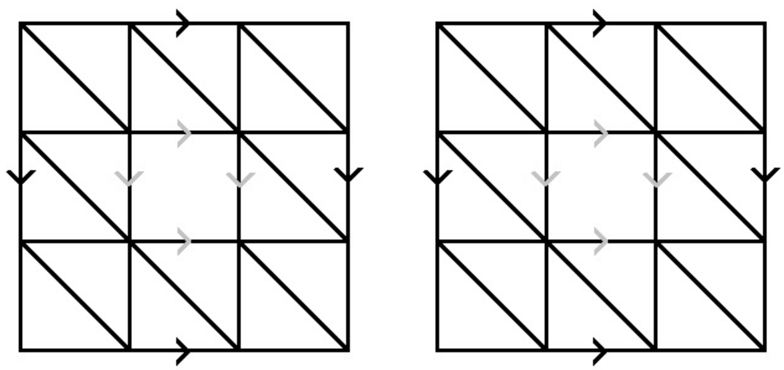
MA109a: HOMEWORK 8 Solutions

1. a)The surface Σ is connected, so we know that

$$b_0 = 1.$$

A triangulation of Σ is given below. The two outer horizontal edges of each annulus are glued together as are the two outer vertical edges. The corresponding inner edges of each annulus are glued to each other. There are 14 vertices, 48 edges and 32 faces, so we calculate that the Euler number is -2.



We can orient the 2-simplices so that each 1-simplex is in the boundary of exactly two 2-simplices with opposite signs. Then by Lemma 5.14 we have that $H_2(\Sigma)$ is generated by the cycle $[\Sigma] = \sum \Delta_i$ where the sum ranges over all 2-simplices which shows that

$$b_2 = 1.$$

Finally,[1]

$$-2 = \chi(\Sigma) = b_0 - b_1 + b_2$$

so

$$b_1 = 4.$$

b) No. As shown in Homework 7, graphs have $b_2 = 0$.

2. Let D^+ and D^- be neighborhoods of the two hemispheres of S^n. They are both homeomorphic to D^n and $D^+ \cap D^- \simeq S^{n-1}$. The relevant part of the Mayer-Vietoris sequence is

$$0 \to H_n(S^n) \to H_{n-1}(S^{n-1}) \to 0.$$

[1]Alternatively, you can deduce this using the Mayer–Vietoris Sequence.

So by induction

$$b_n(S^n) = b_1(S^1) = 1$$

for all $n > 0$.

3. Suppose $f : D^n \to D^n$ has no fixed points. Then we can define a map $g : D^n \to \partial D^n$ where $g(x)$ is the point at which the ray from x through $f(x)$ meets the boundary. As in the 2-dimensional case, g is continuous and fixes the boundary. That is,

   $$g \circ i = id$$

   where $i : S^{n-1} \hookrightarrow D^n$ is the inclusion of the boundary. By functoriality we have

   $$g_* \circ i_* = (g \circ i)_* = id_* = id_{H_n(S^n)}$$

   where $*$ denotes the induced map on n'th homology. But $H_n(S^n)$ is nontrivial so the identity map cannot factor through the trivial group $H^n(D^n)$.

4. a) By definition,

   $$\chi(C(K)) = \sum (-1)^i \text{rank } C_i = \text{rank } C_0 - \text{rank } C_1 + \text{rank } C_2 = V - E + F.$$

   b) We showed on Homework 6 that

   $$\xi(C) = \sum (-1)^i b_i$$

   where b_i is the i'th Betti number. Triangulable spaces that are homeomorphic have isomorphic homology groups and hence the same Betti numbers. So

   $$\chi(C(L)) = \chi(C(K)).$$

   c) The 2-sphere has $b_0 = b_2 = 1$ and $b_1 = 0$. So given a polyhedrom K with $|K| \simeq S^2$,

   $$V - E + F = \chi(C(K)) = \chi(S^2) = 2$$