1.6} \\ &\leq_B \preccurlyeq^2_{M_\omega} \text{ by Prop. 5.1.4} \\ &\leq_B \preccurlyeq^2_{M_2} \text{ by Prop. 5.1.7}\end{aligned}$$

$\square$

The quasi-order $\preccurlyeq^2_{M_2}$ is easier to work with than $\preccurlyeq_\omega$, as both the monoid and the space being acted on are simpler. Using $\preccurlyeq^2_{M_2}$, we will find another universal countable Borel quasi-order, this one of a more combinatorial nature.

5.2 A quasi-order on trees

In this section, we will reduce $\preccurlyeq^2_{M_2}$ to $\preccurlyeq^{tree}_2$, the quasi-order on descriptive-set-theoretic trees defined in chapter 2. This has the advantage of moving us away from working with monoids and towards more classical areas of mathematics. We must first make a few intermediate reductions.

Definition 5.2.1. *The quasi-order* $\preccurlyeq^s_2$ *(the* s *is for "suffix") on* $\mathcal{P}(M_2)$ *is defined by*

$$A \preccurlyeq^s_2 B \iff (\exists m \in M_2)\ Am = B^m$$

where

$$B^m = B \cap M_2 m.$$

Remark 5.2.2. Note that if we made a similar definition for a group M, then we would always have that B^m, since in this case $Mm = M$. So this definition is only interesting when dealing with a monoid.

If we identify $\mathcal{P}(M_2)$ with 2^{M_2}, then this quasi-order is the same as $\preccurlyeq^2_{M_2}$. Writing it in this way brings out the fact that knowing a set $A \in \mathcal{P}(M_2)$ and that $A \preccurlyeq^2_{M_2} B$ only gives partial information about B. This differs from E_∞, the analogous equivalence relation, since knowing $A \in \mathcal{P}(\mathbb{F}_2)$ and that $A\, E_\infty\, B$ gives information about all of B.

Next, we modify this quasi-order slightly, in order to make it somewhat easier to work with.

Definition 5.2.3. *The quasi-order $\preccurlyeq^p_2$ (the p is for "prefix") on $\mathcal{P}(M_2)$ is defined by*

$$A \preccurlyeq^p_2 B \iff (\exists m \in M_2)\ mA = B_m$$

where

$$B_m = B \cap mM_2.$$

As before, this definition is only interesting when working with a monoid.

Theorem 5.2.4. $\preccurlyeq^s_2 \sim_B \preccurlyeq^p_2$

Proof. Every nontrivial element $w \in M_2$ may be written as $w = a^{n_0}b^{m_0}\dots a^{n_k}b^{m_k}$, where $n_i, m_j \in \mathbb{N}$, and only n_0 or m_k may be 0. Define $\bar{w} = b^{m_k}a^{n_k}\dots b^{m_0}a^{n_0}$, and $\bar{e} = e$. Then the bijection $f\colon M_2 \to M_2$ defined by $f(w) = \bar{w}$ induces a Borel bijection $f^*\colon \mathcal{P}(M_2) \to \mathcal{P}(M_2)$ such that if $Am = B^m$, then $\bar{m}f^*(A) = f^*(B)_{\bar{m}}$. Similarly, if $wf^*(A) = f^*(B)_w$, then $A\bar{w} = B^{\bar{w}}$. Thus f^* is a Borel reduction from $\preccurlyeq^s_2$ to $\preccurlyeq^p_2$. Since f^* is its own inverse, we see that it is also a Borel reduction from $\preccurlyeq^p_2$ to $\preccurlyeq^s_2$. $\square$

One can view M_2 as the complete binary tree $2^{<\mathbb{N}}$, with each word in M_2 corresponding to a node in the tree. From this point of view, when looking at $A \subseteq M_2$, we see that A_m is simply the set of words in A which are above the node corresponding

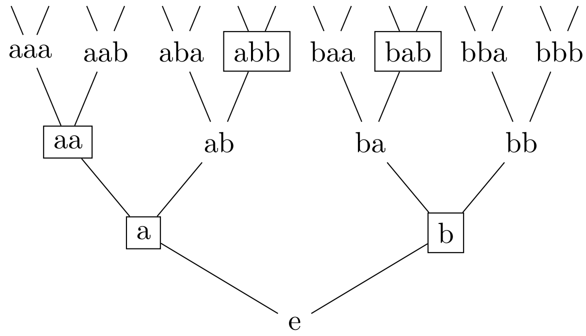

Figure 5.1: The set $A = \{a, b, aa, abb, bab, \ldots\}$ in the binary tree corresponding to M_2. Note that, for example, $A_{ba} = \{bab, \ldots\}$ is the set of words in A above ba.

to m. (See figure 5.1.) This natural interpretation of one of the sets involved in $\preccurlyeq_2^p$ in terms of trees leads us to consider the quasi-order $\preccurlyeq_X^{tree}$ from Definition 2.1.6. Recall that for a countable discrete space X, a tree on X is a (non-empty) collection of finite sequences of elements of X which is closed under initial segments. Let $\Lambda(X)$ be the Borel set of infinite trees in $Tr(X)$, the Polish space of trees on X.

Note that if we have $A, B \in \mathcal{P}(M_2)$ and $m \in M_2$ such that $mA = B_m$, and furthermore A, B are both infinite trees on $\{a, b\}$, then m witnesses that $A \preccurlyeq_{\{a,b\}}^{tree} B$. If A or B is not a tree, then it does not make sense to compare them using $\preccurlyeq_{\{a,b\}}^{tree}$, but this is only a minor difficulty, as we will see in the next proof.

Theorem 5.2.5. $\preccurlyeq_2^p \leq_B \preccurlyeq_3^{tree} \restriction \Lambda(3)$

Proof. Given $A \in \mathcal{P}(M_2)$, we define the tree $T_A \in Tr(3)$ as follows. We start with the complete binary tree $2^{<\mathbb{N}}$, and add to it the sequence $\hat{w}^\frown 2$ iff $w \in A$, where $\hat{w}$ is the sequence in $2^{<\mathbb{N}}$ corresponding to the word w in M_2. This collection is closed under subsequences and so is a tree. Clearly it is infinite. Define T_A to be this collection of sequences.

Suppose that $A \preccurlyeq_2^p B$. Then there exists $m \in M_2$ such that $mA = B_m$. First note that $2^{<\mathbb{N}}$ is contained in both T_A and $(T_B)_{\hat{m}}$. Next suppose that $w \in M_2$. Then

$$\begin{aligned} \hat{w}^\frown 2 \in T_A &\iff w \in A \\ &\iff m^\frown w \in B \\ &\iff \widehat{m^\frown w}^\frown 2 = \hat{m}^\frown \hat{w}^\frown 2 \in T_B \\ &\iff \hat{w}^\frown 2 \in (T_B)_{\hat{m}} \end{aligned}$$

So $T_A = (T_B)_{\hat{m}}$.

Conversely, suppose that $T_A = (T_B)_\alpha$ for some $\alpha \in 3^{<\omega}$. If α contains a 2, then $(T_B)_\alpha$ is $\{\emptyset\}$ or $\emptyset$, since the only sequences in T_B containing 2 are leaves of the tree. However, T_A is infinite. So $\alpha \in 2^{<\omega}$, which means that there is a word $w \in M_2$ such that $\hat{w} = \alpha$. Now

$$\begin{aligned} wx \in B &\iff \widehat{wx}^\frown 2 \in T_B \\ &\iff \hat{x}^\frown 2 \in (T_B)_{\hat{w}} \\ &\iff \hat{x}^\frown 2 \in T_A \\ &\iff x \in A, \end{aligned}$$

so $wA = B_w$. Thus the map $t\colon \mathcal{P}(M_2) \to Tr(3)$ sending A to T_A is a Borel reduction.

□

Finally, we will show that $\preccurlyeq_2^{tree}$ is universal.

Corollary 5.2.6. $\preccurlyeq_2^p \leq_B \preccurlyeq_2^{tree} \restriction \Lambda(2)$. *It follows that* $\preccurlyeq_2^{tree}$ *is universal.*

Proof. We define the map $c\colon 3^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ inductively. First let $c(e) = e$, $c(0) = 00$, $c(1) = 01$, and $c(2) = 10$. Now assume that c has been defined for all words of length $\leq n$, and let $w = x^\frown u$, where $x \in \{a, b, c\}$ and $u \in 3^{<\mathbb{N}}$ has length n. Define $c(w) = c(x)^\frown c(u)$. Given $t(A) \in Tr(3)$, where $t\colon M_2 \to Tr(3)$ is the Borel reduction from $\preccurlyeq_2^p$ to $\preccurlyeq_3^{tree} \restriction \Lambda(3)$ which was defined in the previous proof, apply c and close the resulting set under initial segments.

Suppose that $t(A) \preccurlyeq_3^{tree} t(B)$, so there exists $u \in 3^{<\mathbb{N}}$ (in fact, $u \in 2^{<\mathbb{N}}$) such that

$t(A) = t(B)_u$. Then

$$\begin{aligned} c(w) \in c(t(A)) &\iff w \in t(A) \\ &\iff u^\frown w \in t(B) \\ &\iff c(u^\frown w) = c(u)^\frown c(w) \in c(t(B)) \end{aligned}$$

Hence $c(t(A)) = c(t(B))_{c(u)}$ and thus $c(t(A)) \preccurlyeq_2^{tree} c(t(B))$.

Now suppose that $c(t(A)) \preccurlyeq_2^{tree} c(t(B))$, and so there exists $w \in 2^{<\mathbb{N}}$ such that $c(t(A)) = c(t(B))_w$. Suppose that w is not in the image of c. Then we either have $c(f(B))_w = \emptyset$, which is impossible, or w is an initial segment of odd length of something in the image of c. If w ends in a 0, then $100 \in c(t(B))_w$, but this is not in $c(t(A))$. If w ends in a 1, then $00 \notin c(t(B))_w$, but $00 \in c(t(A))$. Thus w is in the image of c, say $w = c(u)$. Then

$$\begin{aligned} u^\frown v \in t(B) &\iff c(u)^\frown c(v) \in c(t(B)) \\ &\iff c(v) \in c(t(A)) \\ &\iff v \in t(A) \end{aligned}$$

Thus $t(A) = t(B)_u$, and so $t(A) \preccurlyeq_3^{tree} t(B)$. □

5.3 Universal quasi-orders from group theory

We have seen that E_∞ is the same as the quasi-order $\preccurlyeq^2_{\mathbb{F}_2}$, and so our universal quasi-order $\preccurlyeq^2_{M_2}$ is a natural generalization of E_∞. At this point, we will turn our attention to other quasi-orders which can be seen as generalizations of E_∞. The most obvious generalization is the quasi-order $\subseteq^{\mathbb{F}_2,t}_{\mathcal{P}(\mathbb{F}_2)}$ on $\mathcal{P}(\mathbb{F}_2)$ defined by

$$A \subseteq^{\mathbb{F}_2,t}_{\mathcal{P}(\mathbb{F}_2)} B \iff (\exists g \in \mathbb{F}_2)\ gA \subseteq B.$$

Replacing the $\subseteq$ symbol on the right-hand side of the definition with the $=$ symbol gives E_∞. Unfortunately for our purposes, the above quasi-order is clearly not countable, and in fact has been shown to be a universal K_σ quasi-order (see Louveau-Rosendal [25]). Consequently, $\subseteq^{\mathbb{F}_2,t}_{\mathcal{P}(\mathbb{F}_2)}$ is much more complex than any countable Borel quasi-order. So we instead consider the quasi-order $\preccurlyeq^{\mathbb{F}_2}_t$ from Definition 2.2.1.

For any group G, let $\Omega(G)$ be the set of infinite subsets of G. In order to show that $\preccurlyeq_t^{\mathbb{F}_2}$ is a universal countable Borel quasi-order, we will reduce $\preccurlyeq_2^{tree} \restriction \Lambda(2)$ to $\preccurlyeq_t^{\mathbb{F}_2} \restriction \Omega(\mathbb{F}_2)$.

