Classical edge colorings

Theorem (Vizing, Gupta)

Let $G = (V, E)$ be a (simple) graph of degree bounded by $\Delta < +\infty$. Then there is a map $c : E \rightarrow [\Delta + 1]$ such that $c(e) \neq c(f)$ whenever $e \cap f \neq \emptyset$.\(^1\)

\(^1\)All graphs in this talk are assumed to have uniformly bounded degree, and $[k] = \{1, \ldots, k\}$. 
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(a) such a map $c$ is called a proper edge coloring,
(b) chromatic index of $G$, $\chi'(G)$, is the smallest number of colors needed for a proper edge coloring of $G$,
(c) Vizing’s theorem $\Rightarrow \chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$, where $\Delta(G) = \max\{\deg_G(v) : v \in V\}$

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Measurable edge colorings

A Borel graph $\mathcal{G}$ is a triplet $(V, \mathcal{B}, E)$, where $(V, \mathcal{B})$ is a standard Borel space, $(V, E)$ is a graph and $E$ is a Borel subset of $[V]^2$ (the set of unordered pairs of $V$ endowed with the Borel structure inherited from $V \times V$).

A proper Borel edge coloring of $\mathcal{G}$ with $k$ colors is a Borel map $c : E \to [k]$ that is a proper edge coloring.
Measurable edge colorings

A Borel graph $\mathcal{G}$ is a triplet $(\mathcal{V}, \mathcal{B}, E)$, where $(\mathcal{V}, \mathcal{B})$ is a standard Borel space, $(\mathcal{V}, E)$ is a graph and $E$ is a Borel subset of $[\mathcal{V}]^2$ (the set of unordered pairs of $\mathcal{V}$ endowed with the Borel structure inherited from $\mathcal{V} \times \mathcal{V}$).

A proper Borel edge coloring of $\mathcal{G}$ with $k$ colors is a Borel map $c : E \to [k]$ that is a proper edge coloring.

- the Borel chromatic index of $\mathcal{G}$, $\chi'_B(\mathcal{G})$, is the smallest number of colors needed for a proper Borel edge coloring of $\mathcal{G}$.
Theorem (Kechris–Solecki–Todorčević)

Let $\mathcal{G}$ be a Borel graph such that $\Delta(\mathcal{G}) < +\infty$. Then $\chi'_B(\mathcal{G}) \leq 2\Delta(\mathcal{G}) - 1$.

- Laczkovich gave an example of 2-regular acyclic Borel bipartite Borel graph $\mathcal{G}$ such that $\chi'_B(\mathcal{G}) = 3$.

Theorem (Marks)

For every $\Delta > 2$ and every $k \in \{\Delta, ..., 2\Delta - 1\}$, there is a $\Delta$-regular acyclic Borel bipartite Borel graph $\mathcal{G}$ such that $\chi'_B(\mathcal{G}) = k$.

- In particular, Vizing’s theorem fails in the Borel context.
Measurable analogues of Vizing’s theorem

Question (Abért)

*Can every $\Delta$-regular graphing without multiple edges be properly edge colored by $\Delta + 1$ colors?*
Measurable analogues of Vizing’s theorem

Question (Abért)

Can every $\Delta$-regular graphing without multiple edges be properly edge colored by $\Delta + 1$ colors?

Question (Marks)

Given any $\Delta$-regular Borel graph $\mathcal{G}$ on a standard Borel probability space $(V, \mu)$, must there be a $\mu$-measurable edge coloring of $\mathcal{G}$ with $\Delta + 1$ colors?
Measurable analogues of Vizing’s theorem

Question (Abért)

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Question (Kechris–Marks, Problem 6.13)

*Let $\mathcal{G}$ be a Borel graph on a Polish space with $\Delta(\mathcal{G}) < +\infty$. Is it true that $\chi'_M(\mathcal{G}) \leq \Delta + 1$? Is it true that $\chi'_{BM}(\mathcal{G}) \leq \Delta + 1$?*
Measurable setting

Let \( G = (V, \mathcal{B}, E) \) be a Borel graph and \( \mu \) be a Borel probability measure on \((V, \mathcal{B})\).

