Invariant random subgroups and action versus representation maximality

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1 Introduction

Let G be a countably infinite group and (X, μ) a standard non-atomic probability space. We denote by $A(G, X, \mu)$ the space of measure preserving actions of G on (X, μ) with the weak topology. If $\mathbf{a}, \mathbf{b} \in A(G, X, \mu)$, we say that $\mathbf{a}$ is **weakly contained** in $\mathbf{b}$, in symbols $\mathbf{a} \preceq \mathbf{b}$, if $\mathbf{a}$ is in the closure of the set of isomorphic copies of $\mathbf{b}$ (i.e., it is in the closure of the orbit of $\mathbf{b}$ under the action of the automorphism group of (X, μ) on $A(G, X, \mu)$; see [K]). We say that $\mathbf{a} \in A(G, X, \mu)$ is **action-maximal** if for all $\mathbf{b} \in A(G, X, \mu)$ we have $\mathbf{b} \preceq \mathbf{a}$. Such $\mathbf{a}$ exist by a result of Glasner-Thouvenot-Weiss, Hjorth, see [K, Theorem 10.7]).

Now let H be a separable, infinite-dimensional Hilbert space and denote by $\mathrm{Rep}(G, H)$ the space of unitary representations of G on H with the weak topology (see [K, Appendix H]). For $\pi, \rho \in \mathrm{Rep}(G, H)$ we denote by $\pi \preceq \rho$ the usual relation of **weak containment** of representations (see [BHV], [K, Appendix H]). We say that $\pi \in \mathrm{Rep}(G, H)$ is **representation-maximal** if for all $\rho \in \mathrm{Rep}(G, H)$ we have $\rho \preceq \pi$. It is easy to check that such π exist.

For any action $\mathbf{a} \in A(G, X, \mu)$, let $\kappa^{\mathbf{a}}$ be the associated representation on $L^2(X, \mu)$, called the **Koopman representation**, and by $\kappa_0^{\mathbf{a}}$ its restriction to the orthogonal of the constant functions (see [K, page 66]). Then we have

$$\mathbf{a} \preceq \mathbf{b} \implies \kappa_0^{\mathbf{a}} \preceq \kappa_0^{\mathbf{b}}$$

but the converse fails, see [K, pages 66 and 68] and also [CK, page 155] for examples. However in all these examples the actions $\mathbf{a}, \mathbf{b}$ were not both ergodic and this led to the following question.

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Problem 1.1. *If* $\mathbf{a}, \mathbf{b} \in A(G, X, \mu)$ *are free, ergodic, does* $\kappa_0^{\mathbf{a}} \preceq \kappa_0^{\mathbf{b}}$ *imply* $\mathbf{a} \preceq \mathbf{b}$*?*

We provide a negative answer below. The proof is based on a result about invariant random subgroups of $G = \mathbf{F}_\infty$, the free group on a countably infinite set of generators, which might be of independent interest.

If I is a countable set and α is an action of a countable group G on I, we will write $\mathbf{s}_\alpha$ for the corresponding **generalized shift action** on 2^I with the usual product measure, given by $(\mathbf{s}_\alpha(g) \cdot f)(i) = f(\alpha(g)^{-1} \cdot i)$. If $I = G/H$, for some $H \leq G$, we will write $\tau_{G/H}$ for the left-translation action of G on G/H and $\mathbf{s}_{G/H}$ instead of $\mathbf{s}_{\tau_{G/H}}$. If H is trivial, we write $\mathbf{s}_G$ instead of $\mathbf{s}_{G/H}$.

We also let λ_α be the representation on $\ell^2(I)$ given by $(\lambda_\alpha(g) \cdot f)(i) = f(\alpha(g)^{-1} \cdot i)$. Note that $\lambda_{\tau_{G/H}}$ is the usual **quasi-regular representation** of G on $\ell^2(G/H)$, which we will denote by $\lambda_{G/H}$.

We call a subgroup $H \leq G$ with $[G : H] = \infty$ **action-maximal** if $\mathbf{s}_{G/H}$ is action-maximal and **representation-maximal** if $\lambda_{G/H}$ is representation-maximal. It was shown in [K1] that there are H which are action-maximal and also H which are representation-maximal, for any non-abelian free group G.

An **invariant random subgroup (IRS)** of G is a probability Borel measure on $\mathrm{Sub}(G)$, the compact space of subgroups of G, which is invariant under the (continuous) action of G on $\mathrm{Sub}(G)$ by conjugation. Denote by $\mathcal{M}_G \subseteq \mathrm{Sub}(G)$ the set of all $H \leq G$ that are both action-maximal and representation-maximal. We show the following:

Theorem 1.1. *Let* $G = \mathbf{F}_\infty$*. Then there exists an IRS of* G *which is supported by* $\mathcal{M}_G$*.*

Using this and the result of Dudko-Grigorchuk [DG, Proposition 8], we then prove the following:

Theorem 1.2. *Let* $G = \mathbf{F}_\infty$*. Then there exists a free, ergodic* $\mathbf{a} \in A(G, X, \mu)$ *such that* $\mathbf{a}$ *is not action-maximal but* $\kappa_0^{\mathbf{a}}$ *is representation-maximal.*

Let $\mathbf{a}$ be as in Theorem 1.2. Since $G = \mathbf{F}_\infty$ does not have property (T), the free, ergodic actions $\mathbf{b} \in A(G, X, \mu)$ are dense in $A(G, X, \mu)$ (see [K, Theorems 12.2 and 10.8]), so there is a free, ergodic $\mathbf{b} \in A(G, X, \mu)$ such that $\mathbf{b} \not\preceq \mathbf{a}$. On the other hand $\kappa_0^{\mathbf{b}} \preceq \kappa_0^{\mathbf{a}}$, thus we have a negative answer to Problem 1.1.

