INTRODUCTION TO GEOMETRY AND TOPOLOGY

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1 Topological spaces

It is traditional to start by saying that topology is 'rubber geometry'. More precisely, topology is the study of continuous maps, like the ones you learned about in calculus. We will only care about the properties of spaces that are preserved by continuous maps.

1.1 Definitions and basic examples

At first the definitions we give will seem very abstract. Hopefully, it will quickly become clear that we are providing a framework in which we can talk about the facts about continuity that you learned in calculus or analysis.

Definition 1.1. A *topological space* is a set X with a *topology* $\tau \subseteq 2^X$. If $U \in \tau$ then U is called *open*. The topology τ is required to satisfy the following properties.

1. The union of any set of open sets is open.
2. The intersection of any finite union of open sets is open.
3. The empty subset $\varnothing$ and the whole set X are both open.

You can think of a topology as encoding an idea of proximity or nearness. The more open sets that contain two points x and y, the closer we think of x and y as being.

Example 1.2. We can endow any set X with the *discrete* topology $\tau = 2^X$.

This is the *finest* topology we can impose on X—that is, the one with the most open sets. All the points are at a distance from each other.

Example 1.3. Any set X can be given the *indiscrete* topology $\tau = \{\varnothing, X\}$.

This is the *coarsest* topology we can give X—the one with the fewest open sets. All the points are very close to each other.

Neither the discrete topology nor the indiscrete topology is very interesting. If X is finite then in principle we can specify a topology on X by writing down its elements. For infinite X, however, we will need a different way of specifying which topology we are talking about. We will do so by writing down collections of subsets and insisting that they must be open.

Definition 1.4. Let $\sigma \subseteq 2^X$ be any subset. Then σ *generates* a topology τ, which is the coarsest topology in which every element of σ is open. We can construct τ explicitly. Let

$$\beta = \{ \bigcap_{U \in I} U \mid I \subseteq \sigma,\ |I| < \infty \}$$

the set of all *finite* intersections of elements of σ. Now let

$$\tau = \{ \bigcup_{U \in J} U \mid J \subseteq \beta \}$$

the set of all *arbitrary* unions of elements of β.

We say that σ is a *sub-basis* for τ. The collection β is a *basis* for τ, because it has the property that $U, V \in \beta$ implies that $U \cap V \in \beta$. (This is not quite the same as the definition of a basis given in Vassiliev.)

Exercise 1.5. Check that τ really is a topology.

We can define a topology by specifying a sub-basis.

Example 1.6. For any $x \in \mathbb{R}$ and $r > 0$, let

$$B_r(x) = \{y \in \mathbb{R} \mid |y - x| < r\}.$$

Now let

$$\sigma = \{B_r(x) \mid x \in \mathbb{R},\ r > 0\}$$

and let τ be the topology on $\mathbb{R}$ generated by σ. This is the *metric* topology on $\mathbb{R}$.

Unless otherwise stated, we always endow $\mathbb{R}$ with the metric topology. Similarly, we can put a metric topology on $\mathbb{R}^n$ for any $n > 0$, generated by a sub-basis which consists of all sets of the form $B_r(x) = \{y \in \mathbb{R}^n \mid ||y - x|| < r\}$. Here $|| \cdot ||$ denotes the standard norm on $\mathbb{R}^n$, in which the length of a vector is given by the usual Pythagorean formula.

We will see that the metric topology on $\mathbb{R}$ encodes all the information about continuity and convergence of sequences that you are used to from real analysis.

Exercise 1.7. Check that $U \subseteq \mathbb{R}^n$ is open if and only if for every $x \in U$ there exists $r > 0$ such that

$$B_r(x) \subseteq U.$$

Definition 1.8. Let (x_n) be a sequence in X. We say that (x_n) *converges to* x (and write $\lim_{n\to\infty} x_n = x$) if, for every open set U that contains x, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for every $n \geq N$.

Exercise 1.9. Check that this definition of convergence of a sequence coincides with the usual one in $\mathbb{R}$.

Definition 1.10. A subset $F \subseteq X$ is called *closed* if its complement $X \setminus F$ is open.

Exercise 1.11. Check that $F \subseteq \mathbb{R}$ is closed if and only if the limit of any convergent sequence in F is also contained in F.

Be careful! Sets can be both open and closed. For instance, in the discrete topology every subset is both open and closed.

Definition 1.12. A map $f : Y \to X$ from one topological space to another is called *continuous* if the preimage of every open set is open. That is, if U is an open subset of X then the preimage

$$f^{-1}U = \{y \in Y \mid f(y) \in U\}$$

is an open subset of Y.

Exercise 1.13. In the case when $X = Y = \mathbb{R}$, check that this definition of continuity coincides with the usual definition of continuity in terms of ϵ and δ.

Exercise 1.14.
1. Let X be a space endowed with the discrete topology and let Y be any topological space. Which maps $X \to Y$ are continuous?
2. Let X be any topological space and let Y be a space endowed with the indiscrete topology. Which maps $X \to Y$ are continuous?

We can write down many more examples by simply drawing them in the plane. To do this, we will need the notion of induced topology.

