MA109a: HOMEWORK 6 Solutions

1. Let $B_i = im(\partial_{i+1})$ and $Z_i = ker(\partial_i)$. For each i we have short exact sequences $0 \to Z_i \to C_i \to B_{i-1}$ and $0 \to B_i \to Z_i \to H_i$ so

$$rk(C_i) = rk(Z_i) + rk(B_{i-1})$$

$$rk(Z_i) = rk(B_i) + rk(H_i).$$

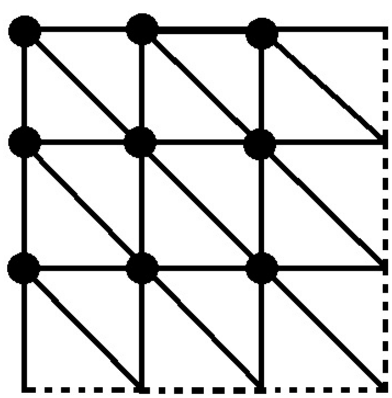
Combining,

$$rk(C_i) = rk(H_i) + rk(B_{i-1}) + rk(B_i)$$

so

$$\begin{aligned}\sum(-1)^i rk(C_i) &= \sum(-1)^i rk(H_i) + \sum(-1)^i[rk(B_{i-1}) + rk(B_i)] \\ &= \sum(-1)^i rk(H_i).\end{aligned}$$

2. See the image below for a simplicial decomposition of the torus. Note that this is not the most efficient one. In fact the minimal number of vertices required is 7.



The simplicial chain complex associated to this decomposition has

$$C_0 \simeq \mathbb{R}^9, C_1 \simeq \mathbb{R}^{27}, \text{and } C_0 \simeq \mathbb{R}^{18}.$$

We prove in a manner identical to that of Thm 5.13 that $H_0(C_*) \simeq \mathbb{R}$. $H_0(C_*) \simeq C_0/im(\partial_1)$. Let $A \subset C_0$ be the subspace of chains where the coefficients sum to one. That is

$$A := \{\Sigma a_i v_i | \Sigma a_i = 0\}.$$

We claim that $im(\partial_1) = A$ so that $H_0(C_*) \simeq \mathbb{R}$. Clearly $\partial C_1 \subseteq A$ since the boundary of any 1-simplex consists of two 0-simplices with opposite signs. To see the opposite inclusion, let $c = \Sigma a_i v_i \in C_0$ be a 0-cycle with $\sum a_i = 0$.

Fixing a vertex v_0 this is equivalent to $a_0 = -\sum_{i\neq 0} a_i$. Since T is connected, for each vertex v_i we can choose a 1-chain b_i with $\partial b_i = v_0 - v_i$. Then letting $b = \sum b_i$, $\partial b = c$.

Similarly, $H_2(C_*) \simeq R$. Orient the 2-simplices counterclockwise. Then each 1-simplex appears as the boundary of two 2-simplices with opposite sign. So if $c = \Sigma a_i \Delta_i$ is a 2-chain then $\partial c = 0$ implies that $a_i = a_j$ if Δ_i and Δ_j are neighboring 2-simplices. Since T is connected, this implies that $H_2(C_*) \simeq \mathbb{R}$ is the subspace of C_2 generated by 2-chains where each simplex has the same coefficient.

Now by problem 1 we know that

$$rk(H_0) - rk(H_1) + rk(H_2) = 9 - 27 + 18 = 0.$$

Hence $rk(H_1) = 2$. But since we're working with coefficients in R the chain groups and homology groups are all $\mathbb{R}$-vector spaces. Vector spaces are always free and are determined by their free ranks. So $H_1 \simeq \mathbb{R}^2$.

3. See Hatcher, Lemma 2.1 on p.105.

4. A 1-dimensional simplicial polyhedron is a graph. Suppose that K has n connected components, v vertices and e edges. By Theorem 5.13, $H_0 \simeq \mathbb{Z}^n$ and from problem 1,

$$v - e = rk(H_0) - rk(H_1).$$

So

$$rk(H_1) = n - v + e.$$

We claim that H_1 has no torsion so that

$$H_1 \simeq \mathbb{Z}^{n-v+e}.$$

This is obvious since $H_1 = ker(\partial_1)$ and $\partial_1 : C_1 \to C_0$ is $\mathbb{Z}$-linear. If $\partial(c) \neq 0$ for a 1-chain c then $\partial(nc) = n\partial(c) \neq 0$ for all nonzero n.

5. Suppose that K has n vertices and edges. Fix a vertex as the basepoint and label the vertices $\{0, ..., n-1\}$ consistent with a choice of orientation. We can then represent a simplicial map as a function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Note that this is a map of sets, not a homomorphism of groups.

   A simplicial homeomorphism is determined by a choice of $f(0)$ and whether it is orientation-preserving or reversing. So there are $2n$ simplicial maps $K \to K$ that are homeomorphisms.

   A choice of map that is not a homeomorphism is given by:

   (a) a choice of $f(0)$, for which there are n options.

(b) given $f(i)$, we can have $f(i+1) = f(i)$ if f collapses the corresponding edge, $f(i+1) = f(i)+1$ if the orientation of that edge is preserved, or $f(i+1) = f(i)-1$ if the orientation of that edge is reversed.

Further, the number of edges whose orientations are preserved must be equal to the number of edges whose orientations are reversed. Suppose m is this number. Then there are $\binom{n}{m}\binom{n-m}{m}$ such maps. In total, there are

$$2n + n \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n}{m}\binom{n-m}{m}$$

simplicial maps $K \to K$.