In the next example, W will have two path components, so $H_0(W) \cong \mathbb{Z}^2$ and the bottom row looks like

$$\cdots \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0 \ .$$

Again, the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ has image isomorphic to $\mathbb{Z}$, so its kernel is $\mathbb{Z}$ and the map into $H_0(W)$ has image $\mathbb{Z}$. In this situation, we can replace the bottom row with $\mathbb{Z} \to 0$.

Example 6.9. The 2-torus T can be covered by two annuli U and V, each of which is homotopy equivalent to a circle; their intersection W is homotopy equivalent to the disjoint union of two circles. Therefore we know that $H_k(U) = H_k(V)$ is isomorphic to $\mathbb{Z}$ if $k = 0, 1$ and 0 otherwise. Similarly, $H_k(W)$ is isomorphic to $\mathbb{Z}^2$ if $k = 0, 1$ and 0 otherwise. Now the Mayer–Vietoris Sequence gives us that

$$\cdots \to H_2(U) \oplus H_2(V) \to H_2(T) \to H_1(W) \to H_1(U) \oplus H_1(V) \to H_1(T) \to \mathbb{Z} \to 0$$

and, again filling in the groups that we know, we have

$$0 \to H_2(T) \xrightarrow{\partial_*} \mathbb{Z}^2 \xrightarrow{i_*} \mathbb{Z}^2 \xrightarrow{j_*} H_1(T) \xrightarrow{\partial_*} \mathbb{Z} \to 0 \ .$$

First, we will try to get as far as we can without understanding the actual maps. Using Lemma 6.2 and the paragraph before it, we see that for any term in an exact sequence of the form

$$\cdots \xrightarrow{f} A \xrightarrow{g} \cdots$$

it follows that rank A = rank im f + rank im g. So at $H_2(T)$ we get

$$b_2(T) = \text{rank im } \partial_*$$

while at $H_1(T)$ we get

$$b_1(T) = \text{rank im } \partial_* + \text{rank } \ker \partial_* = 1 + \text{rank im } j_* \ .$$

Looking at the remaining terms, we get

$$2 = \text{rank im } \partial_* + \text{rank im } i_*$$

and

$$2 = \text{rank im } i_* + \text{rank im } j_*$$

so, solving for the Betti numbers, we get

$$b_2(T) = 2 - \text{rank im } i_* = \text{rank im } j_* = b_1(T) - 1 \ .$$

Now we need some extra input. For instance, Lemma 5.14 tells us that $H_2(T) \cong \mathbb{Z}$, so $b_2(T) = 1$ and $b_1(T) = 2$. Again, in this instance we could have used Euler characteristic to get this far. But if we actually understand the maps, we can do better. For instance, the map $i_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by the matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

whose kernel is isomorphic to $\mathbb{Z}$ and whose image is $\langle(1, -1)\rangle \subseteq \mathbb{Z}^2$. As $\mathbb{Z}^2/\langle(1, -1)\rangle \cong \mathbb{Z}$ has no torsion, it follows that $H_1(T)$ has no torsion, so $H_1(T) \cong \mathbb{Z}^2$.

6.4 Some applications

In Question 2 of homework 8, you are asked to prove that $b_n(S^n) = 1$ for all n.

Example 6.10. Suppose $m > n$. Then $b_{m-1}(S^{m-1}) = 1$ whereas $b_{m-1}(S^{n-1}) = 0$ so, as singular homology (and therefore Betti numbers) are homotopy invariants, it follows that S^{m-1} and S^{n-1} are not homotopy equivalent. But $\mathbb{R}^m \smallsetminus \{0\} \simeq S^{m-1}$ and $\mathbb{R}^n \smallsetminus \{0\} \simeq S^{n-1}$ so we can conclude that $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.

You go on to prove the Brouwer Fixed Point Theorem in Question 3.

Theorem 6.11. *Let D be the closed unit disc in $\mathbb{R}^n$. Any continuous map $f : D^n \to D^n$ has a fixed point.*

One interesting application is a weak version of the Perron–Frobenius Theorem.

Corollary 6.12. *If A is an $n \times n$ matrix with every entry strictly positive then A has an eigenvector in the positive quadrant of $\mathbb{R}^n$.*

Proof. Let $X \subseteq \mathbb{R}^n$ be the positive quadrant and let $Y = S^{n-1} \cap X$, a disc. Let $p : X \smallsetminus \{0\} \to Y$ be the natural projection map given by $x \mapsto x/||x||$. Then $p \circ A$ restricted to Y defines a continuous map $Y \to Y$ which has a fixed point v. This is an eigenvector of A. $\square$