Definition 1.15. Let $Y \subseteq X$ be a subset. The *induced topology* on Y is the topology

$$\tau' = \{U \cap Y \subseteq Y \mid U \in \tau\}$$

that consists of all intersections of Y with open sets in X. If Y is endowed with the induced topology, we say that Y is a *subspace* of X.

Usually, when we draw a picture of a space, we think of it as having the induced topology from $\mathbb{R}^2$.

At last, we can be more precise about what we mean when we say that topology is 'rubber geometry'. The idea is that we will only be interested in the properties of our topological space that are seen by the open sets. We will not concern ourselves with more rigid properties, like the distance between two points or the angle between two lines.

Definition 1.16. If $f : X \to Y$ is a continuous map of topological spaces with a continuous inverse $f^{-1} : Y \to X$ then f is called a *homeomorphism*. If a homeomorphism exists then we say that X and Y are *homeomorphic*.

In topology we only care about the homeomorphism class of a space. If two spaces are homeomorphic, then we will think of them as the same.

Example 1.17. The function $(0, \infty) \to (0, 1)$ defined by

$$x \mapsto \tanh x$$

is a homeomorphism.

So we see that the usual notion of distance in $\mathbb{R}$ is not a topological property, meaning that it is not preserved by homeomorphism.

Exercise 1.18. Show that the boundaries of the unit square and the unit circle, both endowed with the induced topology from $\mathbb{R}^2$, are homeomorphic.

Again, the topology does not see the 'rigid' properties of the square, like the angles at the corners.

To prove that two spaces are homeomorphic, we simply write down a homeomorphism between them. It is much harder to prove that two spaces are not homeomorphic. For instance, how could we show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic? Much of the remainder of the course will be spent talking about ways of proving that spaces are not homeomorphic.

1.2 Operations on topological spaces

In this section, we will see some ways of building new spaces from old.

Definition 1.19 (The product)**.** Let X and Y be topological spaces. There is a basis on $X \times Y$ consisting of all subsets of the form $U \times V$ where U is an open subset of X and V is an open subset of Y. This generates a topology on $X \times Y$ called the *product topology.*

In other words, the product topology on $X \times Y$ is the coarsest topology such that the natural projections onto the two factors, $X \times Y \to X$ and $X \times Y \to Y$, are continuous.

Example 1.20. The product topology on $\mathbb{R} \times \mathbb{R}$ is equal to the metric topology on $\mathbb{R}^2$.

Definition 1.21 (The quotient)**.** Suppose X is a topological space and $\sim$ is an equivalence relation on X. Let $Y = X/\sim$, the set of equivalence classes, and let $q : X \to Y$ be the canonical quotient map, which sends each $x \in X$ to its equivalence class. The topology on Y that consists of every $V \subseteq Y$ such that $q^{-1}V$ is open in X is called the *quotient* topology.

So the quotient topology is the finest topology on Y such that the projection q is continuous. The quotient topology is extremely useful, as it allows us to glue spaces together.

Example 1.22. Suppose X is a topological space and $A \subseteq X$ and $B \subseteq X$ are subspaces, and $\phi : A \to B$ is a homeomorphism. We can define an equivalence relation on the disjoint union X by insisting that $a \sim \phi(a)$ for all $a \in A$ (and x is equivalent only to itself if $x \notin A \cup B$). The resulting space

$$Z = X/\sim$$

is said to be constructed from X by gluing A to B.

Usually we will apply this construction when A and B are disjoint—that is, $A \cap B = \varnothing$. In this case, the partition induced by the equivalence relation $\sim$ is easy to describe: if $x \in X \setminus A \cup B$ then the equivalence class of x is $\{x\}$; if $x \in A$ then the equivalence class of x is $\{x, \phi(x)\}$; and if $x \in B$ then the equivalence class of x is $\{x, \phi^{-1}(x)\}$.

Example 1.23. We can now build many examples by cutting and pasting.

1. The circle can be constructed by gluing the two ends of the interval $[0, 1]$ together.
2. The annulus can be constructed by gluing two opposite sides of the square $[0, 1] \times [0, 1]$ together.
3. The Möbius band is constructed by gluing two opposite sides of the square together with a flip.
4. The torus is constructed by gluing the two boundary circles of an annulus together. You can picture the torus as the surface of a bagel.
5. The Klein bottle is constructed by gluing the two boundary circles of an annulus together with a flip.

We can picture all these by drawing squares and indicating which sides are identified.

We can also use the quotient topology to build nasty examples.

Example 1.24. Define an equivalence relation on $\mathbb{R}$ by $x \sim y$ if and only if there exists $\lambda > 0$ such that $\lambda x = y$. Now $\mathbb{R}/\sim$ has two points; one point is open and the other is not.

The suspension is one of the most important examples of this kind of construction.

Definition 1.25 (The suspension)**.** Define an equivalence relation on $X \times [-1, 1]$ by requiring that $(x, \theta) \sim (y, \phi)$ if and only if either $x = y$ and $\theta = \phi$, or $\theta = \phi = \pm 1$. The quotient

$$\Sigma X = X \times [-1, +1]/\sim$$

is called the *suspension* of X.

Exercise 1.26. For any $n \geq 0$, let S^n be the unit sphere in $\mathbb{R}^{n+1}$; that is,

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i^2 = 1\}$$

endowed with the induced topology. Show that ΣS^n is homeomorphic to S^{n+1} for all n.