**Definition**

The \( \mu \)-measurable chromatic index of \( G \), \( \chi'_\mu(G) \) is defined as the minimum \( k \in \mathbb{N} \) such that there is a \( \mu \)-null set \( X \subseteq V \) such that \( \chi'_\mathcal{B}(G \upharpoonright (V \setminus X)) = k \).
Measurable setting

Let $\mathcal{G} = (V, B, E)$ be a Borel graph and $\mu$ be a Borel probability measure on $(V, B)$.

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The $\mu$-measurable chromatic index of $\mathcal{G}$, $\chi'_\mu(\mathcal{G})$ is defined as the minimum $k \in \mathbb{N}$ such that there is a $\mu$-null set $X \subseteq V$ such that $\chi'_B(\mathcal{G} \upharpoonright (V \setminus X)) = k$.

- equivalently:
  
  (a) there is a $\mu$-measurable map $c : E \rightarrow [k]$ that is a proper edge coloring,

  (b) there is a Borel map $c : E \rightarrow [k]$ that is a proper edge coloring
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- equivalently:
  (a) there is a $\mu$-measurable map $c : E \to [k]$ that is a proper edge coloring,
  (b) there is a Borel map $c : E \to [k]$ that is a proper edge coloring at $\mu$-almost every vertex $v \in V$, 

- the measurable chromatic index of $G$, $\chi'_M(G)$, is defined as $\sup$ of $\chi'_\mu(G)$ over all Borel probability measures $\mu$ on $(V, \mathcal{B})$.

- $\Delta(G) \leq \chi'_\mu(G) \leq \chi'_\mathcal{B}(G) \leq 2\Delta(G) - 1$.
Measurable setting

Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph and $\mu$ be a Borel probability measure on $(V, \mathcal{B})$.

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The $\mu$-measurable chromatic index of $\mathcal{G}$, $\chi'_\mu(\mathcal{G})$ is defined as the minimum $k \in \mathbb{N}$ such that there is a $\mu$-null set $X \subseteq V$ such that $\chi'_\mathcal{B}(\mathcal{G} \upharpoonright (V \setminus X)) = k$.

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▶ the measurable chromatic index of $\mathcal{G}$, $\chi'_M(\mathcal{G})$, is defined as supremum of $\chi'_\mu(\mathcal{G})$ over all Borel probability measures $\mu$ on $(V, \mathcal{B})$. 

\[ \Delta(\mathcal{G}) \leq \chi'_\mu(\mathcal{G}) \leq \chi'_\mathcal{B}(\mathcal{G}) \leq \chi'_M(\mathcal{G}) \leq 2\Delta(\mathcal{G}) - 1 \]
Measurable setting

Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph and $\mu$ be a Borel probability measure on $(V, \mathcal{B})$.

**Definition**

The $\mu$-measurable chromatic index of $\mathcal{G}$, $\chi'_\mu(\mathcal{G})$ is defined as the minimum $k \in \mathbb{N}$ such that there is a $\mu$-null set $X \subseteq V$ such that $\chi'_\mathcal{B}(\mathcal{G} \upharpoonright (V \setminus X)) = k$.

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$\Delta(\mathcal{G}) \leq \chi'(\mathcal{G}) \leq \chi'_\mu(\mathcal{G}) \leq \chi'_M(\mathcal{G}) \leq \chi'_\mathcal{B}(\mathcal{G}) \leq 2\Delta(\mathcal{G}) - 1$
Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph and $\mu$ be a Borel probability measure on $(V, \mathcal{B})$.

- We say that $\mu$ is $\mathcal{G}$-invariant if

$$
\int_A \deg_B(v) \, d\mu(v) = \int_B \deg_A(w) \, d\mu(w)
$$

holds for every two Borel sets $A, B \subseteq V$, where $\deg_Y(v) = |\{w \in Y : \{v, w\} \in E\}|$. 

In this case, the quadruple $(V, \mathcal{B}, E, \mu)$ is called a **graphing**.
Graphings

Let $G = (V, B, E)$ be a Borel graph and $\mu$ be a Borel probability measure on $(V, B)$.