We employ below the following notation:

If α is an action of G on I and $S \subseteq G$, we write $\alpha(S) = \{\alpha(g) : g \in S\} \subseteq \mathrm{Sym}(I)$. For $G = \mathbf{F}_\infty$, we let $g_0, g_1, \ldots$ be free generators of G and let $G_n = \langle g_0, g_1, \ldots, g_n \rangle \leq G$.

If x is a real number, we write $\lfloor x \rfloor$ for the largest integer less than or equal to x. If x, y are real numbers and $\epsilon > 0$, we write $x \approx_\epsilon y$ to mean $|x - y| < \epsilon$. Finally, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$

For the rest of the paper, $G = \mathbf{F}_\infty$.

2 Proof of Theorem 1.1

The structure of the proof is as follows. In Subsection 2.1 we state three lemmas. Temporarily assuming these lemmas, in Subsection 2.2 we give the main argument establishing Theorem 1.1. Then in Subsection 2.3 we prove the lemmas from Subsection 2.1.

Recall that for $\mathbf{a} \in A(G, X, \mu)$, we have $\mathbf{a} \preceq \mathbf{b}$ if and only if $\mathbf{a}$ lies in the closure of the isomorphic copies of $\mathbf{b}$. In particular, $\mathbf{b}$ is action-maximal if and only if the isomorphic copies of $\mathbf{b}$ are dense in $A(G, X, \mu)$. We will use these equivalences without comment several times in the sequel.

2.1 Statements of lemmas

The first lemma provides a general method for constructing invariant random subgroups.

Lemma 2.1. *Let α be an action of G on a countably infinite set I. Suppose there is an increasing sequence of non-empty finite subsets $(F_n)_{n=0}^\infty$ of I such that $\bigcup_{n=0}^\infty F_n = I$ and F_n is $\alpha(G_n)$-invariant. Let θ_n be the probability measure on $\mathrm{Sub}(G)$ given by the pushfoward of the uniform measure on F_n under the map $v \mapsto \mathrm{stab}_\alpha(v)$ (where $\mathrm{stab}_\alpha(v)$ is the stabilizer of v in α). Let θ be any weak-star limit point of the θ_n. Then θ is an invariant random subgroup of G.*

In order to state the second lemma, we need the following definition.

Definition 2.1. *Let α be an action of G on a finite set V and let n be such that all $\alpha(g_k), k > n$, act trivially. Let β be an action of G on a countably infinite set I. Let $Q \subseteq I$ be a finite set. We will say that α (relative to n)* ***appears in*** *β* ***within*** *Q if there is a $\beta(G_n)$-invariant set $W \subseteq Q$ and a bijection $\phi : V \to W$ such that $\phi(\alpha(g) \cdot v) = \beta(g) \cdot \phi(v)$ for all $v \in V$ and*

$g \in G_n$. *We will say that* α ***appears in*** β *if it appears within some finite subset of* I.

Note that if α appears in β as above, then $\mathbf{s}_{\alpha\restriction G_n}$ is a factor of $\mathbf{s}_{\beta\restriction G_n}$.

Lemma 2.2. *There exists a sequence of finite sets* $(V_n)_{n=1}^{\infty}$*, with* $|V_n| \to \infty$*, and actions* $(\alpha_n)_{n=1}^{\infty}$ *of* G*, where* α_n *acts transitively on* V_n *so that all* $g_k, k > n$*, act trivially in* α_n*, such that if* β *is a transitive action of* G *on a countably infinite set and* α_n *(relative to* n*) appears in* β *for each* n*, then* $\mathbf{s}_\beta$ *is action-maximal and* λ_β *is representation-maximal.*

Fix a sequence of finite sets V_n and actions α_n of G on V_n, $n \geq 1$, as in Lemma 2.2. Given $f : \mathbb{N} \to \mathbb{N}^+$, $m > 0$, write $C_m(f) = \sum_{n=0}^{m-1}(|V_{f(n)}| + 1)$. We will need a function f with the following properties.