1.3 Properties of topological spaces

We have seen that many of the properties of spaces that we are used to thinking about—such as length, angles or area—are not topological properties. But there are plenty of properties that do make sense in topology. Roughly, a property will be topological if you can define it using only the open sets.

The first such property that we will look at ensures that the topology sees the individual points of the space. A *neighbourhood* of a point x is an open set that contains x.

Definition 1.27. A topological space X is called *Hausdorff* if for every pair of points $x, y \in X$, there is a neighbourhood U of x and V of y such that U and V are disjoint, *i.e.* $U \cap V = \varnothing$.

If X is Hausdorff then the topology can distinguish any point in the space from any other point. For this reason, we think of non-Hausdorff topologies as badly behaved.

Example 1.28. The discrete topology is always Hausdorff. If $|X| > 1$ then the indiscrete topology on X is not Hausdorff.

Example 1.29. The quotient space in Example 1.24 is not Hausdorff.

On the other hand, most of the examples that we have seen are well behaved.

Example 1.30. $\mathbb{R}^n$, the circle, the torus, the sphere, the Möbius band etc. are all Hausdorff.

We will look at two more properties of topological spaces: path connectedness and compactness. Both turn out to be topological generalisations of well known theorems from real analysis.

The next property that we will look at asks whether or not our space is made up of several pieces.

Definition 1.31. We say that a pair of points $x, y \in X$ *belong to the same path component of* X *if* there is a continuous map $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$. Note that this defines an equivalence relation on X; the equivalence classes are called *path components*. If X consists of just one path component then we say that X is *path connected*.

We think of a path connected space as made up of just one piece.

Exercise 1.32. Is the quotient of X under the equivalence relation of belonging to the same path component always discrete?

Whenever we write down a property of a topological space, it is important to check that it really is invariant under homeomorphism. Better still, the image of a path connected space under a continuous map is also path connected. You can think of this as saying that continuous maps never tear topological spaces into pieces.

Lemma 1.33. *If $f : X \to Y$ is a continuous surjection then the number of path components of Y is at most equal to the number of path components of X. In particular, if X is path connected then Y is path connected.*

Exercise 1.34. Prove Lemma 1.33.

As promised, there is a classical theorem from calculus in the background here. Path connectedness is really a topological generalisation of the Intermediate Value Theorem.

Exercise 1.35. Prove that $[0,1] \cup [2,3]$ is not path connected.

Path components gives us our first really useful technique of proving that two spaces are not homeomorphic: if they have a different number of path components then they are not. This can be useful even for path-connected spaces.

Example 1.36. Let Y be a space that consists of three closed intervals identified at an end point—so Y is homeomorphic to the letter 'Y'. Then Y is not homeomorphic to an interval. Why? Because Y contains a point y such that $Y \smallsetminus \{y\}$ has three path components; but an interval contains no such point.

Example 1.37. The real line $\mathbb{R}$ has the property that, for every $x \in \mathbb{R}$, the complement $\mathbb{R} \smallsetminus \{x\}$ has two path components. On the other hand, whenever $n > 1$, the complement of a point in $\mathbb{R}^n$ is path connected. Therefore $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any other n.

Our final property, compactness, is one of the most important topological properties. Roughly, a compact space is small enough to be manageable. It is often much easier to prove theorems about compact spaces than non-compact spaces.

Definition 1.38. A collection of open sets $\mathcal{U} \subseteq \tau$ with the property that

$$X = \bigcup_{U \in \mathcal{U}} U$$

is called an *open cover* of X. If $\mathcal{V} \subseteq \mathcal{U}$ is also an open cover of X then $\mathcal{V}$ is called a *sub-cover* of $\mathcal{U}$.

Definition 1.39. If every open cover of X contains a finite subcover then X is called *compact*.

Be careful! To show that a space is compact, you have to show that *every* open cover contains a finite subcover. It is *not* enough to show that the space has a finite cover. In fact, every space has a finite cover, with just one element!

Example 1.40. If X is discrete then X is compact if and only if X is finite.

Theorem 1.41 (The Heine–Borel Theorem)**.** *A subset Y of $\mathbb{R}^n$ is compact if and only if Y is closed and bounded.*

Example 1.42.

1. $[0,1]$ is compact.
2. $[0,1)$ is not compact.

Like path connectedness, compactness is preserved by continuous maps.

Proposition 1.43. *If $f : X \to Y$ is continuous and surjective and X is compact then Y is compact.*

It follows that many of the examples that we have already seen are compact.

Example 1.44. The torus, the sphere, the Möbius band etc. are all compact.

The definition of compactness may seem strange to you. But in fact it underlies a familiar theorem from real analysis.

Exercise 1.45. Use the Heine–Borel Theorem to deduce the Bolzano–Weierstrass Theorem, that every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

2 Homotopy groups and homotopy equivalence

We have been able to prove that a few spaces are not homeomorphic. But there are still many similar questions that we cannot yet answer.