Every tree on 2 is isomorphic to a tree T on $\{a, b\}$, and these can easily be identified with subsets of $\mathbb{F}_2 = \langle a, b \rangle$. If we take a subset $T \subset \mathbb{F}_2$ corresponding to a tree and multiply it on the left by w^{-1}, then the positive words in $w^{-1}T$ are precisely T_w. Unfortunately, there is no natural way to pick out the positive words from $w^{-1}T$ simply by intersecting it with other shifts of T, and so we instead will define a set based on T for which T_w is easy to find simply by intersecting its shifts. In order to do this, we will look at subsets of $\mathbb{F}_\infty$, the free group on countably many generators. We list the generators of $\mathbb{F}_\infty$ as

$$\{a, b, x_a, x_b, x_{aa}, x_{ab}, x_{ba}, x_{bb}, x_{aaa}, \ldots\}.$$

Using the two generators a, b we identify T with a subset of the group, to which we add the sets $x_w w T_w$ for $w \in T$. Call this new set T'. Note that for all $w \in T$, $w^\frown T_w$ is a subset of T, and so $x_w w T_w \subseteq T'$. Then $T' \cap x_w^{-1} T' = wT_w$, since wT_w is the set of positive words in $x_w^{-1} T'$. We can then multiply by wT_w by w^{-1} to find T_w. However, the map sending T to T' is not a Borel reduction. Although we can now find T_w by intersecting shifts of T', T_w maps to $(T_w)'$, so that is the set we need to find. The following proof addresses this issue.

Theorem 5.3.1. $\preccurlyeq_t^{\mathbb{F}_\infty} \restriction \Omega(\mathbb{F}_\infty)$ *is a universal countable Borel quasi-order.*

Proof. We will construct the reduction in a few steps. We start with trees on $\{a, b\}$, which we then map to trees on $\{a, b, c, d\}$ for technical reasons. Next we define a map $f \colon \{a, b, c, d\}^{<\mathbb{N}} \to \mathcal{P}(\mathbb{F}_\infty)$, which will induce a map $F \colon Tr(\{a, b, c, d\}) \to \mathcal{P}(\mathbb{F}_\infty)$. The composition of these two maps will be our reduction.

If $T \in Tr(\{a, b\})$, define

$$t_a(T) = \{w \in T \mid w^\frown a \notin T\}.$$

Similarly define $t_b(T)$. These sets are elements of T which are "along the edge" of the tree, i.e. some immediate extension of these words is not in the tree. We define

$S\colon Tr(\{a,b\}) \to Tr(\{a,b,c,d\})$ by

$$S(T) = T \cup (t_a(T)^\frown c) \cup (t_b(T)^\frown d) \tag{5.1}$$

where $X^\frown z = \{x^\frown z \mid x \in X\}$. Here S "outlines" the tree using the letters c and d. The following property of S will be important later.

Lemma 5.3.2. *If $T, T' \in Tr(\{a,b\})$ and $S(T) \subseteq S(T')$, then $T = T'$.*

Proof. It is easily seen that $S(T) \subseteq S(T')$ implies $T \subseteq T'$, as

$$S(T) \cap \{a,b\}^{<\mathbb{N}} = T \text{ and } S(T') \cap \{a,b\}^{<\mathbb{N}} = T'.$$

Suppose $w \in \{a,b\}^{<\mathbb{N}} \setminus T$. Then there is some initial segment of $w' \subset w$ (possibly the empty string) and some $x \in \{a,b\}$ such that $w' \in t_x(T)$, i.e. $w = w'^\frown x^\frown t$, where $w' \in T$, $w'^\frown x \notin T$, and $t \in \{a,b\}^{<\mathbb{N}}$. Then $w'^\frown y \in S(T)$ for some $y \in \{c,d\}$, and so $w'^\frown y \in S(T')$. This is only possible if $w'^\frown x$ and all its extensions are not in T', and in particular $w \notin T'$. $\square$

We list the generators of $\mathbb{F}_\infty$ as

$$\{a, b, c, d, x_a, x_b, x_c, x_d, x_{aa}, x_{ab}, x_{ac} \ldots\}$$

i.e. every string in $\{a,b,c,d\}^{<\mathbb{N}}$ (except the empty string) has a unique generator associated to it in addition to generators corresponding to the letters in our trees. The empty string in $\{a,b,c,d\}^{<\mathbb{N}}$ and the identity element in $\mathbb{F}_\infty$ will both be written as e. This should not cause confusion, although both uses will appear close to each other. Finally, we recall that if $A, B \in \mathcal{P}(\mathbb{F}_\infty)$, then $AB = \{ab \mid a \in A, b \in B\}$. We can now define $f\colon \{a,b,c,d\}^{<\mathbb{N}} \to \mathcal{P}(\mathbb{F}_\infty)$ inductively.

$$
\begin{aligned}
f(e) &= \{e\} \\
f(a) &= \{a, x_a a\} \\
f(b) &= \{b, x_b b\} \\
f(c) &= \{c, x_c c\} \\
f(d) &= \{d, x_d d\} \\
f(w) &= \left(\bigcup_{\substack{w=s^\frown t \\ s,t\neq e}} f(s)f(t) \right) \cup \{x_w w\}
\end{aligned}
$$

The idea here is that every set $f(w)$ contains elements which encode the relation of w to its initial segments. Then define $F\colon Tr(\{a,b,c,d\}) \to \mathcal{P}(\mathbb{F}_\infty)$ by

$$F(S) = \bigcup_{w \in S} f(w).$$

There are a few helpful facts to record at this point. The simplest one is that $w \in f(w)$, which follows by a simple induction. The others we record as lemmas.

Lemma 5.3.3. *If $u, v \in \{a,b,c,d\}^{<\mathbb{N}}$ are not equal, the sets $f(u)$ and $f(v)$ are disjoint.*

Proof. Define the function $\Phi\colon \mathbb{F}_\infty \to \{a,b,c,d\}^{<\mathbb{N}}$ as

$$\Phi(g) = \text{the word in } \{a,b,c,d\}^{<\mathbb{N}} \text{ obtained by removing all other letters}$$
$$\text{from the freely reduced representation of } g.$$

By a simple inductive argument we see that for all $w \in \{a,b,c,d\}^{<\mathbb{N}}$, Φ is constant on $f(w)$ and equal to w. Thus the sets are disjoint. $\square$

Lemma 5.3.4. *If a word starting with x_w is in $f(u)$, then $w \subset u$.*

Proof. This follows from an straightforward induction on the length of u. $\square$

Lemma 5.3.5. *If $\gamma \in f(u)$ starts with $x_w w$ and $u = w^\frown t$, then $\gamma = x_w w \lambda$, with $\lambda \in f(t)$.*

Proof. If $t = e$, then $\gamma = x_w w$. Otherwise, there must be some α, β such that $u = \alpha^\frown\beta$ and $\gamma \in f(\alpha)f(\beta)$. We can then split γ into two words, $\gamma = \delta\lambda$, where δ starts with $x_w w$ and $\delta \in f(\alpha)$, while $\lambda \in f(\beta)$. By the previous lemma, $w \subset \alpha$, say $\alpha = w^\frown z$. Then $u = w^\frown z^\frown \beta$. We write $t = z^\frown\beta$. By induction, $\delta = x_w w\delta'$ with $\delta' \in f(z)$. Then $\gamma = x_w w\delta'\lambda$, and $\delta'\lambda \in f(z)f(\beta) \subseteq f(t)$ by definition. □

Recall the definition of the map S in (5.1). We define the map $G\colon Tr(\{a,b\}) \to \mathcal{P}(\mathbb{F}_\infty)$ by

$$G(T) = F(S(T)).$$

Lemma 5.3.6. *For all $w \in \{a,b\}^{<\mathbb{N}}$ and all nonempty $T \in Tr(\{a,b,c,d\})$,*

$$G(T) \cap x_w^{-1}G(T) = wG(T_w)$$

and hence

$$w^{-1}G(T) \cap (x_w w)^{-1}G(T) = G(T_w).$$

Proof. First, we will show that $G(T) \cap x_w^{-1}G(T) \subseteq wG(T_w)$. Every element of $G(T)$ is a positive word in the generators of $\mathbb{F}_\infty$, so any word not starting with x_w will be freely reduced in $x_w^{-1}G(T)$ and begin with x_w^{-1}, and thus not be in $G(T)$. So we need only focus on the words that start with x_w.

Suppose $g \in f(u) \subseteq G(T)$ and $g = x_w\alpha$ for some $\alpha \in \mathbb{F}_\infty$. By our inductive definition, this implies $g = x_w w\beta$ for some $\beta \in \mathbb{F}_\infty$. By Lemma 5.3.4, we must have $u = w^\frown t$ for some $t \in \{a,b,c,d\}^{<\mathbb{N}}$. By Lemma 5.3.5, $\beta \in f(t)$. Also, $w\beta$ is in $f(w)f(t)$, so $w\beta \in G(T)\cap x_w^{-1}G(T)$. In addition, $w\beta \in wG(T_w)$, since $t \in (S(T))_w = S(T_w)$ (since $w \in \{a,b\}^{<\mathbb{N}}$) and so $f(t) \subseteq G(T_w)$. Thus $G(T) \cap x_w^{-1}G(T) \subseteq wG(T_w)$.

If $g \in G(T_w)$, then there is some $u \in S(T_w)$ such that $g \in f(u)$. Then

$$x_w wg, wg \in f(w)f(u) \subseteq G(T)$$

so $wg \in G(T) \cap x_w^{-1}G(T)$. Thus $G(T) \cap x_w^{-1}G(T) \supseteq wG(T_w)$. □

Lemma 5.3.6 shows that for $T, S \in \Lambda(2)$, if $T \preccurlyeq_2^{tree} S$, then $G(T) \preccurlyeq_t^{\mathbb{F}_\infty} G(S)$. Next we check the other direction.

Suppose that $G(T) = g_1 G(T') \cap \ldots \cap g_n G(T')$. We know that $e \in G(T)$ (since T is nonempty), which means that each g_i must be an inverse of an element in $G(T')$, say $g_i^{-1} = h_i \in G(T')$. Fix some $1 \leq i \leq n$ and suppose that $\Phi(h_i) = w$, i.e. $h_i \in f(w)$. If $u \in S(T)$, then $x_u u \in G(T)$. This implies $h_i x_u u \in G(T') \cap f(w^\frown u)$, and in particular the intersection is nonempty, so $w^\frown u \in S(T')$. Thus $S(T) \subseteq S(T')_w$.

If $w \notin \{a, b\}^{<\mathbb{N}}$, then $S(T')_w$ is either empty or a single element, but $S(T)$ is infinite. Thus $w \in \{a, b\}^{<\mathbb{N}}$, and so $S(T')_w = S(T'_w)$. It follows that $S(T) \subseteq S(T'_w)$, and so by Lemma 5.3.2, $T = T'_w$. Thus G is a Borel reduction. This completes the proof of Theorem 5.3.1. $\square$

Corollary 5.3.7. $\preccurlyeq_t^{\mathbb{F}_2} \restriction \Omega(\mathbb{F}_2)$ *is a universal countable Borel quasi-order, and so* $\preccurlyeq_t^{\mathbb{F}_2}$ *is a universal countable Borel quasi-order.*

Proof. Let $\phi\colon \mathbb{F}_\infty \to \mathbb{F}_2$ be a monomorphism. Then ϕ induces a map

$$\Phi\colon \Omega(\mathbb{F}_\infty) \to \Omega(\mathbb{F}_2)$$
$$A \mapsto \{\phi(a) \mid a \in A\}$$

If $A, B \in \Omega(\mathbb{F}_\infty)$ and there exist $g_1, \ldots, g_n \in \mathbb{F}_\infty$ such that

$$A = g_1 B \cap \ldots \cap g_n B$$

then $\Phi(A) = \phi(g_1)\Phi(B) \cap \ldots \cap \phi(g_n)\Phi(B)$.

Conversely, suppose that

$$\Phi(A) = h_1 \Phi(B) \cap \ldots \cap h_n \Phi(B). \qquad (*)$$

If some h_i is not in the image of ϕ, then $h_i \Phi(B)$ is disjoint from any set in the image of Φ, and so the right hand side cannot equal the left hand side unless $\Phi(A) = \emptyset$, which is impossible. This implies that every h_i in $(*)$ is in the image of ϕ. It follows that $A = \phi^{-1}(h_1)B \cap \ldots \cap \phi^{-1}(h_n)B$. $\square$

Remark 5.3.8. The above proof shows that if G is any countable group containing $\mathbb{F}_2$ as a subgroup, then $\preccurlyeq_t^{G} \restriction \Omega(G)$ is a universal countable Borel quasi-order.