Lemma 2.3. *There exists a function* $f : \mathbb{N} \to \mathbb{N}^+$ *such that:*

(i) for every $n \geq 1$ *there exists positive integer* $K = K_n$ *such that for all* j *there is* l *with* $\left\lfloor \frac{j}{K} \right\rfloor = \left\lfloor \frac{l}{K} \right\rfloor$ *and* $f(l) = n$*,*

(ii) for every $\epsilon > 0$*, there exists* $t > 0$*, such that for all* $m > 0$ *we have*

$$\frac{1}{C_m(f)} \sum_{n=1}^{t} (|V_n| + 1) \cdot \left|\left\{j \in \{0, \ldots, m-1\} : f(j) = n\right\}\right| > 1 - \epsilon.$$

2.2 Main argument

Let, for $n \geq 1$, α_n and V_n be as in Lemma 2.2 and let f be as in Lemma 2.3. Choose a pairwise disjoint sequence of finite sets W_n, $n \geq 0$, such that $|W_n| = |V_{f(n)}|$. Define an action of α of G on $\bigcup_{n=0}^{\infty} W_n$ by identifying W_n with $V_{f(n)}$ and letting G act on W_n according to $\alpha_{f(n)}$. Let $\{u_n\}_{n=0}^{\infty}$ be an enumeration of a countably infinite set disjoint from the W_n. We now modify α to obtain a new action β of G on $I = \left(\bigcup_{n=0}^{\infty} W_n\right) \cup \{u_n\}_{n=0}^{\infty}$. We will have that $\beta(g_k)$ agrees with $\alpha(g_k)$ on W_n when $k \in \{0, \ldots, f(n)\}$.

For each n, choose a point $w_n \in W_n$ and let $\beta(g_{f(n)+1})$ transpose w_n with u_n. Let $(l_n)_{n=0}^{\infty}$ be a strictly increasing sequence of indices such that $\max(n, f(0), \ldots, f(n+1)) + 1 < l_n$. Let $\beta(g_{l_n})$ transpose w_n and w_{n+1}.

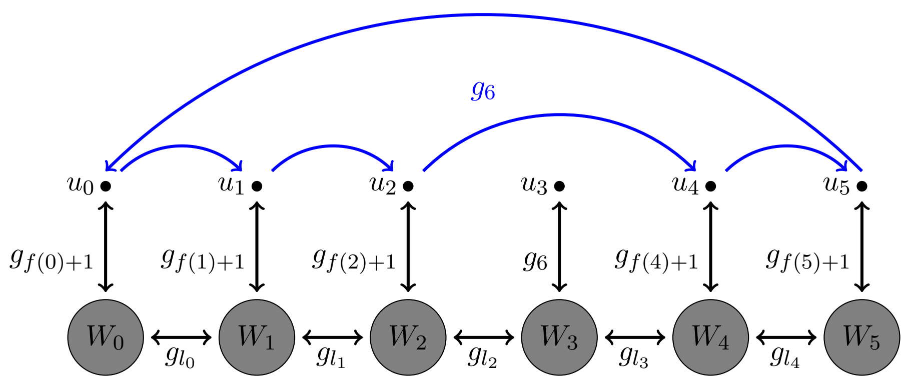
Fix $n \geq 1$. We now define how $\beta(g_n)$ acts on $\{u_j\}_{j=0}^{\infty}$. For $k \in \mathbb{N}$, consider the discrete interval

$$D_{n,k} = \{k \cdot n, \ldots, (k+1) \cdot n - 1\}.$$

We would like to have $\beta(g_n)$ make a cycle out $\{u_j, j \in D_{n,k}\}$ for each k. Unfortunately, we cannot achieve that exactly since there may by some $j \in D_{n,k}$ for which $f(j) + 1 = n$, and in this case we will have already used g_n to link W_j with u_j. Thus for each k, we will let $\beta(g_n)$ make a cycle out of the set

$$\{u_j : j \in D_{n,k} \text{ and } f(j) + 1 \neq n\},$$

making no modification to the action of $\beta(g_n)$ on those u_j for which $f(j) + 1 = n$. We will call these cycles the top cycles of $\beta(g_n)$. We have the following picture of β, where $n = f(3) + 1 = 6$ and we consider the interval $D_{6,0}$.



Finally β is defined trivially for all other points. Clearly β acts transitively. Write for $m > 0$, $\left(\bigcup_{k=0}^{m-1} W_k\right) \cup \{u_0, \ldots, u_{m-1}\} = T_m$ and for $m \geq 0$, $T_{m!} = F_m$. Thus F_m is invariant under $\beta(G_m)$. For each m, define a measure θ_m on $\mathrm{Sub}(G)$ be letting θ_m be the pushforward of the uniform measure on F_m under the map $v \mapsto \mathrm{stab}_\beta(v)$. Let θ be a weak-star limit point of θ_m. By Lemma 2.1, θ is an invariant random subgroup of G.

We claim that θ is supported on $\mathcal{M}_G$. Let $(Q_k)_{k=0}^{\infty}$ be an increasing sequence of finite subsets of G with $\bigcup_{k=0}^{\infty} Q_k = G$. For $H \leq G$, let $Q_k/H = \{gH : g \in Q_k\}$. Write, for $n \geq 1, k \in \mathbb{N}$,

$$A_{n,k} = \big\{H \leq G : \alpha_n \text{ appears in } \tau_{G/H} \text{ within } Q_k/H\big\}.$$

By definition, if $H \in \bigcup_{k=0}^{\infty} A_{n,k}$, then α_n appears in $\tau_{G/H}$. Therefore by Lemma 2.2, we have

$$\bigcap_{n=1}^{\infty} \bigcup_{k=0}^{\infty} A_{n,k} \subseteq \mathcal{M}_G.$$

Thus it suffices to show that for each $n \geq 1$ we have $\sup_{k<\infty} \theta(A_{n,k}) = 1$. Fix n and $\epsilon > 0$. Since the set $A_{n,k}$ is clopen for each k, it is enough to show the following:

Claim 2.1. *There is some $k \in \mathbb{N}$, such that for all $m > 0$, we have $\theta_m(A_{n,k}) > 1 - \epsilon$.*

Let t be large enough that Lemma 2.3(ii) holds for our chosen ϵ. We now define five finite subsets of G.