Question 2.1. *Is S^1 homeomorphic to S^2?*

In this section, we will begin to take a more systematic approach to this sort of problem. The goal is to distinguish S^1 from S^2 using the following intuitive observation: every loop in S^2 can be continuously shrunk to a point; on the other hand, there are loops in S^1 that cannot be shrunk to a point.

2.1 The fundamental group

To be precise about the intuition described above, we need a definition of continuous deformation.

Definition 2.2. Let f_0 and f_1 be continuous maps $X \to Y$. A *homotopy* between them is a continuous map

$$F : X \times [0, 1] \to Y$$

with the property that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$. In this case we say that f_0 and f_1 are *homotopic* and write $f_0 \simeq f_1$.

Note that $\simeq$ is an equivalence relation.

Example 2.3. Any continuous map $f : X \to \mathbb{R}^n$ is homotopic to the constant map $x \mapsto 0 \in \mathbb{R}^n$.

We will see that it is also useful to consider *based* homotopies.

Definition 2.4. Let (X, x_0) and (Y, y_0) be spaces with fixed base points, and suppose that $f_0, f_1 : X \to Y$ are base-point-preserving. A homotopy F between f_0 and f_1 is called *based* if $F(x_0, t) = y_0$ for all t. We will still write $\simeq$ for based homotopy.

Our intuitive idea concerned loops. We therefore introduce the set of all (based) loops in a space.

Definition 2.5. Let (X, x_0) be a space with a base point, and let $*$ be a base point on the circle S^1. The set of all continuous, based (ie base-point-preserving) maps $(S^1, *) \to (X, x_0)$ is called the *loop space* of (X, x_0), and denoted $\Omega(X, x_0)$. Based homotopy defines an equivalence relation on $\Omega(X, x_0)$ and the quotient space is denoted $\pi_1(X, x_0)$.

Example 2.6. For any n and any $x_0 \in \mathbb{R}^n$, $\pi_1(\mathbb{R}^n, x_0)$ has one element.

Consider two paths in a topological space, ie two continuous maps

$$\alpha, \beta : [0,1] \to X.$$

If $\alpha(1) = \beta(0)$ then we can define a new path by concatenating them, as follows.

$$(\alpha \cdot \beta)(\theta) = \begin{cases} \alpha(2\theta) & 0 \le \theta \le \frac{1}{2} \\ \beta(2\theta - 1) & \frac{1}{2} \le \theta \le 1 \end{cases}$$

We can think of a loop in $\Omega(X, x_0)$ as a continuous map $\gamma : [0,1] \to X$, subject to the requirement that $\gamma(0) = \gamma(1) = x_0$. This is because the circle S^1 is homeomorphic to the interval $[0,1]$ with the end points 0 and 1 identified; we choose to take $0 = 1$ as our base point $*$ on S^1.

Therefore, because we work with based loops, concatenation defines a binary operation on $\Omega(X, x_0)$.

Exercise 2.7. If $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$ then $\alpha \cdot \beta \simeq \alpha \cdot \beta$.

It follows immediately that concatenation descends to a well defined binary operation on $\pi_1(X, x_0)$.

Theorem 2.8. *Concatenation of loops defines a group operation on* $\pi_1(X, x_0)$.

Proof. The identity element is the homotopy class of the constant loop $1 : t \mapsto x_0$. We need to check that

$$1 \cdot \gamma \simeq \gamma \simeq \gamma \cdot 1$$

for any based loop γ.

Let $\phi : [0,1] \to [0,1]$ be defined as follows.

$$\phi(\theta) = \begin{cases} 0 & 0 \le \theta \le \frac{1}{2} \\ 2\theta - 1 & \frac{1}{2} \le \theta \le 1 \end{cases}$$

There is a homotopy Φ between ϕ and the identity map.

$$\Phi(\theta, t) = t\theta + (1-t)\phi(\theta)$$

Now $1 \cdot \gamma = \gamma \circ \phi$, and $\gamma \circ \Phi$ is a based homotopy from $1 \cdot \gamma$ to γ. One can prove similarly that $\gamma \simeq \gamma \cdot 1$.

Given $\gamma \in \Omega(X, x_0)$, define a based loop γ^{-1} as follows.

$$\gamma^{-1}(\theta) = \gamma(1 - \theta)$$

It remains to check that

$$\gamma \cdot \gamma^{-1} \simeq 1 \simeq \gamma^{-1} \cdot \gamma$$

for any γ.

Let $\psi : [0,1] \to [0,1]$ be defined as follows.

$$\psi(\theta) = \begin{cases} 2\theta & 0 \leq \theta \leq \frac{1}{2} \\ 2 - 2\theta & \frac{1}{2} \leq \theta \leq 1 \end{cases}$$

There is a homotopy between ψ and the constant map 0.

$$\Psi(\theta, t) = (1-t)\psi(\theta)$$

Again, $\gamma \cdot \psi = \gamma \cdot \gamma^{-1}$, and $\gamma \cdot \Psi$ is the required based homotopy from $\gamma \cdot \gamma^{-1}$ to 1. The proof that $\gamma^{-1} \cdot \gamma \simeq 1$ is similar.