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Positive results for graphings

Theorem (Csóka–Lippner–Pikhurko)

Let $\mathcal{G} = (V, B, E, \mu)$ be a graphing. Then

$$\chi^\prime_{\mu}(\mathcal{G}) \leq \Delta(\mathcal{G}) + O(\sqrt{\Delta(\mathcal{G})}),$$

and $\chi^\prime_{\mu}(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1$ if $\mathcal{G}$ does not contain odd cycles.
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**Theorem (G–Pikhurko)**

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Theorem (G–Pikhurko)
Let \( \mathcal{G} = (V, B, E, \mu) \) be a graphing. Then \( \chi'_\mu(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1 \).

This answers the question of Abért.
Many new results in recent years (work of Bencs, Bernshteyn, Bowen, Chandgotia, Gao, Hrušková, Jackson, Krophne, Qian, Rozhoň, Seward, Thornton, Tóth, Unger, Weilacher, ...).

For example:

- Bernshteyn extended and applied the method of G–Pikhurko in the context of distributed algorithms.

- Free Borel actions of $\mathbb{Z}^d$ admit proper Borel edge coloring with $2d$ colors (independently by Bencs–Hrušková–Tóth, and Chandgotia–Unger, and G–Rozhoň, and Weilacher).

- Qian and Weilacher found connections of the topological relaxation to computable combinatorics which allowed them to derive an upper bound of $\Delta(G) + 2$ colors for the Baire measurable analogue of Vizing’s theorem.
Main result

Theorem
Let $G = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(G) < +\infty$ and $\mu$ be a Borel probability measure on $(V, \mathcal{B})$. Then $\chi'_\mu(G) \leq \Delta + 1$. 

Ingredients in the proof:
(I) technique of augmenting iterated Vizing chains introduced in G–Pikhurko,
(II) replacing $\mu$ with an equivalent but "tame" measure $\nu$. 
Main result

Theorem
Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < +\infty$ and $\mu$ be a Borel probability measure on $(V, \mathcal{B})$. Then $\chi'_\mu(\mathcal{G}) \leq \Delta + 1$.

Ingredients in the proof:
(I) technique of augmenting iterated Vizing chains introduced in G–Pikhurko,
(II) replacing $\mu$ with an equivalent but “tame” measure $\nu$. 
General strategy

A *partial* Borel edge coloring (of $\mathcal{G}$) is a Borel map $c : \text{dom}(c) \to [\Delta(\mathcal{G}) + 1]$ that satisfies $c(e) \neq c(f)$ whenever $e \cap f \neq \emptyset$ and $e, f \in \text{dom}(c)$, where $\text{dom}(c)$ is a Borel subset of $E$.

If we do not want to specify the domain we write simply $c; E \to [\Delta(\mathcal{G}) + 1]$, and we set $U_c = V \setminus \text{dom}(c)$. 
A **partial** Borel edge coloring (of $G$) is a Borel map $c : \text{dom}(c) \to [\Delta(G) + 1]$ that satisfies $c(e) \neq c(f)$ whenever $e \cap f \neq \emptyset$ and $e, f \in \text{dom}(c)$, where $\text{dom}(c)$ is a Borel subset of $E$.

If we do not want to specify the domain we write simply $c; E \to [\Delta(G) + 1]$, and we set $U_c = V \setminus \text{dom}(c)$.

**Strategy:** Inductively improve given partial Borel coloring.
General strategy

Given $c; E \rightarrow [\Delta(G) + 1]$
General strategy

Given \( c; E \to [\Delta(G) + 1] \)

(a) Assign to each \( e \in U_c \) an **augmenting** connected subgraph (a chain of edges) \( V_c(e) \subseteq \text{dom}(c) \) with the property:

\[ \exists c_e; E \to [\Delta(G) + 1] \text{ such that } \text{dom}(c_e) = \text{dom}(c) \cup \{e\} \text{ and } c \upharpoonright E \setminus V_c(e) = c_e \upharpoonright E \setminus V_c(e). \]
General strategy

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(b) Take a maximal collection $\{V_c(e)\}_{e \in I}$ such that $V_c(e) \cap V_c(f) = \emptyset$ for every $e \neq f \in I$, and augment all $\{V_c(e)\}_{e \in I}$ simultaneously to create $c'; E \to [\Delta(G) + 1]$ such that $\text{dom}(c') = \text{dom}(c) \cup \{e\}_{e \in I}$.
General strategy

Given \( c; E \rightarrow [\Delta(G) + 1] \)

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that \( \text{dom}(c') = \text{dom}(c) \cup \{ e \}_{e \in I} \).