Recall that $E_c(G)$ denotes conjugacy equivalence relation on the standard Borel space $\mathrm{Sg}(G)$ of subgroups of G, i.e. for $A, B \in \mathrm{Sg}(G)$,

$$A\, E_c(G)\, B \iff (\exists g \in G)\, A = gBg^{-1}.$$

In [11], Gao used a simple coding technique to prove the following result.

Theorem 5.3.9 (Gao)**.** *If $G = K * H$, where K has a nonabelian free subgroup and H is nontrivial cyclic, then $E_c(G)$ is a universal countable Borel equivalence relation.*

In light of the relationship between E_∞ and $\preccurlyeq_t^{\mathbb{F}_2}$, it is natural to consider the countable Borel quasi-order $\preccurlyeq_c^G$ from Definition 2.2.4.

Let $\Gamma(G)$ be the standard Borel space of infinite subgroups of G. Then the proof of the following result is a straightforward adaptation of Gao's argument in [11].

Theorem 5.3.10. *Suppose that G is a countable group containing a nonabelian free subgroup and that H is a nontrivial cyclic group. Then $\preccurlyeq_c^{G*H} \restriction \Gamma(G*H)$ is a universal countable Borel quasi-order, and so $\preccurlyeq_c^{G*H}$ is a universal countable Borel quasi-order.*

Proof. Let $h \in H$ be a generator of H. We define the map $K \colon \Omega(G) \to \mathrm{Sg}(G*H)$ by

$$K(A) = \langle xhx^{-1} \colon x \in A \rangle.$$

This map is Borel (in fact, continuous). The basic open sets of $\mathrm{Sg}(G*H)$ are of the form

$$V(a_1, \ldots, a_n, b_1, \ldots, b_m) = \{L \in \mathrm{Sg}(G*H) \mid a_1, \ldots, a_n \in L \wedge b_1, \ldots, b_m \notin L\}.$$

Then $K^{-1}(V(a_1, \ldots, a_n, b_1, \ldots, b_m)$ is an open subset of $\mathcal{P}(G)$, and so is an open subset of $\Omega(G)$. We need only check that K is a Borel reduction from $\preccurlyeq_t^G \restriction \Omega(G)$ to $\preccurlyeq_c^{G*H}$. We will make use of the observation that $K(A) = \mathop{*}_{g \in A} gHg^{-1}$.

If $A, B \in \mathcal{P}(G)$, then clearly $K(A \cap B) \subseteq K(A) \cap K(B)$. We will show that $K(A) \cap K(B) \subseteq K(A \cap B)$. Suppose that $g \in K(A) \cap K(B)$, and so can be written both as $g = x_1 h x_1^{-1} \ldots x_n h x_n^{-1}$ with $x_1, \ldots, x_n \in A$ and as $g = y_1 h y_1^{-1} \ldots y_m h y_m^{-1}$ with $y_1, \ldots, y_m \in B$. Then clearly $x_1 = y_1$, and so multiplying g on the left by $y_1 h^{-1} y_1^{-1} = x_1 h^{-1} x_1^{-1}$ we find that

$$x_2 h x_2^{-1} \ldots x_n h x_n^{-1} = y_2 h y_2^{-1} \ldots y_m h y_m^{-1}.$$

Thus $x_2 = y_2$, and repeating this argument we find that $x_i = y_i$ for $1 \leq i \leq \min n, m$. If for example $m < n$, then we would have the equation

$$x_{m+1}hx_{m+1}^{-1} \dots x_n h x_n^{-1} = e$$

which is absurd. Thus $m = n$, and it follows that $g \in K(A \cap B)$. Thus $K(A \cap B) = K(A) \cap K(B)$.

Also note that if $g \in G$, then $K(gA) = gK(A)g^{-1}$. Thus

$$\begin{aligned} K(g_1 A \cap \dots \cap g_n A) &= K(g_1 A) \cap \dots \cap K(g_n A) \\ &= g_1 K(A) g_1^{-1} \cap \dots \cap g_n K(A) g_n^{-1} \end{aligned}$$

Suppose that $A, B \in \mathcal{P}(G)$ and that $A \preccurlyeq_t^G B$, i.e. there exist $g_1, \dots, g_n \in G$ such that $A = g_1 B \cap \dots \cap g_n B$. Then

$$K(A) = K(g_1 B \cap \dots \cap g_n B) = g_1 K(B) g_1^{-1} \cap \dots \cap g_n K(B) g_n^{-1}.$$

Thus $A \preccurlyeq_t^G B$ implies that $K(A) \preccurlyeq_c^{G*H} K(B)$.

Next, suppose that $A, B \in \mathcal{P}(G)$ and that $K(A) \preccurlyeq_c^{G*H} K(B)$, so there exist $\gamma_1, \dots, \gamma_n \in G * H$ such that $K(A) = \gamma_1 K(B) \gamma_1^{-1} \cap \dots \cap \gamma_n K(B) \gamma_n^{-1}$. For each $x \in A$ and $1 \leq i \leq n$, let $w_{x,i} \in K(B)$ be the element such that $xhx^{-1} = \gamma_i w_{x,i} \gamma_i^{-1}$. Clearly for each $i = 1, \dots, n$ the map $x \mapsto w_{x,i}$ is an injection.

Note that xhx^{-1} is a reduced word in $G * H$. For $1 \leq i \leq n$, we may assume that γ_i is a reduced word in $G * H$, and that $w_{x,i} \in K(B)$ can be written as

$$w_{x,i} = z_1 h^{\epsilon_1} z_1^{-1} \dots z_k h^{\epsilon_k} z_k^{-1} \ (z_j \in B, \epsilon_j \in \{\pm 1\}).$$

If we reduce this word, then we obtain that

$$w_{x,i} = u_1 h^{m_1} u_2 h^{m_2} \dots u_t h^{m_t} u_{t+1}$$

where $m_j \in \mathbb{Z} \setminus \{0\}$, $u_j \in G$ and the product $u_1 u_2 \dots u_j \in B$ for $1 \leq j \leq t+1$. Furthermore, $w_{x,i}$ is never the trivial word.

The equation $xhx^{-1} = \gamma_i w_{x,i} \gamma_i^{-1}$ implies that starting with the right-hand side, there is a cancellation procedure which eventually leads to the left-hand side. In any such procedure, there must be some occurrence of h in the right-hand side which is never cancelled. We call this the preserved occurrence of h. Let $\Delta_i \subseteq A$ be the set of elements $x \in A$ for which the preserved occurrence of h in some cancellation procedure is in the original expression for $w_{x,i}$.

We claim that $A \setminus \Delta_i$ is finite for each $1 \leq i \leq n$. If $x \in A \setminus \Delta_i$, then the preserved occurrence of h is in either γ_i or γ_i^{-1}. Suppose that $x_1, x_2 \in A \setminus \Delta_i$ are both words such that the preserved occurrence of h is in γ_i. Then the preserved occurrence of h must be the first h in γ_i, since γ_i is assumed to be reduced. Thus $\gamma_i = khu$ for some $k \in G, u \in G * H$, and this gives us the two equations

$$x_1 h x_1^{-1} = khuw_{x_1,i}\gamma_i^{-1}$$
$$x_2 h x_2^{-1} = khuw_{x_2,i}\gamma_i^{-1}$$

which implies that $x_1 = k = x_2$. Thus there is at most one element in $A \setminus \Delta_i$ such that the preserved occurrence of h is in γ_i. A similar argument shows that there is at most one element in $A \setminus \Delta_i$ such that the preserved occurrence of h is in γ_i^{-1}. So $|A \setminus \Delta_i| \leq 2$ for $1 \leq i \leq n$.

As A is infinite, this implies that each Δ_i must also be infinite. If we fix some $x_{0,i} \in \Delta_i$ for $i = 1, \ldots, n$, then we can write

$$\begin{aligned} x_{0,i} h x_{0,i}^{-1} &= \gamma_i w_{x_{0},i} \gamma_i^{-1} \\ &= \gamma_i u_{x_0,i}(z_{x_0,i} h z_{x_0,i}^{-1}) v_{x_0,i} \gamma_i^{-1} \end{aligned}$$

with $z_{x_0,i} \in B$, $u_{x_0,i}, v_{x_0,i} \in K(B)$, and the displayed h is the preserved occurrence in some cancellation procedure. This implies that $x_{0,i} = \gamma_i u_{x_0,i} z_{x_0,i}$, and $x_{0,i}^{-1} = z_{x_0,i}^{-1} v_{x_0,i} \gamma_i^{-1}$. Let $\beta_i = x_{0,i} z_{x_0,i}^{-1} \in G$. Then $\gamma_i = \beta_i u_{x_0,i}^{-1}$. Thus

$$\begin{aligned} K(A) &= \beta_1 u_{x_0,1}^{-1} K(B) u_{x_0,1} \beta_1^{-1} \cap \ldots \cap \beta_n u_{x_0,n}^{-1} K(B) u_{x_0,n} \beta_n^{-1} \\ &= \beta_1 K(B) \beta_1^{-1} \cap \ldots \cap \beta_n K(B) \beta_n^{-1} \\ &= K(\beta_1 B \cap \ldots \cap \beta_n B) \end{aligned}$$

and so $A = \beta_1 B \cap \ldots \cap \beta_n B$, with each $\beta_i \in G$, and so $A \preccurlyeq_t^G B$, as desired. $\square$

The following result is an immediate consequence of Theorem 5.3.10.

Corollary 5.3.11. *If $n \geq 3$, then $\preccurlyeq_c^{\mathbb{F}_n} \upharpoonright \Gamma(\mathbb{F}_n)$ is a universal countable Borel quasi-order.*

Finally, the proof of the following result is a straightforward adaptation of the proof of Proposition 1 of Thomas-Velickovic [39].

Corollary 5.3.12. *$\preccurlyeq_c^{\mathbb{F}_2} \upharpoonright \Gamma(\mathbb{F}_2)$ is a universal countable Borel quasi-order.*

Proof. Recall that a subgroup H of a group G is said to be malnormal if $gHg^{-1} \cap H = \{1\}$ for all $g \in G \setminus H$, and that $\mathbb{F}_3$ can be embedded as a malnormal subgroup of $\mathbb{F}_2$. Arguing as in Corollary 5.3.7, we see this embedding induces a Borel reduction from $\preccurlyeq_c^{\mathbb{F}_3} \upharpoonright \Gamma(\mathbb{F}_3)$ to $\preccurlyeq_c^{\mathbb{F}_2} \upharpoonright \Gamma(\mathbb{F}_2)$. $\square$

5.4 Embeddability of countable groups

Our ultimate goal is to show that embeddability of finitely generated groups is a universal countable Borel quasi-order. The techniques we will use in the proof are easier to understand in the more general setting of arbitrary countable groups. With this in mind, we first turn our attention to the embeddability relation for countable groups, $\sqsubseteq_{Gp}$. In chapter 3 we showed that there is a standard Borel space of countable groups and that $\sqsubseteq_{Gp}$ is an analytic quasi-order. In this section, we will show the following:

Theorem 5.4.1. *$\sqsubseteq_{Gp}$ is a universal analytic quasi-order.*

Corollary 5.4.2. *The bi-embeddability relation for countable groups $\equiv_{Gp}$ is a universal analytic equivalence relation.*

This is in contrast with the isomorphism relation for countable groups $\cong_{Gp}$, which is known to be universal among all analytic equivalence relations induced by a Borel action of S_∞. (This is due to Mekler in [28].) However, such equivalence relations are known not to be universal among all analytic equivalence relations. Several natural equivalence relations are Borel bireducible with $\cong_{Gp}$, including isomorphism of countable graphs, isomorphism of countable lattices, and isomorphism of countable linear orderings. For more about such equivalence relations, see Chapter 13 in [12].

Clearly $\equiv_{Gp}$-classes are unions of $\cong_{Gp}$-classes, and so it is natural to ask if there is a Borel way to select a particular $\cong_{Gp}$ class from each $\equiv_{Gp}$ class. Such a selection would be a Borel reduction of $\equiv_{Gp}$ to $\cong_{Gp}$, but by Theorem 5.4.1 we know $\cong_{Gp} <_B \equiv_{Gp}$, and hence there is no such selection.