- Let $S_1 \subseteq G$ consist of $\{1_G\}$ together with every word in the generators $g_0, \ldots, g_t$ with length at most $\max_{1 \leq j \leq t} |V_j|$. If $f(j) \leq t$, this choice will allow us to pass between points in W_j using an element of S_1.
- Let $S_2 = \{1_G, g_0, \ldots, g_{t+1}\}$. If $f(j) \leq t$, this choice will allow us to pass to u_j from some point in W_j using an element of S_2.
- Let S_3 consist of all words in the generators g_K, g_{2K}, g_{3K} of length at most $3K$, where $K = K_n$ is the number provided by Lemma 2.3(i) for our fixed n. We will explain this choice later.
- Let $S_4 = \{g_{n+1}\}$. If $f(l) = n$, we will use g_{n+1} to pass from u_l to some point in W_l.
- Let S_5 consists of all words in the generators $g_1, \ldots, g_n$ of length at most $|V_n|$. If $f(l) = n$, this choice will allow us to pass between any two points of W_l using an element of S_5.

Let k be large enough that Q_k contains $S_5 \cdot S_4 \cdot S_3 \cdot S_2 \cdot S_1$. We assert that the following implies Claim 2.1.

Claim 2.2. *If $v \in W_j \cup \{u_j\}$ and $f(j) \leq t$, then α_n appears in $\tau_{G/\mathrm{stab}_\beta(v)}$ within $Q_k/\mathrm{stab}_\beta(v)$.*

Indeed, suppose Claim 2.2 holds and let $m > 0$. Note that $C_{m!}(f)$ defined as in Lemma 2.3 is exactly $|T_{m!}|$. Thus we have

$$\begin{aligned}
\theta_m(A_{n,k}) &= \frac{1}{|T_{m!}|} \cdot \left|\left\{v \in T_{m!} \colon \mathrm{stab}_\beta(v) \in A_{n,k}\right\}\right| && (2.1)\\
&\geq \frac{1}{|T_{m!}|} \cdot \left|\left\{v \in T_{m!} : v \in W_j \cup \{u_j\} \text{ and } f(j) \leq t\right\}\right| && (2.2)\\
&= \frac{1}{|T_{m!}|} \sum_{n=1}^{t} (|V_n| + 1) \cdot \left|\left\{j \in \{0, \ldots, m! - 1\} : f(j) = n\right\}\right| && (2.3)\\
&> 1 - \epsilon, && (2.4)
\end{aligned}$$

where

- (2.1) follows from the definition of θ_m,
- (2.2) follows from (2.1) by Claim 2.2,
- (2.3) follows from (2.2) since $|W_j| = |V_{f(j)}|$,
- (2.4) follows from (2.3) by Lemma 2.1(ii).

Thus it remains to establish Claim 2.2.

Fix j with $f(j) \leq t$. By our choice of K, there is some l such that $\lfloor j/K \rfloor = \lfloor l/K \rfloor$ and $f(l) = n$. Fix $v \in W_j \cup \{u_j\}$. Write $H = \mathrm{stab}_\beta(v)$ and let $P = \{gH : \beta(g) \cdot v \in W_l\}$. Since $\beta(G_n)$ acts on W_l according to α_n, it follows that α_n appears in $\tau_{G/H}$ within P. Therefore it is enough to show that $P \subseteq Q_k/H$, or equivalently $W_l \subseteq \beta(Q_k) \cdot v$. The idea is that we have chosen k large enough that we can reach any point in W_l from v using the β action of a word from Q_k.

By our choice of S_1, if $v \in W_j$ there is an element $\gamma \in S_1$ such that $\beta(\gamma) \cdot v = w_j$ where w_j is the point in W_j connected to u_j. The connection between w_j and u_j is made by $\beta(g_{f(j)+1})$. We have $g_{f(j)+1} \in S_2$ since $f(j) \leq t$. Thus $u_j = \beta(\gamma) \cdot v$, where $\gamma \in S_2 \cdot S_1$.

Note that our assumption on l guarantees that l lies between the same pair of multiples of K as j does. We would like to say that this allows us to pass from u_j to u_l using $\beta(g_K)^i$ for some $i \in [-K, K]$. However, there is the minor issue of the points u_d which are skipped the top cycles of $\beta(g_K)$. We can easily overcome this obstacle by noting that for any d, at most one of $\beta(g_K), \beta(g_{2K})$ and $\beta(g_{3K})$ skips over u_d, and therefore there is a word γ' in g_K, g_{2K}, g_{3K} of length at most $3K$ such that $\beta(\gamma') \cdot u_j = u_l$. We have $\gamma' \in S_3$.