Finally, to prove associativity we need to construct a homotopy from $(\alpha \cdot \beta) \cdot \gamma$ to $\alpha \cdot (\beta \cdot \gamma)$, where $\alpha, \beta, \gamma \in \Omega(X, x_0)$. Once again, we shall see that we can do this simply by reparameterizing the loop. Define a based loop δ as follows.

$$\delta(\theta) = \begin{cases} \alpha(3\theta) & 0 \leq \theta \leq \frac{1}{3} \\ \beta(3\theta - 1) & \frac{1}{3} \leq \theta \leq \frac{2}{3} \\ \gamma(3\theta - 2) & \frac{2}{3} \leq \theta \leq 1 \end{cases}$$

This is the path that traces out α, then β, then γ, taking equal time over each. Let

$$\lambda(\theta) = \begin{cases} \frac{4}{3}\theta & 0 \leq \theta \leq \frac{1}{2} \\ \frac{1}{3} + \frac{2}{3}\theta & \frac{1}{2} \leq \theta \leq 1 \end{cases}$$

and let

$$\mu(\theta) = \begin{cases} \frac{2}{3}\theta & 0 \leq \theta \leq \frac{1}{2} \\ -\frac{1}{3} + \frac{4}{3}\theta & \frac{1}{2} \leq \theta \leq 1 \end{cases}$$

and observe that $(\alpha \cdot \beta) \cdot \gamma = \delta \circ \lambda$ whereas $\alpha \cdot (\beta \cdot \gamma) = \delta \circ \mu$. Then

$$\Lambda(\theta, t) = (1-t)\lambda(\theta) + t\mu(\theta)$$

is a homotopy from λ to μ, and $\delta \cdot \Lambda$ is a based homotopy between the two different representations of $\alpha \cdot \beta \cdot \gamma$, as required. □

Because of this, $\pi_1(X, x_0)$ is called *the fundamental group of* X. To define the fundamental group we had to choose a base point; fortunately, this choice is not too important as long as X is path connected.

Lemma 2.9. *Suppose X is path connected. For any two $x, y \in X$, we have $\pi_1(X, x) \cong \pi_1(X, y)$.*

Proof. Fix α a path starting at x and ending at y and let $\bar{\alpha}$ be the same path running in the opposite direction, from y to x. Consider the map $\hat{f}_\alpha : \Omega(X, y) \to \Omega(X, x)$ defined by concatenation as follows.

$$\gamma \mapsto \alpha \cdot \gamma \cdot \bar{\alpha}$$

(Strictly speaking, there is an associativity problem with this definition. But, just as in the proof of associativity of the fundamental group above, this concatenation is well defined up to homotopy.) This descends to a map $f_\alpha : \pi_1(X, x) \to \pi_1(X, y)$, which is easily checked to be a homomorphism. It is also easy to check that f_α and $f_{\bar{\alpha}}$ are inverses, as required. □

A little care is needed here. The isomorphism is not canonical, but the different possibilities only differ by conjugation. So we see that, as long as X is path connected, it makes sense to talk about its fundamental group, without reference to a base point. When we do this, we will just write $\pi_1(X)$.

Best of all, the construction of the fundamental group is functorial. Let $f : (X, x_0) \to (Y, y_0)$ be a continuous based map. Consider the map

$$\Omega(X, x_0) \to \Omega(Y, y_0)$$

defined by

$$\gamma \mapsto f \circ \gamma$$

for any based loop γ. Note that if $\alpha \simeq \beta$ then $f \circ \alpha \simeq f \circ \beta$, so this map descends to a map of fundamental group.

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

Note that $(fg)_* = f_* g_*$.

Exercise 2.10. Check that f_* is a homomorphism.

In particular, if X and Y are homeomorphic then their fundamental groups are isomorphic. We have associated an algebraic *invariant* to the isomorphism class of X. The study of such algebraic invariants is called *algebraic topology.*

We have already calculated some fundamental groups.

Example 2.11. For any n, $\pi_1(\mathbb{R}^n) \cong 1$.

Definition 2.12. If X is path connected and $\pi_1(X) \cong 1$ then we say that X is *simply connected.*

Example 2.13. Let $n \geq 2$. The proof of Proposition 1.14 of Hatcher explains how to homotope any loop $\gamma : S^1 \to S^n$ so that it becomes non-surjective. So we can assume that the image of γ is contained in $S^n \setminus \{x\}$ for some x. But $S^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^{n-1}$. (To see this use, for instance, stereographic projection.) Therefore any loop in S^n is homotopic to a point and $\pi_1(S^n) \cong 0$.

For this to be an interesting invariant, we had better find a space with non-trivial fundamental group.

2.2 The fundamental group of the circle

We have seen that $\pi_1(S^n)$ is trivial whenever $n \neq 1$. In this subsection, we will compute $\pi_1(S^1)$. The main theorem of this subsection is the following.

Theorem 2.14. *$\pi_1(S^1) \cong \mathbb{Z}$.*

Intuitively, we can think of the based homotopy class of a loop in S^1 as determined by its *winding number*—the number of times that it winds round the circle, taking direction into account.

Definition 2.15. The *n-torus* is the direct product of n copies of S^1.

Exercise 2.16. The torus defined in Subsection 1.2 is homeomorphic to $S^1 \times S^1$.

Exercise 2.17. If $x \in X$ and $y \in Y$ then

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y).$$

Therefore, Theorem 2.14 enables us to easily compute the fundamental group of any torus.