Before we prove Theorem 5.4.1, we need to make a few definitions. We will write $\mathcal{C}$ for the set of countable graphs whose vertex set is $\mathbb{N}$. By identifying each graph with its edge relation, we see that $\mathcal{C}$ is a closed subset of $2^{\mathbb{N}^2}$ and so is a Polish space.

Definition 5.4.3. *If $S, T \in \mathcal{C}$, then we write $S \sqsubseteq_{\mathcal{C}} T$ if S embeds into T, i.e. there exists $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, $(m, n) \in S \Leftrightarrow (f(m), f(n)) \in T$.*

In [25], it was shown that $\sqsubseteq_{\mathcal{C}}$ is a universal analytic quasi-order. Thus to show that $\sqsubseteq_{Gp}$ is universal, we need only show that $\sqsubseteq_{\mathcal{C}}$ Borel reduces to it. To do this, we will use small cancellation techniques to create groups that encode the edge relations of graphs. In particular, we will need the following two theorems, which were discussed in Chapter 4.

Theorem 4.1.2. *Let F be a free group. Let R be a symmetrized subset of F and N its normal closure. If R satisfies $C'(\lambda)$ for some $\lambda \leq 1/6$, then every non-trivial element $w \in N$ contains a subword s of some $r \in R$ with $|s| > (1 - 3\lambda)|r| \geq \frac{1}{2}|r|$.*

Theorem 4.1.3. *Suppose that $G = \langle x_1, x_2, \dots \mid R\rangle$ is such that R is a symmetrized subset of $\langle x_1, x_2, \dots\rangle$ satisfying the $C'(1/6)$ small cancellation condition. If w represents a word of finite order in G, then there is some $r \in R$ of the form $r = v^n$ such that w is conjugate to a power of v.*

With these theorems in hand, we can proceed to the proof of Theorem 5.4.1.

Proof of Theorem 5.4.1. Let $T \in \mathcal{C}$ and let $v_0, v_1, \dots$ be an enumeration of the vertices of T. Then G_T is defined to be the group with generators $v_0, v_1, \dots$ and relators

- v_i^7 for all $i \in \mathbb{N}$
- $(v_i v_j)^{11}$ if $(v_i, v_j) \in T$
- $(v_i v_j)^{13}$ if $(v_i, v_j) \notin T$

Let R_T be the symmetrization of the set of defining relations for G_T. Note that if T is any graph, then R_T obviously satisfies the $C'(1/6)$ condition. Now suppose that $S, T \in \mathcal{C}$ are such that S embeds into T, say via the map f. Then f extends to a group homomorphism from G_S to G_T, as it sends the relations of G_S to relations of G_T.

To see that it is an embedding, let $\alpha = v_{i_1}^{k_1} v_{i_2}^{k_2} \ldots v_{i_n}^{k_n}$ be a word in the generators of G_S, so that

$$f(\alpha) = f(v_{i_1})^{k_1} f(v_{i_2})^{k_2} \ldots f(v_{i_n})^{k_n}$$

and suppose that $f(\alpha) = 1$ in G_T. Then by the $C'(1/6)$ condition, $f(\alpha)$ must contain more than $1/2$ of a relation in R_T. Note that any such relation must involve only generators in the image of the graph embedding $f\colon S \to T$. Suppose that $f(\alpha)$ contains more than half of a relation of the form $f(v_i)^{\pm 7}$. Since f is one-to-one, this cannot happen unless α already contained more than half of $v_i^{\pm 7}$.

Suppose that $f(\alpha)$ contains more than $1/2$ of the relation $(f(v_i)f(v_j))^k$, where the value of k depends on whether or not $(f(v_i), f(v_j)) \in T$. Since

$$(v_i, v_j) \in S \Leftrightarrow (f(v_i), f(v_j)) \in T$$

it must be the case that $(v_i v_j)^k \in R_S$, and α already contained more than $1/2$ of $(v_i v_j)^k$. Thus $f(\alpha)$ does not contain more than $1/2$ of a relation in R_T unless α contains more than $1/2$ of the corresponding relation in R_S. Since every nontrivial element in G_S may be written as a word which does not contain more than $1/2$ of a relation in R_S, every nontrivial element in G_S maps to a nontrivial element in G_T. Thus if S embeds into T, then G_S embeds into G_T.

Conversely, suppose that $\theta\colon G_S \to G_T$ is an embedding. Let $v_0, v_1, \ldots$ enumerate the vertices of S. By Theorem 4.1.3, after adjusting the embedding θ by an inner automorphism of G_T if necessary, we may assume $\theta(v_0) = t_0^k$ for some k such that $|k| < 7$, where t_0 is some vertex of T, since $\theta(v_0)$ must have order 7. Let $v_j \neq v_0$ be some vertex of S. Again by Theorem 4.1.3, we find that $\theta(v_j) = u t_j^l u^{-1}$ for some l such that $|l| < 7$, where $u \in G_T$ and t_j is some vertex of T. Unfortunately we cannot eliminate u by an inner automorphism without possibly changing the value of $\theta(v_0)$. Note that $\theta(v_0 v_j) = t_0^k u t_j^l u^{-1}$. We may assume that u is freely reduced and does not start with any power of t_0. To see this, note that if u began with t_0^m, then we would be able to follow θ by the inner automorphism corresponding to t_0^m without changing the value of $\theta(v_0)$. Thus $\theta(v_0v_j) = t_0^k u t_j^l u^{-1}$ is cyclically reduced. Since v_0v_j is a torsion element, so is $\theta(v_0v_j)$. By Theorem 4.1.3, the only cyclically reduced torsion elements in G_T are cyclically reduced conjugates of the relations in R_T, i.e. cyclic permutations of the relations in R_T. It immediately follows that $u = 1$, since no such words contain a mix of positive and negative powers. Thus $\theta(v_0v_j) = t_0^k t_j^l$.

From this we find that $t_0 \neq t_j$, since otherwise $\theta(v_0v_j)$ would have order 1 or 7, which is impossible since θ is an embedding and v_0v_j has order 11 or 13. Again, by Theorem 4.1.3, we find that $t_0^k t_j^l$ has finite order only if $k = l = \pm 1$. As the orders of v_0v_j and $\theta(v_0v_j) = t_0^{\pm 1} t_j^{\pm 1}$ are equal, we see that

$$(v_0, v_j) \in S \iff (t_0, t_j) \in T.$$

Let $v_m \neq v_n$ be arbitrary vertices in S. Repeating the above argument with v_0 and v_m, as well as v_0 and v_n, we find there are inner automorphisms ψ_1, ψ_2 of G_T, corresponding to conjugating by suitable powers of t_0, such that $\psi_1(\theta(v_m)) = t_m^{\pm 1}$ and $\psi_2(\theta(v_n)) = t_n^{\pm 1}$, where $t_m \neq t_0$ and $t_n \neq t_0$. A priori it may be the case that, for example, $\psi_1(\theta(v_n)) = t_0^k t_n^{\pm 1} t_0^{-k}$, with $k \neq 0$. But then $\psi_1(\theta(v_mv_n)) = t_m^{\pm 1} t_0^k t_n^{\pm 1} t_0^{-k}$ is cyclically reduced, and by Theorem 4.1.3 it has infinite order, which is impossible. Thus $\psi_1 = \psi_2$, and so $\psi_1(\theta(v_mv_n)) = t_m^{\pm 1} t_n^{\pm 1}$, and the above argument shows $t_m \neq t_n$ and that

$$(v_m, v_n) \in S \iff (t_m, t_n) \in T.$$

As v_m and v_n were arbitrary, the function $g\colon S \to T$ defined by $g(v_i) = t_i$ for all $i \in \mathbb{N}$ is an embedding. Thus $\sqsubseteq_{\mathcal{G}} \leq_B \sqsubseteq_{Gp}$, which establishes the result.

$\square$

5.5 Embeddability of finitely generated groups

We now turn our attention to the embeddability relation for finitely generated groups.

Definition 5.5.1. *Let $\mathcal{G}$ denote the Polish space of finitely generated groups. If $A, B \in \mathcal{G}$,* *then we write $A \preccurlyeq_{em} B$ if and only if there is a group embedding from A into B. We write $\equiv_{em}$ for the associated equivalence relation.*

Recall from section 3.2 that $\preccurlyeq_{em}$ is a countable Borel quasi-order. We will show that in fact it is universal by reducing $\preccurlyeq_2^{tree}$ to $\preccurlyeq_{em}$. Given a tree $T \in Tr(2)$, our general strategy is to define a finitely generated group G_T with subgroups corresponding to the trees T_w for $w \in 2^{<\mathbb{N}}$. We will start with two generators and then add relations to this group according to the nodes present in T. As in the previous section, these additional relations will allow us to control the embeddings that exist between two of these groups and thus ensure that $T \mapsto G_T$ is a Borel reduction. Thus we will have shown:

Theorem 5.5.2. *$\preccurlyeq_{em}$ is a universal countable Borel quasi-order.*

In order to define the relations of G_T, we first define the following two homomorphisms:

$$\begin{aligned} f_0 \colon \mathbb{F}_2 &\to \mathbb{F}_2 \\ x &\mapsto x^5 y \\ y &\mapsto y^5 x \end{aligned} \qquad\qquad \begin{aligned} f_1 \colon \mathbb{F}_2 &\to \mathbb{F}_2 \\ x &\mapsto x^2 yxyx \\ y &\mapsto y^2 xyxy \end{aligned}$$

We also define f_e to be the identity map. For any element $w \in 2^{<\mathbb{N}}$, we define f_w to be the corresponding composition of f_0 and f_1, e.g. $f_{01} = f_0 \circ f_1$ and $f_{110} = f_1 \circ f_1 \circ f_0$. In other words, if we can write w as $u^\frown v$, then $f_w = f_u \circ f_v$. The associativity of function composition ensures that this is well-defined.

These maps are not chosen entirely at random. One basic property of the maps is that for all $u \in 2^{<\mathbb{N}}$, the first letter of $f_u(a)$ is different for each $a \in \{x^{\pm 1}, y^{\pm 1}\}$, and the same is true for the last letter. This can be established through an easy induction on the length of u. If $u = e$, then this is immediate, and for $u = 0$ or $u = 1$, we quickly check that it holds. Now suppose that this is true for u. Then for $i \in \{0, 1\}$, consider $f_{u^\frown i}(a) = f_u(f_i(a))$. We have already seen the first and last letters of $f_i(a)$ are different for each a. By assumption, f_u takes the first and last letters of $f_i(a)$ to words with first and last letters different from those of $f_i(b)$ for any $b \neq a$ with $b \in \{x^{\pm 1}, y^{\pm 1}\}$, and this completes the induction.

With this established, a similar induction shows that every f_u takes freely reduced words to freely reduced words. In fact, every f_u takes cyclically reduced words to cyclically reduced words, since for $a, b \in \{x^{\pm 1}, y^{\pm 1}\}$, the first letter of $f_u(a)$ is the inverse of the last letter of $f_u(b)$ only if $b = a^{-1}$, by the uniqueness of the last letters.

Given $T \in Tr(2)$, we define

$$G_T = \langle x, y \mid \{(f_w(x))^{59}, (f_w(y))^{61} \mid w \in T\}, \{(f_w(x))^{67}, (f_w(y))^{71} \mid w \notin T\}\rangle.$$

The numbers in the exponents were chosen to be relatively prime and so that the relations satisfy small cancellation conditions, and they have no significance beyond that. We will eventually show that the map $T \mapsto G_T$ is a Borel reduction from $\preccurlyeq_2^{tree}$ to $\preccurlyeq_{em}$. To see that this is the case, we proceed by a series of lemmas.

Lemma 5.5.3. *Let $T \in Tr(2)$. If R_T denotes the symmetrization of the defining relations for G_T, then R_T satisfies the $C'(1/8)$ small cancellation condition.*

Proof. We only need to check the positive relations, since they satisfy the $C'(1/8)$ condition iff their inverses do, and there is no overlap between a positive word and a negative word.

