MA109a Homework 5: Solutions

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Note that this solution set is only an outline. Solutions that give only as much detail as the ones here will likely not receive full credit.

1

a) $\tau : S^n \to \mathbb{R}P^n$ is a covering map where τ is the quotient map corresponding to the equivalence relation on S^n that associates antipodal points. So $\mathbb{Z}/2\mathbb{Z}$ acts by deck translations on S^n where the $1 \in \mathbb{Z}/2\mathbb{Z}$ takes points to their antipodes; this is the entire deck group since for any $p \in \mathbb{R}P^n$, $\pi^{-1}(p)$ consists of two points. For $n > 1$, S^n is simply connected so this is the universal cover hence $\pi_1(\mathbb{R}P^n) \simeq \mathbb{Z}/2\mathbb{Z}$. [Prop 3.28]

$\mathbb{R}P^1 \approx S^1$ so $\pi_1(\mathbb{R}P^1) \simeq \mathbb{Z}$. Alternatively, use Prop 3.28 as above.

b) Consider S^n as two copies of D^n glued along their boundaries. The quotient map to $\mathbb{R}P^n$ glues the two hemispheres together, so $\mathbb{R}P^n$ may be constructed as a quotient of D^n by identifying antipodal points of ∂D^n. But this is simply $\mathbb{R}P^{n-1}$. So $\mathbb{R}P^n$ may be constructed inductively by gluing D^n to $\mathbb{R}P^{n-1}$ where the attaching map is the same $\tau : \partial D^n = S^{n-1} \to \mathbb{R}P^{n-1}$ as above. So a cell decomposition has a single point as the 0-skeleton and one i-cell for each $i \leq n$ with attaching map τ.

The fundamental group of a cell complex depends only on the 2-skeleton [Cor 4.10] so for $n > 1$, $\pi_1(\mathbb{R}P^n) \simeq \pi_1(\mathbb{R}P^2)$. The 1-skeleton is S^1 so the fundamental has one generator $< a >$. To find the relations consider the identity map $f : S^1 \to D^2$. Composing with the attaching map, $\tau \circ f$ is twice a generator of the fundamental group of the 1-skeleton. Thus $\pi_1(\mathbb{R}P^n) \simeq < a | a^2 > \simeq \mathbb{Z}/2\mathbb{Z}$.

2

Cut a hole out of X that does not intersect either of the poles. The resulting space X' is homotopic to S^1. The identity map $S^1 \to \partial X' \approx S^1$ is nulhomotopic so by van Kampen's theorem, $\pi_1(X) \simeq \pi_1(X') \simeq \mathbb{Z}$.

3

$\mathbb{T}^2$: Call the 0-cell, 1-cells, and 2-cell v, a and b, and D respectively. The only possible attaching maps for a and b to v result in the figure 8. Choose orientations for a, b, and $\partial D \approx S^1$. Divide ∂D into 4 segments and map them in order to a, b, a, b where the first two segments are attached with an orientation-preserving homeomorphism and the last two are attached with orientation-reversing homeomorphisms.

$\mathbb{R}^2$: Put a 0-cell at each integer point and attach 1-cells horizontally and vertically to form a grid. Glue 2-cells into each square.

5

a) The cell decomposition with one 0-cell, two 1-cells and one 2-cell results in $< a, b|aba^{-1}b >$.

b) This is easiest with a picture. Construct the torus from two cylinders with their ends attached with an orientation-reversing homeomorphism. The quotient map to the torus identifies the two cylinders and changes the orientation of the edges.

c) The plane covers the torus and the torus covers the Klein bottle. These spaces are locally path connected and locally simply connected so the composition demonstrates that the plane is the universal cover of the Klein bottle. Use [Cor 3.29].