HOMEWORK 8

1. Let T be the 2-torus and let T' be the space obtained by removing a small open disc D from T; the resulting space T' has a boundary circle C. Take two copies T'_1 and T'_2 of T', with boundary circles C_1 and C_2 respectively. The *surface of genus two* Σ is the space constructed by gluing T'_1 and T'_2 together using a homeomorphism $C_1 \to C_2$.

   (a) Compute the Betti numbers of Σ.

   (b) Is Σ homotopy equivalent to a graph?

2. Prove that $b_n(S^n) = 1$ for any n. (Here $b_n(X)$ is the nth Betti number of a space X and S^n is the n-dimensional sphere.)

3. Prove the n-dimensional Brouwer Fixed Point Theorem.

   Theorem *Let D^n be the closed unit disc in $\mathbb{R}^n$. If $f : D^n \to D^n$ is a continuous map then there exists $x \in D^n$ such that $f(x) = x$.*

   Hint: Follow the strategy in the 2-dimensional case, but use singular homology instead of the fundamental group.

4. This question is about the *Euler characteristic* of 2-dimensional polyhedra. Let K be a 2-dimensional simplicial polyhedron with V vertices (0-simplices), E edges (1-simplices) and F 2-simplices (faces).

   (a) Let $C(K)$ be the simplicial chain complex of K. Prove that $V - E + F = \chi(C(K))$.

   (b) If L is another simplicial polyhedron with $|L| \approx |K|$, prove that $\chi(C(L)) = \chi(C(K))$.

   (c) Suppose now that $|K| \approx S^2$. Prove that $V - E + F = 2$.