We begin with an easy case. Suppose that $w \in 2^{<\mathbb{N}}$ and consider $f_w(x)^{n_x}$ and $f_w(y)^{n_y}$, where n_x and n_y denote the appropriate exponent, which depends on whether $w \in T$. As $f_w(x)$ and $f_w(y)$ do not start with the same letter, they do not have a common initial segment. We must also consider common initial segments of cyclic permutations of these two words, since we had to add the cyclic permutations of $f_w(x)^{n_x}$ and $f_w(y)^{n_y}$ to R_T to make sure that it was symmetrized.

A picture of sorts helps in the analysis. Before any sort of cyclic permutation, the two words are just $f_w(x)$ and $f_w(y)$ repeated some number of times, so they can naturally be seen as being split into blocks. The example of $f_{00}(x)^{n_x}$ is shown in figure 5.2. When a word is cyclically permuted a bit, the blocks at the beginning and end are truncated, as in figure 5.3. Now we cannot determine which word is a power of $f_w(x)$ and which is a power of $f_w(y)$ just by looking at the first letter of the words as before.

$$\underbrace{\underbrace{x^5yx^5yx^5yx^5yx^5yy^5x}_{f_{00}(x)}\underbrace{x^5yx^5yx^5yx^5yx^5yy^5x}_{f_{00}(x)}\dots\underbrace{x^5yx^5yx^5yx^5yx^5yy^5x}_{f_{00}(x)}}_{n_x \text{ times}}$$

Figure 5.2: $f_{00}(x)^{n_x}$, with its "blocks" shown

$$yx^5yx^5yx^5yy^5x\underbrace{\underbrace{x^5yx^5yx^5yx^5yx^5yy^5x}_{f_{00}(x)}\dots\underbrace{x^5yx^5yx^5yx^5yx^5yy^5x}_{f_{00}(x)}}_{n_x-1 \text{ times}}x^5yx^5$$

Figure 5.3: $f_{00}(x)^{n_x}$ after being cyclically permuted

Let r_1 be a cyclic permutation of $f_w(x)^{n_x}$ and r_2 be a cyclic permutation of $f_w(y)^{n_y}$ and let $M = \min\{|r_1|, |r_2|\}$. Suppose that $r_1 = st_1$ and $r_2 = st_2$ with s maximal. Permuting both r_1 and r_2 leftwards by $< |f_w(x)|$ letters, we get r_1^* and r_2^*, with $r_1^* = f_w(x)^{n_x}$. If the f_w-blocks in r_2^* also line up correctly, i.e. $r_2^* = f_w(y)^{n_y}$, then we know that r_1^* and r_2^* disagree at their first letter. Thus $|s| < |f_w(x)| < \frac{1}{8}M$ if $r_1 \neq r_2$.

Suppose that r_2^* is not in alignment, i.e. it is not $f_w(y)^{n_y}$. Then for r_1^* and r_2^* to agree for $\geq |f_w(x)|$ letters, $f_w(x)$ must be a subword of $f_w(y)^2$ containing letters from each copy of $f_w(y)$. We will show that this cannot happen. In fact we can prove a slightly more general result which will be useful later.

Lemma 5.5.4. *Let $u \in 2^{<\mathbb{N}}$, $a, b, c \in \{x^{\pm 1}, y^{\pm 1}\}$. Suppose $f_u(a)$ is a subword of $f_u(bc)$. Then $f_u(a)$ is not a subword of $f_u(bc) = f_u(b)f_u(c)$ containing letters from both $f_u(b)$ and $f_u(c)$. In other words, $f_u(a)$ must equal $f_u(b)$ or $f_u(c)$. It follows that if $\alpha, \beta \in \mathbb{F}_2$ are nontrivial and $f_u(\alpha)$ is a subword of $f_u(\beta)$, then α is a subword of β.*

Proof. We prove this inductively. It is easily checked to be true in the case that $u \in \{0, 1\}$. Now suppose that it is true for all u with $|u| < n$. Then if $u' = u^\frown i$ with $|u'| = n$ and $i \in \{0, 1\}$ we may write $f_{u'}(a) = f_u(f_i(a))$ and $f_{u'}(bc) = f_u(f_i(bc))$. By assumption, the f_u-blocks in $f_{u'}(a)$ line up with the f_u-blocks in $f_{u'}(bc)$, and since $f_{u'}(a)$ is a subword of $f_{u'}(bc)$, it follows that $f_i(a)$ is a subword of $f_i(bc)$. This implies that $f_i(a)$ equals $f_i(b)$ or $f_i(c)$. Thus we find $f_{u'}(a)$ equals $f_{u'}(b)$ or $f_{u'}(c)$. $\square$

This tells us that r_1^* and r_2^* agree for $< |f_w(x)|$ letters if they are out of alignment and so $|s| < 2|f_w(x)| < \frac{1}{8}M$. In fact, the same reasoning shows that two different cyclic permutations of $f_w(x)^{n_x}$ or of $f_w(y)^{n_y}$ also overlap for less than $\frac{1}{8}M$ letters.

Now we consider the case when $w, v \in T$ are distinct and $a, b \in \{x, y\}$. Let r_1 be a cyclic permutation of $(f_w(a))^{n_a}$ and r_2 be a cyclic permutation of $(f_v(b))^{n_b}$, and let $M = \min\{|r_1|, |r_2|\}$. If $v = e$ and $w \neq e$, then we observe that no cyclic permutation of $(f_w(a))^{n_a}$ agrees with $(f_e(b))^{n_b} = b^{n_b}$ for more than 6 letters, which is less than 1/8 of the length of either word. If w and v begin with different symbols, then one of r_1 and r_2 will be a cyclic permutation of a word in x^5y and y^5x, while the other will be a cyclic permutation of a word in x^2yxyx and y^2xyxy. Then the biggest possible common initial segment between r_1 and r_2 is xyx^3 or yxy^3, which is less than 1/8 of the length of either word.

So we may assume that w and v start with the same symbols. Suppose that $w = u^\frown w'$ and $v = u^\frown v'$, with u maximal. Taking our cue from the notation for the greatest common divisor, we will write this as $u = (w, v)$. This should not cause confusion, as there are no ordered pairs (or greatest common divisors!) in what follows. Then up to some truncated bits at the beginning and end, r_1 and r_2 are both words in $f_u(x)$ and $f_u(y)$, and so we are in a situation very similar to our first case, except that now r_1 and r_2 contain a mix of $f_u(x)$- and $f_u(y)$-blocks, rather than just being conjugates of a power of one or the other. See figure 5.4 for a picture.

$$y\, \underbrace{x^5y}_{f_0(x)}\, \underbrace{x^5y}_{f_0(x)}\, \underbrace{x^5y}_{f_0(x)}\, \underbrace{y^5x}_{f_0(y)}\, \underbrace{x^5y}_{f_0(x)}\, \underbrace{x^5y}_{f_0(x)}\, \underbrace{x^5y}_{f_0(x)}\, \underbrace{x^5y}_{f_0(x)} \ldots \underbrace{x^5y}_{f_0(x)}\, \underbrace{x^5y}_{f_0(x)}\, \underbrace{x^5y}_{f_0(x)}\, \underbrace{y^5x}_{f_0(y)}\, \underbrace{x^5y}_{f_0(x)}\, x^5$$

Figure 5.4: $f_{00}(x)^{n_x}$ after being cyclically permuted, with $f_0(x)$- and $f_0(y)$-blocks shown

Suppose that r_1 and r_2 are both made up entirely of f_u-blocks, i.e. there are no truncated f_u-blocks at their beginning and end. Because $f_u(x)$ and $f_u(y)$ start with different letters, we see that if r_1 and r_2 agree on the beginning of a block, then they agree for the entire block. So the largest common initial segment s which r_1 and r_2 share is made up of entire f_u-blocks. We have $r_1 = f_u(\alpha)$, $r_2 = f_u(\beta)$, and $s = f_u(\gamma)$ with α, β, γ words in x and y, and so by Lemma 5.5.4, we find that γ is a common initial segment of α and β. Furthermore, α and β are related to $f_w(a^{n_a})$ and $f_v(b^{n_b})$ as follows.

We know that $r_1 = f_u(\alpha)$ is a cyclic permutation of $f_w(a^{n_a}) = f_u(f_{w'}(a^{n_a}))$, and both r_1 and $f_w(a^{n_a})$ can be viewed as words made up entirely of f_u-blocks. Further, Lemma 5.5.4 implies that r_1 must be the result of cyclically permuting whole f_u-blocks, since whatever the first f_u-block of r_1 is, it cannot meet two of the f_u-blocks in $f_w(a^{n_a})$. It follows that α is a cyclic permutation of $f_{w'}(a^{n_a})$. Similarly, β is a cyclic permutation of $f_{v'}(b^{n_b})$.

So γ is a common initial segment of a cyclic permutation of $f_{w'}(a^{n_a})$ and a cyclic permutation of $f_{v'}(b^{n_b})$. This brings us back to the earlier cases. If w' and v' are both nontrivial words, then they start with different symbols, which implies that $|\gamma| \leq 5$, and so s is made up of at most 5 f_u-blocks. On the other hand, r_1 and r_2 are made up of at least $6 \cdot \min\{|f_{w'}(a^{n_a})|, |f_{v'}(b^{n_b})|\}$ f_u-blocks, and so $|s| < \frac{1}{8}M$. If $w' = e$ and $v' \neq e$ or vice versa, then $|\gamma| \leq 6$ and we still find that $|s| < \frac{1}{8}M$. If $w' = e$ and $v' = e$, then γ is empty unless $a = b$, which implies that $r_1 = r_2$.

This leaves only the out of alignment cases to deal with. As before, we may permute r_1 and r_2 leftwards by $< |f_u(x)|$ letters to get r_1^* and r_2^*, with r_1^* a product of f_u-blocks. If r_2^* is not a product of f_u-blocks, then Lemma 5.5.4 tells us that r_1^* and r_2^* agree for $< |f_u(x)|$ letters, and so r_1 and r_2 agree for $< 2|f_u(x)|$ letters, which is $< 1/8$ of the length of each word. If r_2^* is a product of f_u-blocks, then we are in the previous case, and we have seen that either $r_1^* = r_2^*$ or the corresponding common initial segment between them consists of at most 6 f_u-blocks. This implies that $|s| < 7|f_u(a)| < \frac{1}{8}M$. We have finally exhausted all of the cases and have shown that R_T satisfies the $C'(1/8)$ condition, as desired. □

The following lemma was shown to be true in the course of the above proof. We record it here separately for ease of future reference.

Lemma 5.5.5. *If $f_w(\alpha)$ is a cyclic permutation of $f_w(\beta)$ and β is cyclically reduced, then α is a cyclic permutation of β.*

Lemma 5.5.6. *If $T, T' \in Tr(2)$ and there exists $w \in 2^{<\mathbb{N}}$ such that $T = T'_w$, then $G_T \hookrightarrow G_{T'}$.*

Proof. This is obvious if $w = e$, and so we may assume that w is nontrivial. It is easy to see that f_w, viewed as a map from G_T to $G_{T'}$, is a homomorphism, since it will take defining relations in G_T to defining relations in $G_{T'}$. In more detail,

$$\begin{aligned} f_v(x)^{59}, f_v(y)^{61} \in R_T &\iff v \in T \\ &\iff w^\frown v \in T' \\ &\iff f_{w^\frown v}(x)^{59} = f_w(f_v(x)^{59}), \\ &\quad f_{w^\frown v}(y)^{61} = f_w(f_v(y)^{61}) \in R_{T'} \end{aligned}$$

and similar equivalences hold for $f_v(x)^{67}$ and $f_v(y)^{71}$. It remains to show that f_w is an embedding.

We still need to show that nontrivial elements of G_T do not map to the identity in $G_{T'}$. Our map is defined in terms of where it sends words, but we must take into account the relations in our two groups in order to see which words correspond to the identity. As in the previous section, we will show that if $\alpha \in \mathbb{F}_2$ is such that $f_w(\alpha)$ contains more than $1/2$ of a relation in $R_{T'}$, then α contains more than $1/2$ of a relation in R_T, which easily implies the result.