Corollary 2.18. *If T^n is the n-torus then $\pi_1(T^n) \cong \mathbb{Z}^n$.*

Theorem 2.14 also has many profound applications, including Brouwer's Fixed Point Theorem (see the homework).

The proof of Theorem 2.14 is surprisingly tricky. We will sketch it. It serves as an introduction to the concept of a *covering map*, which will be important later in the course.

Exercise 2.19. The quotient group $\mathbb{R}/\mathbb{Z}$, endowed with the quotient topology from $\mathbb{R}$, is homeomorphic to S^1. *(Hint: consider the map from $\mathbb{R}$ to the unit circle in $\mathbb{C}$ defined by $\theta \mapsto e^{2\pi i\theta}$.)*

The map

$$\phi : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \approx S^1$$

is the main tool of the proof. When using this representation of S^1, we will always take the base point $*$ to be the equivalence class $\mathbb{Z}$ in $\mathbb{R}/\mathbb{Z}$, so $\phi^1(*) = \mathbb{Z}$.

The idea is to lift our loops and homotopies from S^1 to $\mathbb{R}$. To be precise, if $f : X \to S^1$ and $\hat{f} : X \to \mathbb{R}$ are continuous maps such that $f = \phi \circ \hat{f}$ then we say that $\hat{f}$ is a *lift* of f. The next theorem gives us a condition under which we can lift.

Theorem 2.20. *Let X be a path connected topological space. For any continuous map $F : X \times [0, 1] \to S^1$ and any choice of lift $\hat{f} : X \to \mathbb{R}$ of $F|_{X\times\{0\}}$ there exists a unique lift $\hat{F} : X \times [0, 1] \to \mathbb{R}$ of F such that $\hat{F}|_{X\times\{0\}} = \hat{f}$.*

We can summarise the statement of the theorem in the following commutative diagram.

$$\begin{tikzcd} & & \mathbb{R} \arrow[d, "\phi"] \\ X \times \{0\} \arrow[r] \arrow[urr, bend left, "\hat{f}"] & X \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\exists! \hat{F}"'] & S^1 \end{tikzcd}$$

We will not prove Theorem 2.20, but a proof is given in Hatcher (p. 30). Only the following property of $\mathbb{R}$, S^1 and ϕ that is used in the proof of Theorem 2.20.

Remark 2.21. Any point $x \in S^1$ has a neighbourhood U_x such that

$$\phi^{-1}(U_x) \approx U_x \times \Delta_x$$

for some discrete space Δ_x. Furthermore, the restriction of the ϕ to $U_x \times \Delta_x$ is just the projection onto the first factor.

Remark 2.21 asserts that ϕ is a *covering map*. We will use two different consequences of Theorem 2.20.

Corollary 2.22. *If $\gamma : [0,1] \to (S^1, *)$ is a path with $\gamma(0) = *$ then there is a unique lift $\hat{\gamma} : [0,1] \to \mathbb{R}$ such that $\hat{\gamma}(0) = 0$.*

Proof. Apply Theorem 2.20 in the case when X is a single point. □

Corollary 2.23. *If $F : [0,1] \times [0,1] \to S^1$ is a homotopy between loops α and β and $\hat{\alpha} : [0,1] \to \mathbb{R}$ is a lift of α then there is a unique lift $\hat{F}$ of F to $\mathbb{R}$ such that $\hat{F}(\theta, 0) = \hat{\alpha}(\theta)$ for all $\theta \in [0,1]$.*

Proof. Apply Theorem 2.20 in the case when $X = [0,1]$. □

Proof of Theorem 2.14. For each integer n, let $\hat{\gamma}_n : [0,1] \to \mathbb{R}$ be the path defined by $\theta \mapsto n\theta$, and let $\gamma_n = \phi \circ \hat{\gamma}_n$. Let $W : \mathbb{Z} \to \pi_1(S^1, *)$ be defined by $n \mapsto \gamma_n$. We need to check that W is a group isomorphism.

First, we need to check that W is a homomorphism. To do this, we need to prove that $\gamma_m \cdot \gamma_n \simeq \gamma_{m+n}$. Lift the concatenation $\gamma_m \cdot \gamma_n$ to a path

$$\widehat{\gamma_m \cdot \gamma_n} : [0,1] \to \mathbb{R}$$

starting at 0. Now

$$\hat{F}(\theta, t) = t\hat{\gamma}_{m+n}(\theta) + (1-t)\widehat{\gamma_m \cdot \gamma_n}(\theta)$$

deforms the lift to $\hat{\gamma}_{m+n}$, without moving the end points. Therefore $\phi \circ \hat{F}$ is the required based homotopy from $\gamma_m \cdot \gamma_n$ to γ_{m+n}.

The proof that W is surjective is similar. Let $\gamma \in \Omega(S^1, *)$. By Corollary 2.22, γ has a unique lift $\hat{\gamma} : [0,1] \to \mathbb{R}$ such that $\hat{\gamma}(0) = 0$. Let $n = \hat{\gamma}(1)$. The homotopy

$$\hat{F}(\theta, t) = t\hat{\gamma}_n(\theta) + (1-t)\hat{\gamma}(\theta)$$

deforms $\hat{\gamma}$ into $\hat{\gamma}_n$, without moving the end points. Therefore, $\phi \circ \hat{F}$ is a homotopy from γ to γ_n. This proves that W is surjective.