(c) Start with \( c_0 = \emptyset \), and iterate this procedure to create a
sequence of partial colorings \( \{ c_n \}_{n=1}^{\infty} \) with the hope that

\[
  c(e) = \lim_{n \to \infty} c_n(e)
\]

is defined off of a \( \mu \)-null set.
Augmenting chains

Let $e = \{x, y\} \in U_c$, and pick

- $\alpha \in m_c(x) = \Delta(G) + 1 \setminus \{c(f) : x \in f\}$ (colors missing at $x$),
- $\beta \in m_c(y) = \Delta(G) + 1 \setminus \{c(f) : y \in f\}$.
(I) Augmenting chains

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- $\beta \in m_c(y) = [\Delta(G) + 1] \setminus \{c(f) : y \in f\}$.

Define $P_c(x, e)$ to be the concatenation of $e$ and the maximal $\alpha/\beta$ path starting at $y$. 
If we are lucky and $P_c(x, e)$ does not come back to $x$, then $P_c(x, e)$ is **augmenting**.
Augmenting chains

Up to a little reshuffling of colors around $x$, this can be always achieved. The augmenting chain is called Vizing’s chain, and it is of the form $W_c(x, e) = F^aP(\alpha/\beta)$. (Augmenting these chains one-by-one gives the proof of Vizing’s theorem for finite graphs.)
(I) Approximate edge colorings

How do augmenting chains connect with the measure $\mu$?
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How do augmenting chains connect with the measure $\mu$?

Proposition

Let $\mathcal{G}$ be a graphing and $c: E \rightarrow [\Delta(\mathcal{G}) + 1]$ be such that $|W_c(x, e)| \geq L + \Delta$ for some $L \in \mathbb{N}$. Then $\mu(U_c) \leq \frac{2\Delta^3}{L}$. 

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**Proposition**

Let $\mathcal{G}$ be a graphing and $c: E \rightarrow [\Delta(\mathcal{G}) + 1]$ be such that $|W_c(x, e)| \geq L + \Delta$ for some $L \in \mathbb{N}$. Then $\mu(U_c) \leq \frac{2\Delta^3}{L}$.

**Proof** Define an auxiliary graph $\mathcal{H}_c$ with vertex set $U_c \sqcup \text{dom}(c)$ and $(e, f) \in \mathcal{H}_c$ if $f \in P$ where $W_c(x, e) = F \cap P$. 
(I) Approximate edge colorings

We have $\deg_{\mathcal{H}_c}(e) \geq L$ for every $e \in U_c$ and $\deg_{\mathcal{H}_c}(f) \leq 2\Delta^3$ for every $e \in \text{dom}(c)$.

The fact that $\mu$ is $\mathcal{G}$-invariant gives

$$L\mu(U_c) \leq \int_{U_c} \deg_{\mathcal{H}_c}(e) = \int_{\text{dom}(c)} \deg_{\mathcal{H}_c}(f) \leq 2\Delta^3.$$
(I) Approximate edge colorings

Elek–Lippner type argument shows that given $d; E \to [\Delta(G) + 1]$, it is always possible to modify colors of at most $O(L)\mu(U_d)$ edges to produce $c; E \to [\Delta(G) + 1]$ such that $|W_c(x, e)| \geq L$ for every $e \in U_c$ and $\text{dom}(d) \subseteq \text{dom}(c)$. 

Proposition (Approximate Vizing for graphings)

For every $\epsilon > 0$, there is a partial Borel proper edge coloring $c; E \to [\Delta(G) + 1]$ such that $\mu(U_c) \leq \epsilon$. 