Since $f(l) = n$, we see that u_l is connected to W_l by $\beta(g_{f(l)+1}) = \beta(g_{n+1})$. Therefore $\beta(g_{n+1}\gamma'\gamma) \cdot v \in W_l$. Since $W_l \subseteq \beta(S_5) \cdot \beta(g_{n+1}\gamma'\gamma) \cdot v$, we have that $W_l \subseteq \beta(Q_k) \cdot v$ and we are done.

2.3 Proofs of lemmas

Proof of Lemma 2.1. Let $h_1, \ldots, h_l, k_1, \ldots, k_{l'}, g \in G$ and let $\epsilon > 0$. Let m be large enough that $h_1, \ldots, h_l$, $k_1, \ldots, k_{l'}, g$ are words in the generators $\{g_0, \ldots, g_m\}$. Write

$$C = \{H \leq G : h_1, \ldots, h_l \in H \text{ and } k_1, \ldots, k_{l'} \notin H\}.$$

Note that C is a clopen set and therefore there is some $n \geq m$ such that

$$\theta(C) \approx_\epsilon \theta_n(C) \text{ and } \theta(gCg^{-1}) \approx_\epsilon \theta_n(gCg^{-1}). \tag{2.5}$$

Noting that F_n is $\alpha(\langle g, h_1, \ldots, h_l, k_1, \ldots, k_{l'}\rangle)$ invariant we have

$$\begin{aligned}
\theta_n(gCg^{-1}) = {} & \frac{1}{|F_n|} \cdot \big|\big\{v \in F_n : \alpha(gh_jg^{-1}) \cdot v = v \text{ for all } j \in \{1, \ldots, l\} \\
& \text{and } \alpha(gk_jg^{-1}) \cdot v \neq v \text{ for all } j \in \{1, \ldots, l'\}\big\}\big| \\
= {} & \frac{1}{|F_n|} \cdot \big|\big\{v \in F_n : \alpha(h_j)\alpha(g^{-1}) \cdot v = \alpha(g^{-1}) \cdot v \text{ for all } j \in \{1, \ldots, l\} \\
& \text{and } \alpha(k_j)\alpha(g^{-1}) \cdot v \neq \alpha(g^{-1}) \cdot v \text{ for all } j \in \{1, \ldots, l'\}\big\}\big| \\
= {} & \frac{1}{|F_n|} \cdot \big|\big\{w \in F_n : \alpha(h_j) \cdot w = w \text{ for all } j \in \{1, \ldots, l\} \\
& \text{and } \alpha(k_j) \cdot w \neq w \text{ for all } j \in \{1, \ldots, l'\}\big\}\big| \\
= {} & \theta_n(C)
\end{aligned}$$

Then from (2.5) we have $\theta(C) \approx_{2\epsilon} \theta(gCg^{-1})$. $\square$

Proof of Lemma 2.2. It is clearly enough to find such V_n, α_n such that for any β as in that lemma, $\mathbf{s}_\beta$ is action-maximal and another sequence, also denoted below by V_n, α_n, such that for any β as in that lemma, λ_β is representation-maximal. Then by interlacing these two sequences, we have a sequence that achieves both goals.

Case 1: We first find the sequence for which the appropriate $\mathbf{s}_\beta$ is action-maximal. By [K1, Theorem 5.1], there is a countably infinite set J and a transitive action α of G on J such that $\mathbf{s}_\alpha$ is action-maximal. Identify (X, μ) with 2^J carrying the usual product measure. For a finite set $T \subseteq J$ and $\rho \in 2^T$, write

$$N_\rho = \big\{x \in 2^J : x(v) = \rho(v) \text{ for all } v \in T\big\}.$$

For $n \geq 1$, $\epsilon > 0$ and a finite set $T \subseteq J$, let $U_{n,\epsilon,T}$ be the set of all $\mathbf{c} \in A(G, X, \mu)$ such that

$$\mu\big(\mathbf{s}_\alpha(g_k) \cdot N_\rho \cap N_\sigma\big) \approx_\epsilon \mu\big(\mathbf{c}(g_k) \cdot N_\rho \cap N_\sigma\big), \forall \sigma, \rho \in 2^T,\ k \in \{0, \ldots, n-1\}.$$

Observe that the collection of all $U_{n,\epsilon,T}$ is a neighborhood basis at $\mathbf{s}_\alpha \in A(G, X, \mu)$. Let $(T_n)_{n=1}^\infty$ be an increasing sequence of finite subsets of J with $\bigcup_{n=1}^{\infty} T_n = J$. Write $U_n = U_{n,2^{-n-|T_n|},T_n}$. Then the sets U_n form a neighborhood basis at $\mathbf{s}_\alpha$. Note that for each $n \geq 1$ and each $k \in \{0, \ldots, n-1\}$, we can extend $\alpha(g_k) \restriction \left(T_n \cup \bigcup_{j=0}^{n-1} \alpha(g_j) \cdot T_n\right)$ to a permutation of J which is trivial on the complement of a finite set containing $T_n \cup \bigcup_{j=0}^{n-1} \alpha(g_j) \cdot T_n$. Hence for each $n \geq 1$, we can find an action $\widehat{\alpha}_n$ of G on J with the following properties:

(I) $\widehat{\alpha}_n(g_k) \cdot v = \alpha(g_k) \cdot v$, if $k \in \{0, \ldots, n-1\}$ and $v \in T_n$.

