Exercise 1.14.
1. Let X be a space endowed with the discrete topology and let Y be any topological space. Which maps $X \to Y$ are continuous?
2. Let X be any topological space and let Y be a space endowed with the indiscrete topology. Which maps $X \to Y$ are continuous?

We can write down many more examples by simply drawing them in the plane. To do this, we will need the notion of induced topology.

Definition 1.15. Let $Y \subseteq X$ be a subset. The *induced topology* on Y is the topology

$$\tau' = \{U \cap Y \subseteq Y \mid U \in \tau\}$$

that consists of all intersections of Y with open sets in X. If Y is endowed with the induced topology, we say that Y is a *subspace* of X.

Usually, when we draw a picture of a space, we think of it as having the induced topology from $\mathbb{R}^2$.

At last, we can be more precise about what we mean when we say that topology is ‘rubber geometry’. The idea is that we will only be interested in the properties of our topological space that are seen by the open sets. We will not concern ourselves with more rigid properties, like the distance between two points or the angle between two lines.

Definition 1.16. If $f : X \to Y$ is a continuous map of topological spaces with a continuous inverse $f^{-1} : Y \to X$ then f is called a *homeomorphism*. If a homeomorphism exists then we say that X and Y are *homeomorphic*.

In topology we only care about the homeomorphism class of a space. If two spaces are homeomorphic, then we will think of them as the same.

Example 1.17. The function $(0, \infty) \to (0, 1)$ defined by

$$x \mapsto \tanh x$$

is a homeomorphism.

So we see that the usual notion of distance in $\mathbb{R}$ is not a topological property, meaning that it is not preserved by homeomorphism.

Exercise 1.18. Show that the boundaries of the unit square and the unit circle, both endowed with the induced topology from $\mathbb{R}^2$, are homeomorphic.

Again, the topology does not see the ‘rigid’ properties of the square, like the angles at the corners.

To prove that two spaces are homeomorphic, we simply write down a homeomorphism between them. It is much harder to prove that two spaces are not homeomorphic. For instance, how could we show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic? Much of the remainder of the course will be spent talking about ways of proving that spaces are not homeomorphic.

1.2 Operations on topological spaces

In this section, we will see some ways of building new spaces from old.

Definition 1.19 (The product)**.** Let X and Y be topological spaces. There is a basis on $X \times Y$ consisting of all subsets of the form $U \times V$ where U is an open subset of X and V is an open subset of Y. This generates a topology on $X \times Y$ called the *product topology*.

In other words, the product topology on $X \times Y$ is the coarsest topology such that the natural projections onto the two factors, $X \times Y \to X$ and $X \times Y \to Y$, are continuous.

Example 1.20. The product topology on $\mathbb{R} \times \mathbb{R}$ is equal to the metric topology on $\mathbb{R}^2$.

Definition 1.21 (The quotient)**.** Suppose X is a topological space and $\sim$ is an equivalence relation on X. Let $Y = X/\sim$, the set of equivalence classes, and let $q : X \to Y$ be the canonical quotient map, which sends each $x \in X$ to its equivalence class. The topology on Y that consists of every $V \subseteq Y$ such that $q^{-1}V$ is open in X is called the *quotient* topology.

So the quotient topology is the finest topology on Y such that the projection q is continuous. The quotient topology is extremely useful, as it allows us to glue spaces together.

Example 1.22. Suppose X is a topological space and $A \subseteq X$ and $B \subseteq X$ are subspaces, and $\phi : A \to B$ is a homeomorphism. We can define an equivalence relation on the disjoint union X by insisting that $a \sim \phi(a)$ for all $a \in A$ (and x is equivalent only to itself if $x \notin A \cup B$). The resulting space

$$Z = X/\sim$$

is said to be constructed from X by gluing A to B.

Usually we will apply this construction when A and B are disjoint—that is, $A \cap B = \varnothing$. In this case, the partition induced by the equivalence relation $\sim$ is easy to describe: if $x \in X \smallsetminus A \cup B$ then the equivalence class of x is $\{x\}$; if $x \in A$ then the equivalence class of x is $\{x, \phi(x)\}$; and if $x \in B$ then the equivalence class of x is $\{x, \phi^{-1}(x)\}$.

Example 1.23. We can now build many examples by cutting and pasting.

1. The circle can be constructed by gluing the two ends of the interval $[0, 1]$ together.
2. The annulus can be constructed by gluing two opposite sides of the square $[0, 1] \times [0, 1]$ together.
3. The Möbius band is constructed by gluing two opposite sides of the square together with a flip.

4. The torus is constructed by gluing the two boundary circles of an annulus together. You can picture the torus as the surface of a bagel.
5. The Klein bottle is constructed by gluing the two boundary circles of an annulus together with a flip.

We can picture all these by drawing squares and indicating which sides are identified.