Suppose that $\alpha \in \mathbb{F}_2$ and that $f_w(\alpha) = 1$ in $G_{T'}$. Then $f_w(\alpha)$ contains more than half of a relation $r \in R_{T'}$. We know that r is a cyclic permutation of some $f_v(a^{n_a})$, where $v \in 2^{<\mathbb{N}}$, $a \in \{x^{\pm 1}, y^{\pm 1}\}$, and $n_a \in \{59, 61, 67, 71\}$. Let $u = (w, v)$, so that $w = u^\frown w'$ and $v = u^\frown v'$. Then $f_w(\alpha) = f_u(f_{w'}(\alpha))$, and r is a cyclic permutation of $f_u(f_{v'}(a^{n_a}))$. By assumption, the subword of r that both words contain must be big enough to contain an entire f_u-block. Thus Lemma 5.5.4 tells us that the f_u-blocks of r and $f_w(\alpha)$ must line up. The f_u-blocks are uniquely identified by their first or last letters, so once $f_w(\alpha)$ and r agree for part of an f_u-block, they agree on the whole thing, unless r begins and ends with a truncated f_u-block. In this case, cyclically permuting r until it is made up of f_u-blocks will "complete" the f_u-block at one end of r. This new word is also a relation which agrees with $f_w(\alpha)$ for at least as long as r did, since either only one end of r was in $f_w(\alpha)$ and cyclically permuting increases the length of the word the two agree on, or r was a subword of $f_w(\alpha)$ and this cyclic permutation is also a subword of $f_w(\alpha)$.

Thus we may assume that $r = f_u(\omega)$ for some $\omega \in \mathbb{F}_2$, and that r and $f_w(\alpha)$ share a subword of the form $f_u(\gamma)$, where $\gamma \in \mathbb{F}_2$. By Lemma 5.5.5, we know that ω is a cyclic permutation of $f_{v'}(a^{n_a})$. Then γ is a subword of a cyclic permutation of $f_{v'}(a^{n_a})$ and a subword of $f_{w'}(\alpha)$. If w' and v' are both nontrivial, then they begin with different symbols, and so $|\gamma| \leq 5$. But then

$$\begin{aligned}
\frac{|f_u(\gamma)|}{|f_v(a^{n_a})|} &= \frac{|\gamma|}{|f_{v'}(a^{n_a})|} \\
&\leq \frac{5}{|f_{v'}(a^{n_a})|} \\
&< 1/2
\end{aligned}$$

which is a contradiction. If $v' = e$ but $w' \neq e$, then virtually the same inequalities hold since $f'_w(\alpha)$ does not contain any letter to a power greater than 6, and again we get a contradiction. Thus $w' = e$, meaning $w = u$, and so α contains $> 1/2$ of a cyclic permutation of $f_{v'}(a^{n_a})$. Further, since $w^\frown v' \in T' \Leftrightarrow v' \in T$, it follows that $f_{v'}(a^{n_a}) \in R_T$. Thus if $f_w(\alpha) = 1$ in $G_{T'}$, then α contains $> 1/2$ of a word in R_T, as desired. $\square$

The proof of the converse of Lemma 5.5.6 will depend on the following two lemmas.

Lemma 5.5.7. *Suppose $\alpha, \beta \in \mathbb{F}_2$ are cyclically reduced, and $w, v \in 2^{<\mathbb{N}}$. If r_1 is a cyclic permutation of $f_w(\alpha)$ and r_2 is both a cyclic permutation of $f_v(\beta)$ and a subword of r_1, then $v \subset w$ or $w \subset v$.*

Moreover, if $w = v^\frown w'$ then a cyclic permutation of $f_{w'}(\alpha)$ contains a cyclic permutation of β, and if $v = w^\frown v'$, then a cyclic permutation of α contains a cyclic permutation of $f_{v'}(\beta)$.

Proof. The result is trivial if $w = e$ or $v = e$, so we may assume that w and v are nontrivial. Let $u = (w, v)$, with $w = u^\frown w'$ and $v = u^\frown v'$. If $u = e$, then w and v begin with different symbols, which is impossible, since $|r_1|, |r_2| > 5$, the length of the longest possible agreement between r_1 and r_2. So u is nontrivial, and r_1 is a cyclic permutation of $f_u(f_{w'}(\alpha))$, while r_2 is a cyclic permutation of $f_u(f_{v'}(\beta))$. The f_u-blocks of each word must line up, by Lemma 5.5.4. Further, since r_2 is a subword of r_1, any truncated bits of f_u-blocks at the ends of r_2 are duplicated in r_1. So we can permute r_1 and r_2 the same amount to get $r_1^* = f_u(\gamma)$ and $r_2^* = f_u(\omega)$, words composed entirely of f_u-blocks, with r_2^* contained in r_1^*. By Lemma 5.5.5, we know γ is a cyclic permutation of $f_{w'}(\alpha)$ and ω is a cyclic permutation of $f_{v'}(\beta)$. In addition, Lemma 5.5.4 implies that ω is a subword of γ.

If w' and v' are nontrivial, then they start with different symbols, and as above we reach a contradiction. Thus either $w' = e$, and so $w \subset v$ and a cyclic permutation of α contains a cyclic permutation of $f_{v'}(\beta)$, or $v' = e$, so $v \subset w$ and a cyclic permutation of $f_{w'}(\alpha)$ contains a cyclic permutation of β. □

Lemma 5.5.8. *Suppose that $t, u, v \in 2^{<\mathbb{N}}$, and some cyclic permutation of $f_t(x^k)$ is a product of a cyclic permutation of $f_u(x^l)$ and a cyclic permutation of $f_v(y^m)$, with $k, l, m \in \mathbb{Z} \setminus \{0\}$. Then $u = v$, $t = u^\frown 0$, $k = m = \pm 1$, $l = 5k$, and $f_t(x^k) = f_{u^\frown 0}(x^{\pm}1)$ is either*

$$f_u(x^5) f_u(y)$$

or

$$f_u(y^{-1}) f_u(x^{-5}).$$

Proof. By Lemma 5.5.7, either $t \subset u$ or $u \subset t$. If $t \subset u$ and $u = t^\frown u'$, then by the previous lemma, we find that a cyclic permutation of x^k contains a cyclic permutation of $f_{u'}(x^l)$. This is impossible unless $u' = e$. So we may assume that $u \subset t$ and $t = u^\frown t'$. Similarly we find that $v \subset t$ and $t = v^\frown t''$. It follows that $u \subset v$ or $v \subset u$.

Suppose that $u \subset v$ and $v = u^\frown v'$. We know that the f_u-blocks in $f_u(x^l)$ and $f_u(f_{v'}(y^m))$ must line up with those in $f_t(x^k)$. This means in particular that a truncated f_u-block at the end of the cyclic permutation of $f_u(x^l)$ must be completed by a truncated f_u-block at the beginning of the cyclic permutation of $f_u(f_{v'}(y^m))$, and vice versa. So we can assume that the cyclic permutations we are considering are made up of complete

f_u-blocks. Then by Lemma 5.5.4 we obtain that a cyclic permutation of $f_{t'}(x^k)$ is a product of x^l and a cyclic permutation of $f_{v'}(y^m)$.

Now, $f_{t'}(x^k)$ is composed of $f_{v'}$-blocks, which must line up with those in the cyclic permutation of $f_{v'}(y^m)$. So it must be the case that x^l is a cyclic permutation of $f_{v'}$-blocks. But this can only happen if $v' = e$, meaning $u = v$. Similar reasoning applies if $v \subset u$. Thus $u = v$. It is not possible for a truncated $f_u(x)$-block to be completed by a truncated $f_u(y)$-block, or vice versa, and so we must have that the cyclic permutation of $f_t(x^k)$ we started with is either $f_u(x^l)f_u(y^m)$ or $f_u(y^m)f_u(x^l)$.

It follows that $x^l y^m$ is a cyclic permutation of $f_{t'}(x^k)$. This can only happen if $t' = 0$ and k, l, m are as in the statement of the lemma, since if $t' = e$ then x^k contains no occurrences of y, and if $t' \neq 0$ is nontrivial then $f_{t'}(x^k)$ must contain at least two distinct blocks of x's and y's. $\square$

We now take up the converse of Lemma 5.5.6, which will complete the proof of Theorem 5.5.2.

Lemma 5.5.9. *If $T, T' \in Tr(2)$ and $G_T \hookrightarrow G_{T'}$, then $\exists w \in 2^{<\mathbb{N}}$ such that $T = T'_w$.*

Proof. Suppose that $\theta\colon G_T \to G_{T'}$ is a monomorphism. Our main goal is to prove that θ must actually be f_w for some $w \in 2^{<\mathbb{N}}$, up to an inner automorphism of $G_{T'}$. Once we know this, it is easy to recover the relations in each group, and thus to show that $T = T'_w$.

Since $x = f_e(x)$, x has some finite order n_x in G_T. Then $\theta(x)$ has order n_x, and so by Theorem 4.1.3, $\theta(x)$ must be conjugate to a power of some $f_w(x)$, where $w \in T' \Leftrightarrow e \in T$. If we follow θ by an inner automorphism of $G_{T'}$, we may assume that $\theta(x) = (f_w(x))^\delta$ for some nonzero integer δ.

Similarly, $\theta(y)$ is conjugate to a power of some $f_v(y)$ with $v \in T' \Leftrightarrow e \in T$. We find that $\theta(y) = u(f_v(y))^\gamma u^{-1}$. We can assume that u does not contain more than half of an element of $R_{T'}$. We may also follow θ by the inner automorphism corresponding to $f_w(x)$ as necessary to ensure that u does not begin with a power of $f_w(x)$, and this will not change the value of $\theta(x)$. After freely reducing we get that $\theta(y) = u' r u'^{-1}$, where r is a cyclic permutation of $(f_v(y))^\gamma$. To see this, suppose that $u = \alpha\beta^{-1}$, where $(f_v(y))^\gamma = \beta\mu$ and β is the longest subword of u for which this is true. Then $u(f_v(y))^\gamma u^{-1} = \alpha\beta^{-1}\beta\mu\beta\alpha^{-1} = \alpha\mu\beta\alpha^{-1}$. Then $u' = \alpha$ and $r = \mu\beta$. A similar argument works if $(f_v(y))^\gamma$ cancels with u^{-1}. For example, if we had $\theta(y) = xy^{-1}f_0(y)yx^{-1} = xy{-}1(y^5x)yx^{-1}$, then after freely reducing we would find $\theta(y) = xy^4xyx^{-1} = xrx^{-1}$, where $r = y^4xy$.

We now proceed much as in the proof of 5.4.1. Let m_x be the order of $f_0(x)$ in G_T. Since θ is a monomorphism, it must take $f_0(x)$ to a torsion element of order m_x. We know that

$$\begin{aligned}\theta(f_0(x)) &= \theta(x^5y) = (f_w(x))^{5\delta}u'ru'^{-1}\\ &= (f_w(x))^{\delta'}u'ru'^{-1}\end{aligned}$$

where $|\delta'| < \lfloor \frac{n_x}{2} \rfloor$. Note that, as written, this word may not be freely reduced, so we can not necessarily use Theorem 4.1.2 yet. Let $z = \theta(f_0(x))$.

Suppose that z is cyclically reduced as written and that $u' \neq 1$. Then z is a cyclically reduced word with finite order in $G_{T'}$ which contains a mixture of positive and negative letters, which is impossible since the words in $R_{T'}$ are either entirely positive or entirely negative. So either $u' = 1$ or else z is not cyclically reduced.

First suppose that $u' = 1$, so that $z = (f_w(x))^{\delta'}r$. If δ' and γ have the same sign, then z is cyclically reduced as written. By Theorem 4.1.3, z is a cyclic permutation of some $f_t(x^k)$, and so by Lemma 5.5.8, $\theta(x) = f_w(x)^{\pm 1}$ and $\theta(y) = f_w(y)^{\pm 1}$.

Suppose that δ' and γ have opposite signs. There is a (possibly trivial) inner automorphism ψ such that $\psi(z) = sr'$, where s is a cyclic permutation of $f_w(x)^{\delta'}$, and r' is a cyclic permutation of $f_v(y)^\gamma$, and freely reducing sr' will leave a cyclically reduced word. Let $\psi(z) = z'$. Since z' is a torsion element, it must be a cyclic permutation of some $f_t(x^k)$. So if we write z' as the result of freely reducing sr', then its letters must all have the same sign.