Finally, we prove that W is injective. Suppose that $\gamma_m \simeq \gamma_n$, and let F be a based homotopy from γ_m to γ_n. By Corollary 2.23, F has a unique lift $\hat{F}$ such that $\hat{F}(\theta, 0) = \hat{\gamma}_m(\theta)$ for all $\theta \in [0,1]$.

For each t, $F(0,t) = *$ and so $\hat{F}(0,t)$ is an integer. As $\hat{F}$ is continuous and $\hat{F}(0,0) = 0$, it follows that $\hat{F}(0,t) = 0$ for all t. Therefore $\hat{F}(-,1)$ is a lift of γ_n that starts at 0, and hence by uniqueness it is equal to $\hat{\gamma}_n$. Now $\hat{F}(1,-)$ is a continuous, integer-valued path from $\hat{\gamma}_m(1)$ to $\hat{\gamma}_n(1)$, so $m = n$ as required. □

2.3 Homotopy equivalence

We have seen that the fundamental group is a homeomorphism invariant. In fact, there is a weaker notion of equivalence for topological spaces, which extends the idea we already have of topology as 'rubber geometry'.

Definition 2.24. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps. If $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$ then f and g are called *homotopy equivalences*, and we say that X and Y are *homotopy equivalent*. In this case we will write $X \simeq Y$.

Example 2.25. Consider S^n as the unit sphere in $\mathbb{R}^{n+1}$. Let $p : \mathbb{R}^{n+1} \smallsetminus \{0\} \to S^n$ be the natural projection map

$$x \mapsto \frac{x}{||x||}$$

and let $i : S^n \hookrightarrow \mathbb{R}^{n+1} \smallsetminus \{0\}$ be the inclusion map. Clearly $p \circ i = id_{S^n}$. On the other hand,

$$F(x,t) = tx + (1-t)p(x)$$

defines a homotopy from $i \circ p$ to the identity map on $\mathbb{R}^n \smallsetminus \{0\}$.

Exercise 2.26. Recall Example 1.36, in which we saw that the space Y, consisting of three closed intervals identified at a point, is not homeomorphic to a closed interval. Show that Y is homotopy equivalent to a closed interval.

The examples that we have seen above are very special instances of homotopy equivalences.

Definition 2.27. Suppose Y is a subspace of a space X; let $i : Y \to X$ be the inclusion map. A continuous map $p : X \to Y$ is a *retraction* (and Y is called a *retract* of X) if $p(y) = y$ for all $y \in Y$. If, in addition, there is a homotopy P from $i \circ p$ to id_X with the property that $P(y,t) = y$ for all $y \in Y$ and all $t \in [0,1]$ then Y is called a *deformation retract* of X.

Exercise 2.28. Suppose Y is a retract of X and $y \in Y$. Then the inclusion map i induces an injective homomorphism $i_* : \pi_1(Y,y) \to \pi_1(X,y)$.

Exercise 2.29. Suppose Y is a deformation retract of X and $y \in Y$. Then $\pi_1(Y,y) \cong \pi_1(X,y)$.

Example 2.30. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for any $n \neq 2$.

With a little more work we can see that any homotopy equivalence (even if it does not respect base points) preserves the fundamental group. The next lemma shows us that it does not matter too much if homotopies move the base point.

Lemma 2.31. *Let X be a topological space equipped with a base point x. Consider a homotopy $G : X \times [0,1] \to Y$, between continuous maps g_0 and g_1 from X to Y. Let α be the path in Y from $g_0(x)$ to $g_1(x)$, defined by*

$$\alpha(t) = G(x,t)$$

for any $t \in [0,1]$. Then

$$f_\alpha \circ g_{1*} = g_{0*}$$

as homomorphisms $\pi_1(X,x) \to \pi_1(Y, g_0(x))$. (Here f_α is the base-point change isomorphism defined in Lemma 2.9.)

Proof. Let $\gamma \in \Omega(X, x)$. We need to exhibit a homotopy F from $g_0 \circ \gamma$ to $f_\alpha \circ g_1 \circ \gamma$. Consider the map

$$\hat{G} : [0,1] \times [0,1] \to Y$$

defined by $\hat{G}(\theta, t) = G(\gamma(\theta), t)$. Let $\beta_0 : [0,1] \to [0,1] \times [0,1]$ be the path $\theta \mapsto (\theta, 0)$, and let β_1 be the path

$$\beta_1(\theta) = \begin{cases} (3\theta, 0) & 0 \le \theta \le \frac{1}{3} \\ (1, 3\theta - 1) & \frac{1}{3} \le \theta \le \frac{2}{3} \\ (3 - 3\theta, 1) & \frac{2}{3} \le \theta \le 1 \end{cases}$$

that runs round the other three sides of the square. Observe that

$$g_0 \circ \gamma = \hat{G} \circ \beta_0$$

and that

$$f_\alpha \circ g_1 \circ \gamma = \hat{G} \circ \beta_1.$$

Let F be any homotopy from β_0 to β_1 that preserves the end points $(0,0)$ and $(1,0)$. (It is clear that such a homotopy exists. It just pushes β_0 over the square.) Then $\hat{G} \circ F$ is the required based homotopy. □

The following is an easy consequence.

