INTRODUCTION TO GEOMETRY AND TOPOLOGY

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1 Topological spaces

It is traditional to start by saying that topology is 'rubber geometry'. More precisely, topology is the study of continuous maps, like the ones you learned about in calculus. We will only care about the properties of spaces that are preserved by continuous maps.

1.1 Definitions and basic examples

At first the definitions we give will seem very abstract. Hopefully, it will quickly become clear that we are providing a framework in which we can talk about the facts about continuity that you learned in calculus or analysis.

Definition 1.1. A *topological space* is a set X with a *topology* $\tau \subseteq 2^X$. If $U \in \tau$ then U is called *open*. The topology τ is required to satisfy the following properties.

1. The union of any set of open sets is open.
2. The intersection of any finite union of open sets is open.
3. The empty subset $\varnothing$ and the whole set X are both open.

You can think of a topology as encoding an idea of proximity or nearness. The more open sets that contain two points x and y, the closer we think of x and y as being.

Example 1.2. We can endow any set X with the *discrete* topology $\tau = 2^X$.

This is the *finest* topology we can impose on X—that is, the one with the most open sets. All the points are at a distance from each other.

Example 1.3. Any set X can be given the *indiscrete* topology $\tau = \{\varnothing, X\}$.

This is the *coarsest* topology we can give X—the one with the fewest open sets. All the points are very close to each other.

Neither the discrete topology nor the indiscrete topology is very interesting. If X is finite then in principle we can specify a topology on X by writing down its elements. For infinite X, however, we will need a different way of specifying which topology we are talking about. We will do so by writing down collections of subsets and insisting that they must be open.

Definition 1.4. Let $\sigma \subseteq 2^X$ be any subset. Then σ *generates* a topology τ, which is the coarsest topology in which every element of σ is open. We can construct τ explicitly. Let

$$\beta = \{\bigcap_{U \in I} U \mid I \subseteq \sigma,\ |I| < \infty\}$$

the set of all *finite* intersections of elements of σ. Now let

$$\tau = \{\bigcup_{U \in J} U \mid J \subseteq \beta\}$$

the set of all *arbitrary* unions of elements of β.

We say that σ is a *sub-basis* for τ. The collection β is a *basis* for τ, because it has the property that $U, V \in \beta$ implies that $U \cap V \in \beta$. (This is not quite the same as the definition of a basis given in Vassiliev.)

Exercise 1.5. Check that τ really is a topology.

We can define a topology by specifying a sub-basis.

Example 1.6. For any $x \in \mathbb{R}$ and $r > 0$, let

$$B_r(x) = \{y \in \mathbb{R} \mid |y - x| < r\}.$$

Now let

$$\sigma = \{B_r(x) \mid x \in \mathbb{R},\ r > 0\}$$

and let τ be the topology on $\mathbb{R}$ generated by σ. This is the *metric* topology on $\mathbb{R}$.

Unless otherwise stated, we always endow $\mathbb{R}$ with the metric topology. Similarly, we can put a metric topology on $\mathbb{R}^n$ for any $n > 0$, generated by a sub-basis which consists of all sets of the form $B_r(x) = \{y \in \mathbb{R}^n \mid ||y - x|| < r\}$. Here $|| \cdot ||$ denotes the standard norm on $\mathbb{R}^n$, in which the length of a vector is given by the usual Pythagorean formula.

We will see that the metric topology on $\mathbb{R}$ encodes all the information about continuity and convergence of sequences that you are used to from real analysis.

Exercise 1.7. Check that $U \subseteq \mathbb{R}^n$ is open if and only if for every $x \in U$ there exists $r > 0$ such that

$$B_r(x) \subseteq U.$$

Definition 1.8. Let (x_n) be a sequence in X. We say that (x_n) *converges to* x (and write $\lim_{n\to\infty} x_n = x$) if, for every open set U that contains x, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for every $n \geq N$.

Exercise 1.9. Check that this definition of convergence of a sequence coincides with the usual one in $\mathbb{R}$.

Definition 1.10. A subset $F \subseteq X$ is called *closed* if its complement $X \smallsetminus F$ is open.

Exercise 1.11. Check that $F \subseteq \mathbb{R}$ is closed if and only if the limit of any convergent sequence in F is also contained in F.

Be careful! Sets can be both open and closed. For instance, in the discrete topology every subset is both open and closed.

Definition 1.12. A map $f : Y \to X$ from one topological space to another is called *continuous* if the preimage of every open set is open. That is, if U is an open subset of X then the preimage

$$f^{-1}U = \{y \in Y \mid f(y) \in U\}$$

is an open subset of Y.

Exercise 1.13. In the case when $X = Y = \mathbb{R}$, check that this definition of continuity coincides with the usual definition of continuity in terms of ϵ and δ.