Elek–Lippner type argument shows that given \( d; E \to [\Delta(G) + 1] \), it is always possible to modify colors of at most \( O(L)\mu(U_d) \) edges to produce \( c; E \to [\Delta(G) + 1] \) such that \( |W_c(x, e)| \geq L \) for every \( e \in U_c \) and \( \text{dom}(d) \subseteq \text{dom}(c) \).

**Proposition (Approximate Vizing for graphings)**

For every \( \epsilon > 0 \), there is a partial Borel proper edge coloring \( c; E \to [\Delta + 1] \) such that \( \mu(U_c) \leq \epsilon \).
(I) Iterated Vizing chains

Unfortunately, the price that we have to pay for the modification, \( O(L) \), is of the same order as \( \mu(U_c)^{-1} \). Hence iterating this process will not produce \( d(e) = \lim_{n \to \infty} d_n(e) \).
Unfortunately, the price that we have to pay for the modification, $O(L)$, is of the same order as $\mu(U_c)^{-1}$. Hence iterating this process will not produce $d(e) = \lim_{n \to \infty} d_n(e)$.

New idea in $G$–Pikhurko yields $\deg_{\mathcal{H}_c}(e) \geq L^2$ for $e \in U_c$: 
Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < +\infty$. Then the connectivity component relation of $\mathcal{G}$, $F_\mathcal{G}$, is a countable Borel equivalence relation (CBER), i.e., the connectivity component $[v]_\mathcal{G}$ of each $v \in V$ is at most countable.
Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph such that $\Delta(\mathcal{G}) < +\infty$. Then the connectivity component relation of $\mathcal{G}$, $F_\mathcal{G}$, is a countable Borel equivalence relation (CBER), i.e., the connectivity component $[v]_\mathcal{G}$ of each $v \in V$ is at most countable.

**Definition**

Let $\mu$ be a Borel probability measure on $(V, \mathcal{B})$. We say that $\mu$ is $\mathcal{G}$-quasi invariant if $\mu([A]_\mathcal{G}) = 0$, whenever $\mu(A) = 0$. 

(II) Quasi-invariant measures
(II) Quasi-invariant measures

Fundamental tool $\rightsquigarrow$ Radon-Nikodym cocycle

A Borel function $\rho_\mu : F_g \to \mathbb{R}_{>0}$ with the property that

$$\mu(g(C)) = \int_C \rho_\mu(x, g(x)) \, d\mu(x)$$

for every $C \in \mathcal{B}$ and injective Borel map $g : C \to V$ such that $(\nu, g(\nu)) \in F_g$. 
The length of the chain $W_c(x, e)$ is not measured by the number of edges but by the weight

$$\sum_{w \in W_c(x, e)} \rho_\mu(x, w).$$
(II) Vizing chains

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~⇒~ chains of finite weight can be infinite (not that problematic)
The length of the chain $W_c(x,e)$ is not measured by the number of edges but by the weight

$$
\sum_{w \in W_c(x,e)} \rho_{\mu}(x, w).
$$

$\Rightarrow$ chains of finite weight can be infinite (not that problematic)

$\Rightarrow$ chains of large weight can be very short (the main issue for iterated Vizing chains)
(II) Bounded cocycle

Theorem

Let $G$ be a Borel graph and $\mu$ be a $G$-quasi-invariant Borel probability measure on $(V, \mathcal{B})$. Then there is an equivalent Borel probability measure $\nu$ on $(V, \mathcal{B})$ such that

$$\frac{1}{4\Delta} \leq \rho_{\nu}(x, y) \leq 4\Delta$$

for every edge $(x, y) \in E$. 
Theorem

Let $\mathcal{G}$ be a Borel graph and $\mu$ be a $\mathcal{G}$-quasi-invariant Borel probability measure on $(V, \mathcal{B})$. Then there is an equivalent Borel probability measure $\nu$ on $(V, \mathcal{B})$ such that

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for every edge $(x, y) \in E$.

if $\text{dist}_\mathcal{G}(x, y) = k \in \mathbb{N}$, then $\rho_{\nu}(x, y) \leq (4\Delta)^k$, where $\text{dist}_\mathcal{G}$ is the graph distance on $\mathcal{G}$.

chains of weight $L$ have size $\Omega(\log(L))$
(II) Bounded cocycle

Sketch of the argument:

(a) Averaging the cocycle $\rho_\mu$ to define everywhere positive $\Omega \in L^1(\mu)$ such that $\int \frac{1}{\Omega} \, d\mu \, \Omega = \frac{\nu}{\mu}$.