(II) $\widehat{\alpha}_n(g_k)$ acts trivially if $k > n$.

(III) There is a $\widehat{\alpha}_n$-invariant finite set $V_n \subseteq J$ such that $\widehat{\alpha}_n \restriction (J \setminus V_n)$ is trivial and $\widehat{\alpha}_n \restriction V_n$ is transitive.

By (I) we see that $\mathbf{s}_{\widehat{\alpha}_n}(g_k) \cdot N_\rho = \mathbf{s}_\alpha(g_k) \cdot N_\rho$ for all $\rho \in 2^{T_n}$ and $k \in \{0, \ldots, n-1\}$. Therefore $\mathbf{s}_{\widehat{\alpha}_n} \in U_n$. Write $\alpha_n = \widehat{\alpha}_n \restriction V_n$. By (II) all $g_k, k > n$, act trivially in α_n. Observe that (III) implies that $\mathbf{s}_{\widehat{\alpha}_n} \cong \mathbf{s}_{\alpha_n} \times \boldsymbol{\iota}$, where $\boldsymbol{\iota}$ is the trivial action of G on a nonatomic standard probability space. Thus for each $n \geq 1$ there is an isomorphic copy of $\mathbf{s}_{\alpha_n} \times \boldsymbol{\iota}$ in U_n.

Suppose β is a transitive action of G on a countably infinite set such that α_n appears in β for each $n \geq 1$. Note that $\boldsymbol{s}_\beta$ is ergodic (see, e.g., [KT, 2.1]). Then $\mathbf{s}_{\alpha_n \restriction G_n}$ is a factor of $\mathbf{s}_{\beta \restriction G_n}$ and hence $\mathbf{s}_{\alpha_n \restriction G_n} \times (\boldsymbol{\iota} \restriction G_n)$ is a factor of

$\mathbf{s}_{\beta \restriction G_n} \times (\boldsymbol{\iota} \restriction G_n)$. Using the fact that the definition of U_n depends only on G_n, this implies that for each $n \geq 1$ there is an isomorphic copy of $\mathbf{s}_\beta \times \boldsymbol{\iota}$ in U_n. Therefore there is a sequence of isomorphic copies of $\mathbf{s}_\beta \times \boldsymbol{\iota}$ in $A(G, X, \mu)$ which converges to $\mathbf{s}_\alpha$. Since the isomorphic copies of $\mathbf{s}_\alpha$ are dense in $A(G, X, \mu)$, this implies that the isomorphic copies of $\mathbf{s}_\beta \times \boldsymbol{\iota}$ are dense in $A(G, X, \mu)$.

By [T, Theorem 3.11], we see that any ergodic action $\mathbf{d}$ of G is weakly contained in almost every ergodic component of $\mathbf{s}_\beta \times \boldsymbol{\iota}$. In particular, any ergodic action $\mathbf{d}$ of G is weakly contained in $\mathbf{s}_\beta$ and therefore the isomorphic copies of $\mathbf{s}_\beta$ are dense in the ergodic actions. Since G does not have Property (T), [K, Theorem 12.2] implies that the isomorphic copies of $\mathbf{s}_\beta$ are dense in A(G, X, μ).

Case 2: We next find a sequence V_n, α_n, for which the appropriate λ_β is representation-maximal. We start with a transitive action α of G on a countably infinite set J such that λ_α is representation-maximal (see [K1, Theorem 5.5]. Then proceed as in the proof of Case 1 to find V_n, α_n such that for some isomorphic copy σ_n of $\lambda_{\alpha_n} \oplus \infty 1_G$, (σ_n) converges to λ_α, where 1_G is the trivial one-dimensional representation of G and $\infty 1_G$ the direct sum of countably many copies of 1_G, i.e., the trivial representation on a separable, infinite-dimensional Hilbert space. Let now β be as above. Then the isomorphic copies of $\lambda_\beta \oplus \infty 1_G$ converge to λ_α. By a result of Hjorth, see [K, H.7], the irreducible representations are dense in $\mathrm{Rep}(G, H)$. Every irreducible representation π is $\preceq \lambda_\alpha$ and thus $\preceq_Z \lambda_\alpha \preceq_Z \lambda_\beta \oplus \infty 1_G$, where $\preceq_Z$ is weak containment in the sense of Zimmer. Recall that $\sigma \preceq_Z \rho$ iff σ is in the closure of the isomorphic copies of ρ. Also $\sigma \preceq_Z \rho \implies \sigma \preceq \rho$ and for σ irreducible, $\sigma \preceq_Z \rho \iff \sigma \preceq \rho$ (see [BHV, page 397] and [K, page 209]). Then by [AE, Proposition 3.5] π is a subrepresentation of an ultrapower of $\lambda_\beta \oplus \infty 1_G$, which is of course of the form $\lambda^*_\beta \oplus \eta^*$, where λ^* is an ultrapower of λ_β and η^* a trivial representation of G on a Hilbert space H^*. Let H_1 be the space on which this subrepresentation acts, which is a G-invariant subspace of the direct sum of the space of λ^*_β and H^*. Then if $v \in H^*$ and v_1 is its projection on H_1, v_1 is G-invariant, so as π is irreducible, $v_1 = 0$, i.e., $H^* \perp H_1$. Thus H_1 is contained in the space of λ^*_β, i.e., π is a subrepresentation of λ^*_β, so $\pi \preceq_Z \lambda_\beta$. Thus the isomorphic copies of λ_β are dense in $\mathrm{Rep}(G, H)$, i.e., λ_β is representation-maximal. □