Suppose z' and s have letters of the same sign. Then $z'r'^{-1}$ is cyclically reduced and so we have a cyclic permutation of $f_w(x)^{\delta'}$ written as a product of a cyclic permutation of $f_t(x)^k$ and a cyclic permutation of $f_v(y)^\gamma$. By Lemma 5.5.8, we get that $\theta(x) = f_{v\frown 0}(x^{\pm 1}) = f_v((x^5y)^{\pm 1})$, and $\theta(y) = f_v(y^{\mp 1})$. But then either $\theta(xy) = f_v(x^5)$ or

$\theta(xy) = f_v(y^{-1}x^{-5}y) = f_v(y)^{-1}f_v(x^{-5})f_v(y)$. Both of these are torsion elements, but xy is not a torsion element in G_T, which contradicts the fact that θ is an embedding. So suppose that z' and r' have letters of the same sign. Then $z's^{-1}$ is cyclically reduced and so we have a cyclic permutation of $f_v(y)^\gamma$ written as a product of a cyclic permutation of $f_t(x)^k$ and a cyclic permutation of $f_w(x)^{\delta'}$. Arguing as in the proof of Lemma 5.5.8, we find that $w = t$ and that $w \subset v$. Let $v = w^\frown v'$. We obtain that $f_{v'}(y^\gamma) = x^{-\delta'+k}$, which is absurd.

We still must address the case where $u' \neq 1$ and z is not cyclically reduced. This can happen for two reasons. It may be that u' and $f_w(x)$ begin in the same way. We know that u' does not begin with an entire copy of $f_w(x)$, and we have assumed that u' does not further cancel with r, so there is an inner automorphism ψ such that $\psi\theta(f_0(x)) = su''ru''^{-1}$, a cyclically reduced word with s a cyclic permutation of $(f_w(x))^{\delta'}$. If $u'' \neq 1$, then $\psi(\theta(f_0(x))$ contains positive and negative letters, and we have already seen that this is impossible. Thus we must have that $z' = sr$, and as before we see that $\theta(x) = f_w(x)^{\pm 1}$ and $\theta(y) = f_w(y)^{\pm 1}$.

The other possibility is that u' cancels with the end of $f_w(x)$. It cannot cancel with the whole of $f_w(x)$, and so again after following θ by an appropriate inner automorphism ψ we get that $\psi(\theta(f_0(x))) = su''ru''^{-1}$, a cyclically reduced word with s a cyclic permutation of $(f_w(x))^{\delta'}$. This case is treated exactly as in the previous paragraph.

So we have shown that $\theta(x) = f_w(x)^{\pm 1}$ and $\theta(y) = f_w(y)^{\pm 1}$ with the signs matching. If $\theta(x) = f_w(x)$ and $\theta(y) = f_w(y)$, then $\theta = f_w$, and hence

$$\begin{aligned} u \in T &\Leftrightarrow f_u(x^{53}) \in R_T \\ &\Leftrightarrow f_w(f_u(x^{53})) \in R_{T'} \\ &\Leftrightarrow w^\frown u \in T' \end{aligned}$$

Thus $T = T'_w$, as desired. Thus it only remains is to eliminate the undesirable case when $\theta(x) = f_w(x^{-1})$ and $\theta(y) = f_w(y^{-1})$. In this case we have that

$$\begin{aligned}\theta(f_{00}(x)) &= \theta(x^5yx^5yx^5yx^5yx^5yy^5x) \\ &= f_w(x^{-5}y^{-1}x^{-5}y^{-1}x^{-5}y^{-1}x^{-5}y^{-1}x^{-5}y^{-1}y^{-5}x^{-1}) \\ &= f_w((xy^5yx^5yx^5yx^5yx^5yx^5)^{-1})\end{aligned}$$

We will show this is not a torsion element in $G_{T'}$. Since $f_{00}(x)$ is a torsion element in G_T, this implies that θ is not a homomorphism, which is a contradiction.

Let $\alpha = xy^5yx^5yx^5yx^5yx^5yx^5$. It is easy to see that $f_{00}(x)$ is the only torsion element that has length 36 and that contains 26 xs. However, α is not a cyclic permutation of $f_{00}(x)$. It follows that $f_w(\alpha)$ (and hence $\theta(f_{00}(x))$) is not a torsion element, since otherwise it would have to be a cyclic permutation of some $f_t(x^k)$ with $t \in T'$. Thus by Lemma 5.5.7, either $w \subset t$ or $t \subset w$. Suppose that $w \subset t$ and $t = w^\frown t'$. Then α must be a cyclic permutation of $f_{t'}(x^k)$, which we have already seen is impossible. If $t \subset w$ and $w = t^\frown w'$, then x^k must be a cyclic permutation of $f_{w'}(\alpha)$, which is also impossible. Thus we reach a contradiction, eliminating the final undesirable case. □

Proof of Theorem 5.5.2. Lemmas 5.5.6 and 5.5.9 establish that the map $T \mapsto G_T$ is a Borel reduction from $\preccurlyeq_2^{tree}$ to $\preccurlyeq_{em}$. □

Corollary 5.5.10. *$\equiv_{em}$ is a universal countable Borel equivalence relation.*

It may have been possible to prove this corollary without any reference to quasi-orders, by reducing some known universal countable Borel equivalence relation to $\equiv_{em}$. However, it seems that the most natural and direct route to this result is through Theorem 5.5.2. A closer look at the above proof also leads to the following result, which tells us that the bi-embeddability relation on the groups constructed above is much more complicated than the isomorphism relation on these groups.

Corollary 5.5.11. *With the above notation, $G_T \cong G_S \Leftrightarrow T = S$.*

Proof. One direction is trivial. For the other, suppose that $T, S \in Tr(2)$ are such that $G_T \cong G_S$, via $\phi\colon G_T \to G_S$. Then in particular ϕ is an embedding, and by the proof of Lemma 5.5.9, there exists $w \in 2^{<\mathbb{N}}$ such that $T = S_w$. Furthermore, after adjusting by an inner automorphism of G_S if necessary, we can suppose that $\phi = f_w$. We will show that if $w \neq e$, then $f_w \colon G_T \to G_S$ is not surjective. Hence $w = e$ and $G_T = G_S$.

Suppose that $w \in 2^{<\mathbb{N}}$ is nontrivial and that $f_w \colon G_T \to G_S$ is surjective. Then there is some word $\alpha \in \mathbb{F}_2$, which we may assume does not contain more than $1/2$ of a relation in R_T, such that $f_w(\alpha) = x$ in G_S, where x is one of the generators of G_S. This means that $f_w(\alpha)x^{-1} = 1$ in G_S. By the proof of Lemma 5.5.6, we know that $f_w(\alpha)$ does not contain more than $1/2$ of a relation in G_S. By Theorem 4.1.2, for $f_w(\alpha)x^{-1}$ to represent the identity in G_S, it must contain more than $(1-3/8) = 5/8$ of a relation in R_T. But $f_w(\alpha)x^{-1}$ has at most one more letter in common with a relation than $f_w(\alpha)$ does. Since $f_w(\alpha)$ contains less than $1/2$ of a relation in R_T, $f_w(\alpha)x^{-1}$ contains less than $5/8$ of a relation in R_T. This is a contradiction. $\square$

5.6 The structure of the countable Borel quasi-orders under $\leq_B$

The previous results have all been concerned with universal countable Borel quasi-orders, which are above all other countable Borel quasi-orders with respect to $\leq_B$. It is natural to ask about the overall structure of the countable Borel quasi-orders.

Recall that n may be viewed as a discrete space with n elements. We have already seen that if we restrict our attention to equivalence relations, the equality relations $\Delta(n)$ are such that

$$\Delta(1) <_B \Delta(2) <_B \Delta(3) <_B \ldots <_B \Delta(\mathbb{N}).$$

If we also look at quasi-orders on these spaces, then the structure is more complicated. For these spaces, Borel reducibility is embeddability. This is undoubtedly chaotic, but a complete description more properly belongs to combinatorics rather than logic.

In light of the relative chaos that occurs in the lower part of the Borel hierarchy for countable Borel quasi-orders, one might worry that we lose all of the nice dichotomies that hold for countable Borel equivalence relations. Thankfully this is not the case. Suppose that Q is a countable Borel quasi-order on an uncountable Polish space. By the Kuratowski-Ulam theorem, since every section of Q is meager, Q is a meager subset of X^2. Then recall the following theorem of Mycielski in [30].

Theorem 5.6.1. *If X is an uncountable Polish space and R_n is a meager subset of X^n, then there is a perfect subset $P \subseteq X$ such that*

$$\forall p_1, \ldots, p_n \in P\, (p_1, \ldots, p_n) \notin R_n.$$

In particular, if Q is a meager quasi-order on an uncountable Polish space X, then there is a perfect subset $P \subseteq X$ for which none of the elements are Q-comparable, and hence $\Delta(P) \leq_B Q$ by the identity map. Since P is an uncountable standard Borel space in its own right, $\Delta(\mathbb{R}) \sim_B \Delta(P)$, and so $\Delta(\mathbb{R})$ reduces to every meager quasi-order on an uncountable Polish space, including every countable Borel quasi-order.

Also, because symmetry is preserved downwards under $\leq_B$, E_0 is an immediate successor of $\Delta(\mathbb{R})$ within the collection of countable Borel quasi-orders. It is natural to ask whether there are any other immediate successors of $\Delta(\mathbb{R})$. Recall from section 2.4 that if E is a countable Borel equivalence relation on a standard Borel space X with a Borel linear order $\leq$, then $E(\leq)$ is defined to be $E \cap \leq$. Some of these quasi-orders seem to furnish candidates for immediate successors of $\Delta(\mathbb{R})$. We have seen that

$$E(\leq) \leq_B F(\leq) \Rightarrow E \leq_B F. \quad (*)$$

However, the converse is not true in general. Define E on $2^{\mathbb{N}} \times \mathbb{Q}$ by

$$(x, r)\, E\, (y, s) \iff x = y$$

and similarly define F on $2^{\mathbb{N}} \times \mathbb{N}$. Also, let $\leq^{\mathbb{Q}}$ denote the lexicographical order on $2^{\mathbb{N}} \times \mathbb{Q}$, i.e.

$$(x, r) \leq^{\mathbb{Q}} (y, s) \iff x \leq_{2^{\mathbb{N}}} y \vee (x = y \wedge r \leq_{\mathbb{Q}} s)$$

where $\leq_{2^{\mathbb{N}}}$ is the standard lexicographical order on $2^{\mathbb{N}}$ and $\leq_{\mathbb{Q}}$ is the standard order on $\mathbb{Q}$. Similarly define $\leq^{\mathbb{N}}$ to be the lexicographical order on $2^{\mathbb{N}} \times \mathbb{N}$. Clearly $E \sim_B F$, as both equivalence relations are smooth. However, $E(\leq^{\mathbb{Q}}) \not\leq_B F(\leq^{\mathbb{N}})$. To see this, let $x \in 2^{\mathbb{N}}$. Any Borel reduction from $E(\leq^{\mathbb{Q}})$ to $F(\leq^{\mathbb{N}})$ must send $[(x, 0)]_E$ into some $[(y, 0)]_F$ in an order-preserving way, which is impossible.

We now discuss another important property of the $E(\leq)$.

Definition 5.6.2. *Let $\preccurlyeq$ and $\preccurlyeq'$ be quasi-orders on X and Y, respectively. A map $h\colon X \to Y$ is half-order-preserving, or h.o.p., if $x \preccurlyeq y \Rightarrow h(x) \preccurlyeq' h(y)$.*

Definition 5.6.3. *A Borel quasi-order Q on X is Borel linearizable if there is a Borel linear order L on Y and a Borel h.o.p. map $h\colon X \to Y$ (called a linearization map) such that $x\, E_Q\, y \Leftrightarrow h(x) = h(y)$.*

This concept first appeared in [16]. Each $E(\leq)$ is Borel linearizable, as the identity map is a linearization map. In [20], Kanovei proved a dichotomy theorem for Borel quasi-orders which involves the property of Borel linearizability. Define $\leq_0$ on $2^{\mathbb{N}}$ by $x \leq_0 y$ iff $x = y$ or $x\, E_0\, y$ and $x(n) < y(n)$ where n is the largest natural number for which $x(n) \neq y(n)$. Note that this is not the same as $E_0(\leq)$. Kanovei proved the following dichotomy:

Theorem 5.6.4 (Kanovei [20])**.** *Suppose that $\preccurlyeq$ is a Borel quasi-order on $\mathbb{N}^{\mathbb{N}}$. Then exactly one of the following conditions is satisfied:*

i) $\preccurlyeq$ is Borel linearizable,

ii) there exists a continuous 1-1 map $F\colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

$$a \leq_0 b \Rightarrow F(a) \preccurlyeq F(b)$$

while $a \not\!E_0 b$ implies that $F(a)$ and $F(b)$ are $\preccurlyeq$-incomparable.

