Example 2.30. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for any $n \neq 2$.

With a little more work we can see that any homotopy equivalence (even if it does not respect base points) preserves the fundamental group. The next lemma shows us that it does not matter too much if homotopies move the base point.

Lemma 2.31. *Let X be a topological space equipped with a base point x. Consider a homotopy $G : X \times [0,1] \to Y$, between continuous maps g_0 and g_1 from X to Y. Let α be the path in Y from $g_0(x)$ to $g_1(x)$, defined by*

$$\alpha(t) = G(x,t)$$

for any $t \in [0,1]$. Then

$$f_\alpha \circ g_{1*} = g_{0*}$$

as homomorphisms $\pi_1(X,x) \to \pi_1(Y, g_0(x))$. (Here f_α is the base-point change isomorphism defined in Lemma 2.9.)

Proof. Let $\gamma \in \Omega(X,x)$. We need to exhibit a homotopy F from $g_0 \circ \gamma$ to $f_\alpha \circ g_1 \circ \gamma$. Consider the map

$$\hat{G} : [0,1] \times [0,1] \to Y$$

defined by $\hat{G}(\theta,t) = G(\gamma(\theta),t)$. Let $\beta_0 : [0,1] \to [0,1] \times [0,1]$ be the path $\theta \mapsto (\theta, 0)$, and let β_1 be the path

$$\beta_1(\theta) = \begin{cases} (3\theta, 0) & 0 \leq \theta \leq \frac{1}{3} \\ (1, 3\theta - 1) & \frac{1}{3} \leq \theta \leq \frac{2}{3} \\ (3 - 3\theta, 1) & \frac{2}{3} \leq \theta \leq 1 \end{cases}$$

that runs round the other three sides of the square. Observe that

$$g_0 \circ \gamma = \hat{G} \circ \beta_0$$

and that

$$f_\alpha \circ g_1 \circ \gamma = \hat{G} \circ \beta_1.$$

Let F be any homotopy from β_0 to β_1 that preserves the end points $(0,0)$ and $(1,0)$. (It is clear that such a homotopy exists. It just pushes β_0 over the square.) Then $\hat{G} \circ F$ is the required based homotopy. □

The following is an easy consequence.

Proposition 2.32. *If $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$.*

Proof. Let $\phi : X \to Y$ and $\psi : Y \to X$ be a pair of homotopy equivalences. By Lemma 2.31, $\psi_* \circ \phi_*$ is equal to a base-point change isomorphism, in particular a bijection. It follows that ϕ_* is injective and ψ_* is surjective. By symmetry, we also have that ϕ_* is surjective and ψ_* is injective, as required. □

2.4 Higher homotopy groups

The fundamental group is defined as (based) homotopy classes of maps from the circle into a space X. There is nothing particularly special about the circle; one can make a similar definition for any choice of domain. Taking the domain to be a higher-dimensional sphere defines a higher homotopy group.

Definition 2.33. Let n be a non-negative integer and let $*$ be a choice of base point on S^n. Let X be a any topological space, with base point x_0. The set of based homotopy classes of base-point-preserving maps $S^n \to X$ is denoted $\pi_n(X, x_0)$.