Proof of Lemma 2.3. Note that letting for $n \geq 1$, $A_n = f^{-1}(\{n\})$ the statement of the lemma is equivalent to the existence of a partition $\mathbb{N} = \bigsqcup_{n \geq 1} A_n$ with the following properties:

(i) For each $n \geq 1$ there is positive integer K_n such that A_n intersects each interval $I_i^n = [iK_n, (i+1)K_n), i = 0, 1, 2, \ldots$.

(ii) Let $g : \mathbb{N}^+ \to \mathbb{N}^+$ be defined by $g(n) = |V_n| + 1$, where V_n is as in Lemma 2.2. Then we have that for each $\epsilon > 0$, there is $t > 0$, such that for all $m > 0$:

$$\frac{\sum_{n>t}(|A_n \cap m| \cdot g(n))}{\sum_n(|A_n \cap m| \cdot g(n))} < \epsilon,$$

where we identify here m with $\{0, 1, \ldots, m-1\}$.

To construct A_n, K_n, first chose $a_2 < a_3 < \ldots$ to be large enough so that a_n is divisible by 3 and

$$\sum_{n=2}^{\infty} \frac{1}{a_2 \cdots a_n} < \frac{1}{3} \text{ and } \frac{a_n}{3} > \frac{g(n)2^n}{g(n-1)}.$$

We let $A'_1 = \{2i \colon i \in \mathbb{N}\}$ and also put $K_1 = 2, K_n = 2a_2 \cdots a_n$ for $n \geq 2$. We will then inductively define pairwise disjoint $A_2, A_3, \ldots$, which are also disjoint from A'_1, to satisfy (ii) above and so that for $n \geq 2$, A_n has exactly one member in each interval I_i^n as above, and finally we put $A_1 = \mathbb{N} \setminus \bigcup_{n=2}^{\infty} A_n$.

So assume that $A'_1, A_2, \ldots, A_{n-1}$ have been constructed (this is just A'_1, if $n = 2$). To find A_n, so that (i) above is satisfied, it is enough to have for each i,

$$K_n > \frac{3}{2}\left|(A'_1 \cup A_2 \cup \cdots A_{n-1}) \cap I_i^n\right|.$$

But

$$\left|(A'_1 \cup A_2 \cup \cdots A_{n-1}) \cap I_i^n\right| = a_2 \cdots a_n + a_3 \cdots a_n + \cdots + a_{n-1}a_n + a_n,$$

so this follows from $\sum_{n=2}^{\infty} \frac{1}{a_2 \cdots a_n} < \frac{1}{3}$. Also for $i = 0$, we can choose the element of A_n in $[0, K_n)$ to be $\geq \frac{K_n}{3}$.

We finally check that (ii) is satisfied. Fix $\epsilon > 0$ and choose $t > 1$ so that $\sum_{n>t}^{\infty} 2^{-n} < \epsilon$. Consider now any $m > 0$ and $n > t$.

Case 1. $m \geq K_n$. Then for some $s > 1$, we have that $m \in I_{s-1}^n$ and $|A_n \cap m| \leq s$, while

$$\sum_n |A_n \cap m| \cdot g(n) \geq |A_{n-1} \cap m| \cdot g(n-1) \geq (s-1)a_n \cdot g(n-1)$$

so

$$\frac{|A_n \cap m| \cdot g(n)}{\sum_n |A_n \cap m| \cdot g(n)} \leq \frac{s \cdot g(n)}{(s-1) \cdot g(n-1)} \cdot \frac{1}{a_n} < 2^{-n}.$$

Case 2. $m < K_n$. Then either $m \leq \frac{K_n}{3}$ and $|A_n \cap m| = 0$ or $m > \frac{K_n}{3}$ and $|A_n \cap m| \leq 1$, in which case also

$$|A_{n-1} \cap m| \geq \frac{a_n}{3}.$$

So for any $m < K_n$,

$$\frac{|A_n \cap m| \cdot g(n)}{\sum_n |A_n \cap m| \cdot g(n)} \leq \frac{g(n)}{(\frac{a_n}{3})g(n-1)} < 2^{-n}.$$

Thus for any $n > t$, we have

$$\frac{|A_n \cap m| \cdot g(n)}{\sum_n |A_n \cap m| \cdot g(n)} < 2^{-n}$$

and so

$$\frac{\sum_{n>t}(|A_n \cap m| \cdot g(n))}{\sum_n(|A_n \cap m| \cdot g(n))} < \epsilon$$

□

3 Proof of Theorem 1.2

We note that if $\lambda_{G/H}$ is representation-maximal, then H is not amenable. This is because $1_G \preceq \lambda_{G/H}$ implies $\tau_{G/H}$ is amenable (see [KT, Theorem 1.1]).

