5 Homology and polyhedra

5.1 Chain complexes

Definition 5.1. A sequence of homomorphisms of abelian groups

$$\cdots \to C_i \xrightarrow{\partial_i} C_{i-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0$$

is called a *chain complex* if $\partial_{i-1} \circ \partial_i = 0$ for each i. The $\{\partial_i\}$ are called *boundary homomorphisms.*

The elements of the subgroup C_i are called i*-chains.* The elements of the subgroup $Z_i = \ker \partial_i \subseteq C_i$ are called i*-cycles*, and the elements of the subgroup $B_i = \text{im } \partial_{i+1}$ are called i*-boundaries.* The condition translates into the requirement that $B_i \subseteq Z_i$ for each i.

The ith homology group of the chain complex C is defined to be

$$H_i(C) = Z_i/B_i$$

for $i > 0$, and we take $H_0 = C_0/B_0$. These groups are sometimes packaged together into the *total homology group*

$$H_*(C) = H_0(C) \oplus H_1(C) \oplus H_2(C) \oplus \cdots$$

Example 5.2. The sequence

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to 0$$

is a chain complex, for any integer p. The only non-zero homology group is $H_1(C) = \mathbb{Z}/p\mathbb{Z}$.

Most often, our homology groups will be finitely generated. Any finitely generated abelian group can be written as

$$\Gamma = \mathbb{Z}^n \oplus (\bigoplus_j (\mathbb{Z}/p_j\mathbb{Z}))$$

for some integer n and finitely many integers p_j. The subgroup $\bigoplus_j(\mathbb{Z}/p_j\mathbb{Z})$ is precisely the set of elements of Γ with finite order, and is called the *torsion* of Γ. The subgroup $\mathbb{Z}^n$ is not canonically determined, but the integer n can be recovered because the quotient of Γ by the torsion subgroup is isomorphic to $\mathbb{Z}^n$. If

$$H_i(C) = \mathbb{Z}^n \oplus (\bigoplus_j (\mathbb{Z}/p_j\mathbb{Z}))$$

then n is called the ith Betti number of C, denoted $b_i(C)$.

Exercise 5.3. Suppose that each $C_i = \mathbb{Z}^{a_i}$. The *Euler characteristic* of C is the alternating sum

$$\chi(C) = \sum_i (-1)^i a_i.$$

Show that

$$\chi(C) = \sum_i (-1)^i b_i.$$

5.2 Simplicial polyhedra and simplicial homology

Definition 5.4. A *k-dimensional simplex* or *k-simplex* Δ^k is the convex hull of $k+1$ points in general position in $\mathbb{R}^N$, for $N \geq k$.

Example 5.5.

1. A 0-simplex is a point.
2. A 1-simplex is a closed interval.
3. A 2-simplex is a triangle.
4. A 3-simplex is a tetrahedron.

Definition 5.6. An *orientation* on Δ^k is a choice of order for its $k+1$-vertices, up to even permutation.

Any subset of the vertices of Δ^k span a *face* of k, which is itself a simplex. (In particular, the empty set spans the empty face.)