MIDTERM

Instructions

Open book, open notes. You may appeal to results stated in the accompanying class notes, including exercises. References to theorems from other sources will not be accepted. You may have any text book you like open while you take the exam, but you should refer to the class notes, not the text book.

There is a **four-hour** time limit. No credit will be given for work done after four hours.

Turn in the midterm to the usual box before 4pm on Wednesday, November 4^{th}.

1. Show that a space X is compact if and only if it has the following property. For every collection of closed sets $\{C_\alpha\}$ in X such that the intersection of every finite subcollection of C_α's is non-empty, the intersection of all the C_α's is non-empty.

2.

   Definition 1. A topological space X is called *disconnected* if there exist non-empty, disjoint open sets U and V with $X = U \cup V$. (Disjoint means that $U \cap V = \varnothing$.) If X is not disconnected then it is called *connected*.

   The goal of this question is to prove that path connected topological spaces are connected. First, we will prove that $[0, 1]$ is connected.

   Suppose $[0, 1]$ is disconnected, so $[0, 1] = U \cup V$ for U and V non-empty, disjoint, open subsets. Without loss of generality, assume that $0 \in U$. Let

   $$S = \{x \in [0, 1] \mid [0, x] \subseteq U\}$$

   and let $s = \sup S$.

   (a) Use the fact that V is open to prove that S is closed.

   (b) Use the fact that U is open to prove that S is open.

   (c) Deduce that $s = 1$, and conclude that $[0, 1]$ is connected.

   (d) Prove that any path connected space is connected.

3. Prove that a contractible space is path connected.

4. As usual, think of S^n as the unit sphere in $\mathbb{R}^{n+1}$. Let X be any topological space and let $f, g : X \to S^n$ be two continuous maps such that $f(x) \neq -g(x)$ for all $x \in X$.

   (a) Prove that f and g are homotopic in $\mathbb{R}^{n+1} \setminus \{0\}$.

   (b) Deduce that f and g are homotopic in S^n.

5. Prove that the wedge $S^1 \vee S^1$ is not homeomorphic to S^1. You may **not** use the Seifert–van Kampen Theorem.