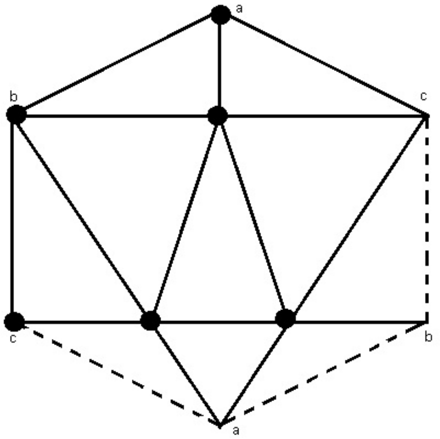
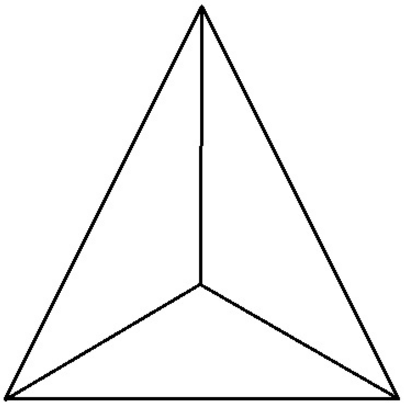
MA109a: HOMEWORK 7 Solutions

1. Here's a triangulation with six vertices:



2. a) Here's a picture:



b)[1] The boundary of a standard 4-simplex is consists of five 0-simplices with each pair of vertices connected by a 1-simplex. That is, it is the complete graph on 5 vertices, K_5. We will show that K_5 cannot be embedded in $\mathbb{R}^2$

Label the vertices of K_5 by $v_1, ..., v_5$. Suppose K_5 could be embedded in $\mathbb{R}^2$. Since no three vertices are collinear, the triangle $v_1v_2v_3$ is a Jordan curve and hence partitions the plane into two regions. Now v_4 must lie in one of these regions so the plane is partitioned into four regions, $R_i, i \in \{1, 2, 3, 4\}$ where R_i is the boundary and j, k, l distinct and not equal to i. Now v_5 must lie in one of

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[1] The proof given here shows that there are no *topological embeddings* (ie continuous injections) of the 1-skeleton of a 4-simplex into $\mathbb{R}^2$. Actually, my intention in asking the question was that you should prove the easier fact that there are no simplicial polyhedra in $\mathbb{R}^2$ that abstractly have the structure of the 1-skeleton of a 4-simplex. The proof is more or less the same as the one given here, but you don't need to invoke the Jordan Curve Theorem (which is obvious for triangles).—Henry

these regions, say R_i, so that v_i and v_5 are separated by the curve $v_jv_kv_l$. Then any curve edge connecting v_5 and v_i must intersect $v_jv_kv_l$ by the Jordan curve theorem.

3. Let $\alpha \in H_i(C_*)$ be represented by a cycle $a \in Z_i = ker(\partial_i)$. Then $f_*(\alpha), f_*(\alpha)$ are the homology classes of $f(a), g(a)$ respectively. Now

$$f(a) - g(a) = \partial'(h(a)) + h(\partial(a)) = \partial'(h(a))$$

using the fact that a is a cycle. But this means that $f(a)$ and $g(a)$ differ by a boundary and hence are in the same homology class.

4. a) We just need to get rid of self-loop and pairs of vertices that are connected by more than one edge. This is a simple inductive argument. If there is an edge in X that has the same vertex as its at both ends, then subdivide that edge. Continue this process until all edges have distinct ends.

Similarly, if there is some pair of edges in X that share both endpoints then subdivide one of those edges. Continue until all pairs of edges share at most one endpoint.

b) Let X' be the graph that results from the process in part (a). We have

$$H_1(X) \simeq H_1(X') \simeq H_1^{\Delta}(X'),$$

where the first isomorphism comes from Cor 5.25 using the fact that X and X' have the same homotopy type, and the second isomorphism comes from Thm 5.24 using the fact that X' is a simplicial polyhedron.

By the previous homework, $H_1^{\Delta}(X') \simeq \mathbb{Z}^{n'-v'+e'}$ where n', e', and v' are the number of components, edges, and vertices of X'. The process in part (a) does not change the number of components and each step adds one vertex and one edge. So $n' - v' - e' = n - v - e$. In summary,

$$H_1(X) \simeq \mathbb{Z}^{n-v+e}$$
$$H_0(X) \simeq \mathbb{Z}^{n}.$$

5. In a previous homework exercise we showed that the once-punctured torus is homotopy equivalent to a graph with one vertex and two edges. So using Cor 5.25 and exercise 4,

$$H_1(T - \{p\}) \simeq \mathbb{Z}^2$$
$$H_0(T - \{p\}) \simeq \mathbb{Z}.$$