(b) Showing that $\rho_\nu(x, y) = \frac{\Omega(y)}{\Omega(x)} \rho_\mu(x, y)$ has the desired properties. Technical but direct computation once we describe $\Omega$. 
(II) Bounded cocycle

(a) Suppose that $\alpha$ is a probability distribution on $n$, i.e.,
$\alpha : n \rightarrow (0, 1]$ such that $\sum_{i=1}^{n} \alpha(i) = 1$. 
(II) Bounded cocycle

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$\alpha : n \to (0, 1]$ such that $\sum_{i=1}^{n} \alpha(i) = 1$.

The cocycle is then defined as $\rho_{\alpha}(i, j) = \frac{\alpha(j)}{\alpha(i)}$. 

How to get the uniform distribution by averaging?

Define $\Omega(k) = \sum_{j=1}^{n} \rho_{\alpha}(k, j) = 1$.

Then we have $R_{\Omega}d\alpha = \frac{1}{n}$ and $\beta$ that satisfies $\frac{1}{n} \Omega = d\beta d\alpha$ is the uniform distribution on $n$.

Indeed, we have $\rho_{\beta}(k, \ell) = \Omega(\ell) \frac{\alpha(\ell)}{\alpha(k)} \rho_{\alpha}(k, \ell) = \frac{\alpha(\ell)}{\alpha(i)}$. 
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\[
\Omega(k) = \sum_{j=1}^{n} \rho_{\alpha}(k, j) = \frac{1}{\alpha(k)}.
\]

Then we have \( \int_{n} \Omega d\alpha = n \) and \( \beta \) that satisfies \( \frac{1}{n} \Omega = \frac{d\beta}{d\alpha} \) is the uniform distribution on \( n \).
(II) Bounded cocycle

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Define

$$\Omega(k) = \sum_{j=1}^{n} \rho_{\alpha}(k, j) = \frac{1}{\alpha(k)}.$$  

Then we have $\int \Omega d\alpha = n$ and $\beta$ that satisfies $\frac{1}{n} \Omega = \frac{d\beta}{d\alpha}$ is the uniform distribution on $n$.

Indeed, we have

$$\rho_{\beta}(k, \ell) = \frac{\Omega(\ell)}{\Omega(k)} \rho_{\alpha}(k, \ell) = \frac{\alpha(k)}{\alpha(\ell)} \frac{\alpha(\ell)}{\alpha(k)} = 1.$$
For $v \in V$, we define

$$
\Omega(v) = \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{\text{dist}(v,w)=k} \frac{1}{\Delta_k} \rho_\mu(v, w).
$$

Need to show that $\Omega \in L^1(\mu)$, in particular, $\Omega(v)$ is finite $\mu$-almost everywhere.

In reality formula is more complicated that is why we get the estimate $2\Delta$ on the next slide.
(II) Bounded cocycle

(b) We have for every edge \((x, y) \in E\)

\[
\Omega(y)\rho_\mu(x, y) = \left( \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{\text{dist}(y, w) = k} \frac{1}{\Delta_k} \rho_\mu(y, w) \right) \rho_\mu(x, y)
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{\text{dist}(y, w) = k} \frac{1}{\Delta_k} \rho_\mu(x, w)
\]

\[
\leq 2\Delta \left( \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{\text{dist}(x, w) = k} \frac{1}{\Delta_k} \rho_\mu(x, w) \right) = 2\Delta \Omega(x).
\]

\[
\sim \rho_\nu(x, y) = \frac{\Omega(y)}{\Omega(x)} \rho_\mu(x, y) \leq 2\Delta
\]
Thank you!