We will use the notion of a random Bernoulli shift over an invariant random subgroup; we refer the reader to [T, Section 5.3] and [AGV, Proposition 45] for details. Let θ be the invariant random subgroup constructed in Theorem 1.1 and let $\mathbf{s}_\theta$ be the θ-random Bernoulli shift. Note that for almost every ergodic component $\mathbf{b}$ of $\mathbf{s}_\theta$, almost all stabilizers of $\mathbf{b}$ lie in $\mathcal{M}_G$ and hence the type of $\mathbf{b}$ is supported on $\mathcal{M}_G$. Fix such an action $\mathbf{b}$. Let (Y, ν) be the underlying space of $\mathbf{b}$.

For $y \in Y$ write $H_y = \mathrm{stab}_{\mathbf{b}}(y)$. By [DG, Proposition 14] we have $\lambda_{G/H_y} \preceq \kappa_0^{\mathbf{b}}$ for ν-almost every $y \in Y$. Since the type of $\mathbf{b}$ is supported on $\mathcal{M}_G$, for ν-almost every y we have that λ_{G/H_y} is representation-maximal and so $\kappa_0^{\mathbf{b}}$ is representation-maximal. Let $\mathbf{a} = \mathbf{b} \times \mathbf{s}_G$. Then $\mathbf{a}$ is free and ergodic and $\kappa_0^{\mathbf{a}}$ is representation-maximal. Suppose, toward a contradiction, that $\mathbf{a}$ were action-maximal.

Let $S \subseteq G^2$ be the collection of all pairs (g,h) such that $\langle g,h\rangle$ is nonamenable. Since λ_{G/H_y} is representation-maximal for ν-almost every $y \in Y$, and so H_y is not amenable, we see that $S \cap H_y^2$ is nonempty for ν-almost every y. Let $\phi : \mathbb{N} \to S$ be an enumeration of S. For $y \in Y$ let $\phi_y = \min\{n : \phi(n) \in H_y^2\}$. Then there is some $k \in \mathbb{N}$ such that $\nu(\{y : \phi_y = k\}) > 0$. Write $A = \{y : \phi_y = k\}$ and let N be the subgroup of G generated by the coordinates of $\phi(k)$. Note that for $y \in A$, we have $N \subseteq H_y$, and so $\mathbf{b} \upharpoonright N$ is trivial on A. By [K, Page 74], since $\mathbf{a}$ is action-maximal for G, we have that $\mathbf{a} \upharpoonright N$ is action-maximal for N. Observe that

$$\mathbf{a} \upharpoonright N = (\mathbf{b} \upharpoonright N) \times (\mathbf{s}_G \upharpoonright N) \cong (\mathbf{b} \upharpoonright N) \times (\mathbf{s}_N)^{\mathbb{N}} \cong (\mathbf{b} \upharpoonright N) \times \mathbf{s}_N.$$

So writing $\mathbf{c} = (\mathbf{b} \upharpoonright N) \times \mathbf{s}_N$, we have that $\mathbf{c}$ is action-maximal for N.

By [T, Theorem 3.12], this implies that any ergodic action $\mathbf{d}$ of N is weakly contained in almost every ergodic component of $\mathbf{c}$. Note that if $y \in A$, then $\boldsymbol{\iota}_{\{y\}} \times \mathbf{s}_N \cong \mathbf{s}_N$ is an ergodic component of $\mathbf{c}$, where by $\boldsymbol{\iota}_{\{y\}}$ we mean the trivial action of N on the one-point space $\{y\}$. Therefore $\mathbf{d} \preceq \mathbf{s}_N$. Since N does not have property (T), the ergodic actions of N are dense in $A(N, X, \mu)$ (see [K, 12.2]), so the isomorphic copies of $\mathbf{s}_N$ are dense in $A(N, X, \mu)$. But by [K, Proposition 13.2] this contradicts the fact that N is nonamenable.

Remark 3.1. *For $G = \mathbf{F}_\infty$, let $\mathbf{a}$ be as in Theorem 1.2. Then for any irreducible π we have $\pi \preceq \kappa_0^{\mathbf{a}}$, so $\pi \preceq_Z \kappa_0^{\mathbf{a}}$. Thus, as the irreducible representations are dense, $\pi \preceq_Z \kappa_0^{\mathbf{a}}$, for all π. Thus there is a free ergodic action $\mathbf{b}$ such that $\kappa_0^{\mathbf{b}} \preceq_Z \kappa_0^{\mathbf{a}}$ but $\mathbf{b} \not\preceq \mathbf{a}$, which is a somewhat stronger negative answer to Problem 1.1.*

Remark 3.2. *It is possible that one could use the techniques developed in this paper to show that Theorem 1.2 also holds for the free groups with finitely many generators $n > 1$ but we have not verified that.*

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