FINAL

Instructions

Open book, open notes. You may appeal to results stated in the accompanying class notes, including exercises. References to theorems from other sources will not be accepted. You may have any text book you like open while you take the exam, but you should refer to the class notes, not the text book.

You may lose points if you do not justify your assertions, either with a proof or with a reference from the notes.

There is a **four-hour** time limit. No credit will be given for work done after four hours.

Turn in the midterm to the usual box before 4pm on Friday, December 11$^{\text{th}}$.

1. Consider $\mathbb{Q}^2$ as a subset of $\mathbb{R}^2$. Prove that the complement $\mathbb{R}^2 \smallsetminus \mathbb{Q}^2$ is path connected.

2. Let X be a compact topological space and let $Y \subseteq X$ be a subset, equipped with the induced topology.

   (a) Prove that if Y is closed then Y is compact.

   (b) Prove that if X is Hausdorff and Y is compact then Y is closed.

   (c) Let $f : X \to X'$ be a continuous bijection from a compact topological space to a Hausdorff topological space. Prove that f is a homeomorphism.

3. Let M_1 and M_2 be Möbius bands, with boundary circles C_1 and C_2 respectively. Let X be the topological space obtained by gluing M_1 and M_2 using a homeomorphism $\phi : C_1 \to C_2$.

   (a) Describe a cell-space structure for X that has only one two-cell. (A good drawing is sufficient.)

   (b) Use this cell-space structure to write down a presentation for the fundamental group of X.

   (c) What is the usual name for the space X?

4. Compute the fundamental group of the real plane $\mathbb{R}^2$ with the two points $(-1, 0)$ and $(+1, 0)$ removed. Be sure to describe exactly any homotopies that you use (although a precise description using a drawing is acceptable).

5. For this question, you may use the fact that $H_1(S^1) \cong \mathbb{Z}$.

   (a) Use the Mayer–Vietoris Sequence to compute the singular homology groups of S^2.

   (b) Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.