Proposition 2.32. *If $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$.*

Proof. Let $\phi : X \to Y$ and $\psi : Y \to X$ be a pair of homotopy equivalences. By Lemma 2.31, $\psi_* \circ \phi_*$ is equal to a base-point change isomorphism, in particular a bijection. It follows that ϕ_* is injective and ψ_* is surjective. By symmetry, we also have that ϕ_* is surjective and ψ_* is injective, as required. □

2.4 Higher homotopy groups

The fundamental group is defined as (based) homotopy classes of maps from the circle into a space X. There is nothing particularly special about the circle; one can make a similar definition for any choice of domain. Taking the domain to be a higher-dimensional sphere defines a higher homotopy group.

Definition 2.33. Let n be a non-negative integer and let $*$ be a choice of base point on S^n. Let X be a any topological space, with base point x_0. The set of based homotopy classes of base-point-preserving maps $S^n \to X$ is denoted $\pi_n(X, x_0)$.

From now on, we will assume that $n > 0$. Just as when $n = 1$, concatenation defines a binary operation on $\pi_n(X, x_0)$.

Definition 2.34. Let X_1 and X_2 be topological spaces and let $x_i \in X_i$ be a choice of base point, for each i. The *wedge* of X_1 and X_2 is defined to be the quotient

$$X_1 \vee X_2 = X_1 \sqcup X_2 / \sim$$

where $\sim$ is the equivalence relation that identifies x_1 and x_2. In other words, $X_1 \vee X_2$ is defined by gluing the base points of X_1 and X_2 together.

Consider the natural embedding of S^{n-1} as the 'equator' in S^n; assume that the base point $*$ lies on the equator. Let S^n/S^{n-1} be the space obtained by crushing S^{n-1} to a point. Observe that

$$S^n/S^{n-1} \approx S^n \vee S^n$$

so there is a natural quotient map $q : S^n \to S^n \vee S^n$. If f_1, f_2 are continuous maps $S^n \to X$, denote by $f_1 \vee f_2$ the map $S^n \vee S^n \to X$ that restricts to f_1 on the first copy and f_2 on the second copy. We can define a binary operation on $\pi_n(X, x_0)$ as follows.

$$f_1 \cdot f_2 = (f_1 \vee f_2) \circ q$$

It is clear that this is well defined: based homotopies of f_1 or f_2 translate to based homotopies of $f_1 \vee f_2$, and thence to based homotopies of $f_1 \cdot f_2$. Furthermore, analogously to in the case when $n = 1$, it is not too hard to prove the following.

Theorem 2.35. *Let X be any topological space.*

1. *The binary operation defines a group structure on $\pi_n(X, x_0)$.*
2. *A path $\alpha : [0,1] \to X$ defines an isomorphism $f_\alpha : \pi_n(X, \alpha(0)) \cong \pi_n(X, \alpha(1))$.*
3. *If X, Y are homotopy equivalent spaces then $\pi_n(X) \cong \pi_n(Y)$ for all n.*

It follows from part (2) of the theorem that it makes sense to write $\pi_n(X)$ for the nth homotopy group of a path-connected space X.

In general, computing the homotopy groups of a space is a highly non-trivial problem.

Definition 2.36. If a space X is homotopy equivalent to a point then X is called *contractible*.

Example 2.37. Real space $\mathbb{R}^n$ is contractible for any n, but S^1 is not. In particular, $\pi_m(\mathbb{R}^n) = 0$ for all m, n.

We will not see any examples of non-trivial higher homotopy groups at this point in the course. Higher homotopy groups have some interesting properties that we do not see in the fundamental group. We shall see that fundamental groups are often non-abelian.

Theorem 2.38. *If $n > 1$ then $\pi_n(X)$ is abelian.*

Proof. Embed S^n into $\mathbb{R}^{n+1}$ so that the equator is the intersection of S^n with the hyperplane $x_0 = 0$ and the base point is $(0, \ldots, 0, 1)$. Consider the rotation that fixes the codimension-two subspace $x_0 = x_1 = 0$ defined by the following matrix.

$$A_t = \begin{pmatrix} \cos \pi t & \sin \pi t & 0 \\ -\sin \pi t & \cos \pi t & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}$$

(Here I_{n-1} denotes the $(n-1)$-dimensional identity matrix.) The map $(x, t) \mapsto A_t x$ defines a homotopy on S^n that exchanges the two path components of $S^n \smallsetminus S^{n-1}$. Now $(f \cdot g) \circ A_t$ is a homotopy from $f \cdot g$ to $g \cdot f$, as required. $\square$

Furthermore, we have the following consequence of part (2) of Theorem 2.35: $\pi_1(X, x_0)$ *acts* on $\pi_n(X, x_0)$, via the assignment $\gamma(\phi) = f_\gamma(\phi)$. In summary, $\pi_n(X)$ is an abelian group with an action of $\pi_1(X)$. In formal algebraic language, this says that $\pi_n(X)$ is a $\pi_1(X)$-*module*.