In particular, $\leq_0$ is not Borel linearizable. Notice that Borel linearizability is preserved downwards under $\leq_B$. To see this, suppose Q, R are Borel quasi-orders on standard Borel spaces X, Y respectively, and R is Borel linearizable. Then there is some standard Borel space Z with a Borel linear order $\leq$ and a h.o.p. map $h\colon Y \to Z$. Suppose that $f\colon X \to Y$ is a Borel reduction from Q to R. Then

$$\begin{aligned} x\, E_Q\, y &\iff f(x)\, E_R\, f(y) \\ &\iff h(f(x)) = h(f(y)) \end{aligned}$$

Thus $h \circ f$ witnesses that Q is Borel linearizable. Consequently, Kanovei's theorem has the following corollary.

Corollary 5.6.5. *If E is any countable Borel equivalence relation, then $\leq_0$ does not Borel reduce to $E(\leq)$.*

Open Problem. Suppose Q is a countable Borel quasi-order, E is a countable Borel equivalence relation, and $Q \leq_B E(\leq)$. Does it follow that $Q \sim_B F(\leq)$ for some countable Borel equivalence relation F?

It is currently not known if any of the $E(\leq)$ Borel reduce to $\leq_0$. Note that $\leq_0$ symmetrizes to $\Delta(2^{\mathbb{N}})$. So all of the examples of "genuine" quasi-orders which we have discussed so far symmetrize to either equality on some standard Borel space or to a universal countable Borel equivalence relation. This leads us to ask if there are any natural quasi-orders Q which are not equivalence relations and for which E_Q is neither smooth nor universal.

There is at least one example. Recall the earlier definition of $\preccurlyeq_t^G$ for a countable group G. If we write E_t^G for the symmetrization of $\preccurlyeq_t^G$ and E_G for the equivalence relation on subsets of G given by the orbits of the shift action, then clearly $E_G \subseteq E_t^G$. In particular, it follows that E_t^G is not smooth. This is because if E, F are countable Borel equivalence relations and $E \subseteq F$, then if F is smooth, so is E. (This is an immediate consequence of Lemma 2.1 in [36].) But for any infinite countable group G, E_G is not smooth (cf. [24, Example 3.1]). So the question is whether or not E_t^G is universal. We will show that $E_t^{\mathbb{Z}}$ is not universal. In fact, we will prove a stronger statement, after first establishing a lemma due to Cherlin which simplifies the author's original argument.

Lemma 5.6.6. *If $A \subseteq \mathbb{Z}$ and $A = \cap_{i=1}^n (A + c_i)$, then $A = A + c_i$ for some i.*

Proof. Let $I = \{k \in \mathbb{Z} \mid A \subseteq A + k\}$. Then $c_i \in I$ for all i, and I is closed under addition. If I contains both positive and negative integers, then $I = d\mathbb{Z}$ for some $d \in \mathbb{Z}$. Hence $d \mid c_i$ for all i. Then $A \subseteq A + d$, $A \subseteq A - d \Leftrightarrow A + d \subseteq A$, and so $A = A + d$. Thus $A = A + c_i$ for all i.

So we may suppose that I contains no negative numbers. We may assume that no $c_i = 0$, since otherwise we are done. Thus all the c_i are positive. Let $d = \gcd(c_1, \ldots, c_n)$.

If $A \subseteq A + d$, then $A + d \subseteq A + c_i$ for all i, and so $A = A + d$. But then $-d \in I$, a contradiction. So A is not contained in $A+d$. We may assume that $0 \in A \backslash (A+d)$, since otherwise we shift A as necessary. Every sufficiently large multiple of d is a positive linear combination of the c_i. Thus $-nd \in A$ for every sufficiently large n. Take n maximal with $-nd \notin A$. Then $-nd \in (A + c_i)$ for all i, a contradiction. $\square$

Theorem 5.6.7. $E_t^{\mathbb{Z}} = E_{\mathbb{Z}}$

Proof. Suppose $A \, E_t^{\mathbb{Z}} \, B$, as witnessed by $a_1, \ldots a_n, b_1, \ldots, b_m \in \mathbb{Z}$, i.e.

$$A = (a_1 + B) \cap \ldots \cap (a_n + B) \tag{5.2}$$

$$B = (b_1 + A) \cap \ldots \cap (b_m + A) \tag{5.3}$$

If $n = 1$ or $m = 1$, then A is a shift of B, so we may assume that $n > 1$ and $m > 1$. If we replace every instance of B on the right hand side of (5.2) with the right hand side of (5.3), and similarly replace the instances of A in (5.3), we get

$$A = \cap_{i,j}(a_i + b_j + A)$$

$$B = \cap_{i,j}(a_i + b_j + B)$$

By Lemma 5.6.6, we have $A = a_i + b_j + A$ for some i, j. Then

$$A \subseteq B + a_i \subseteq A + b_j + a_i = A$$

and so $A = B + a_i$. $\square$

In general $E_t^G \neq E_G$. It is still the case that any sets which are equivalent under E_t^G to sets other than their shifts are contained in translates of themselves, but the arguments used in showing $E_t^{\mathbb{Z}} = E_{\mathbb{Z}}$ do not carry over. For example, there are sets $A, B \in \mathcal{P}(\mathbb{Z}^3)$ such that $A E_t^{\mathbb{Z}^3} B$ and $A \not{E}_{\mathbb{Z}^3} B$. Figure 5.5 shows part of such an A. Each grid is a copy of $\mathbb{Z}^2$, and the copies are indexed by $\mathbb{Z}$. So the element (a, b, c) corresponds to (a, b) in the grid indexed by c. A 1 indicates the corresponding element is in A, while a 0 indicates it is not. All of the grids below what is pictured are entirely 0s. The grids above continue the pattern of moving the lowest nonzero row up by two, and adding an additional 1 to the lowest row. Let $B = A \cap A + (1, 0, 0)$, which is shown in figure 5.6.

It is easy to see that B is not a shift of A. However, $B \cap B + (0, 0, -1) = A + (0, 2, 0)$, and so $A \; E_t^{\mathbb{Z}^3} \; B$.

This implies that $E_t^G \neq E_G$ for any countable group G such that $\mathbb{Z}^3 \leq G$. Suppose that G is a countable group such that $E_t^G \neq E_G$ and H is a countable group which surjects onto G, say via f. Let $A, B \subseteq \mathcal{P}(G)$ be such that $A \; E_t^G \; B$, say via $c_1, \ldots, c_n$ and $d_1, \ldots, d_m$, and $A \not\!\!E_G B$. Then $f^{-1}(A) \; E_t^H \; f^{-1}(B)$ and $f^{-1}(A) \not\!\!E_H f^{-1}(B)$. Let $g_1, \ldots, g_n \in H$ be such that $g_i \in f^{-1}(c_i)$ and $h_1, \ldots, h_m \in H$ be such that $h_i \in f^{-1}(d_i)$. Then

$$f^{-1}(A) = g_1 f^{-1}(B) \cap \ldots \cap g_n f^{-1}(B)$$
$$f^{-1}(B) = h_1 f^{-1}(A) \cap \ldots \cap h_m f^{-1}(A)$$

Suppose that there were some $h \in H$ such that $f^{-1}(A) = hf^{-1}(B)$. Then $A = f(h)B$, a contradiction. This further expands the collection of groups such that $E_t^G \neq E_G$. In particular, any SQ-universal group has this property, including $\mathbb{F}_2$. So while we have seen that $E_t^{\mathbb{F}_2} \sim_B E_\infty$, the two are not equal.

If we restrict the shift action of $\mathbb{F}_2$ on its subsets to the free part of the shift action (i.e. the points which are not fixed by any nontrivial element of $\mathbb{F}_2$), then we obtain the universal treeable equivalence relation $E_{\infty T}$. Recall that $E_0 <_B E_{\infty T} <_B E_\infty$. In the same spirit, we can restrict $\preccurlyeq_t^{\mathbb{F}_2}$ to the free part of the shift action and get a new quasi-order $\preccurlyeq_{ft}^{\mathbb{F}_2}$. It is reasonable to conjecture that the associated equivalence relation is Borel bireducible with $E_{\infty T}$, but the above results make it unlikely that the associated equivalence relation is actually equal to $E_{\infty T}$. If this conjecture holds, it would give another example of a quasi-order whose associated equivalence relation is neither smooth nor universal.

5.7 Future directions

We have already mentioned some of the remaining open questions concerning the structure of the countable Borel quasi-orders under $\leq_B$. Of course it is also natural to ask where specific countable Borel quasi-orders are situated in this structure. In particular, most of the reducibilities from computability theory, such as Turing reducibility $\leq_T$ or

$$\dots$$

$$\begin{array}{ccccccccc}
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
\end{array}$$

$$\begin{array}{ccccccccc}
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
\end{array}$$

$$\begin{array}{ccccccccc}
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\
\dots & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \dots \\
\hline
\end{array}$$

$$\begin{array}{ccccccccc}
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
\end{array}$$

$$\dots$$

Figure 5.5: The set $A \in \mathcal{P}(\mathbb{Z}^3)$

$\cdots$

$$\begin{array}{ccccccccc} \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline \end{array}$$

$$\begin{array}{ccccccccc} \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline \end{array}$$

$$\begin{array}{ccccccccc} \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ \cdots & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \cdots \\ \hline \end{array}$$

$$\begin{array}{ccccccccc} \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \hline \end{array}$$

$\cdots$

Figure 5.6: The set $B = A \cap A + (1, 0, 0)$

1-reducibility $\leq_1$, are countable Borel quasi-orders. The equivalence relations associated with them have been the subject of a great deal of work in descriptive set theory; for example, see [6].

Open Problem. Are any of $\leq_1, \leq_T$, etc. universal countable Borel quasi-orders?

It is open whether the associated countable Borel equivalence relations $\equiv_1$ or $\equiv_T$ are universal. We have seen that universal countable Borel quasi-orders give rise to universal countable Borel equivalence relations, so a positive answer to any part of this question would settle the problem for the corresponding equivalence relation. It is possible for a non-universal countable Borel quasi-order to symmetrize to a universal countable Borel equivalence relation, for example a quasi-order which was already an equivalence relation. So we ask the following.

Open Problem. How many countable Borel quasi-orders are there up to Borel bireducibility which symmetrize to a universal countable Borel equivalence relation?

There are interesting open questions about the relationship between the biembeddability relation $\equiv_{em}$ for finitely generated groups and the isomorphism relation $\cong_{\mathcal{G}}$ for finitely generated groups. As both relations are universal there are Borel reductions in each direction. However, both universality results ultimately rely upon the Lusin-Novikov theorem, which gives no information about the resulting Borel reductions.

It is therefore natural to ask if there are such Borel reductions which are group-theoretic in nature, meaning maps which only use group-theoretic constructions such as semidirect products, wreath products, HNN extensions, etc. All of these constructions induce continuous maps on the space $\mathcal{G}$ of finitely generated groups, so this raises the question of whether there is a continuous reduction between the two relations.

Open Problem. Is there a continuous reduction from $\cong_{\mathcal{G}}$ to $\approx_{\mathcal{G}}$, or in the opposite direction?

Currently the only results ruling out continuous reductions between countable Borel equivalence relations E, F such that $E \leq_B F$ are due to Thomas. In particular, we have the following.

Theorem 5.7.1 (Thomas [36]). *Suppose G is a countable subgroup of* $\mathrm{Sym}(\mathbb{N})$*, the group of bijections of* $\mathbb{N}$*. Let E_G be the orbit equivalence relation of the group on* $2^{\mathbb{N}}$*. Then $\equiv_T$ does not continuously reduce to E_G.*

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