

## HOMEWORK 8 SOLUTIONS

1. Note that

$$\begin{aligned}x_s(s, v) &= \alpha'(s) + vb'(s) = t(s) + v\tau n(s) \\x_v(s, v) &= b(s)\end{aligned}$$

which are linearly independent.

Along  $\alpha$ , i.e. where  $v = 0$ , the tangent space is spanned by  $t(s)$  and  $b(s)$ . Note that  $\alpha''(s)$  is perpendicular to these so it is geodesic.

2. a) We know that for all  $u, v \in T_p(S^2)$  we have

$$\langle dP_\alpha(u), dP_\alpha(v) \rangle = \langle u, v \rangle .$$

In particular,  $dP_\alpha$  takes an orthonormal basis to an orthonormal basis, so it's an isometry.

b) For any  $\theta$  use the path in problem 4 of homework 7.

3. Let  $L : S^2 \rightarrow S^2$  be an isometry. Then the length of a geodesic between any  $p, q \in S^2$  is preserved. But this length is the angle in  $\mathbb{R}^3$  between  $p$  and  $q$  so in particular we have that  $p \cdot q$  is preserved. Now pick an orthonormal basis in  $\mathbb{R}^3$ . This is sent under  $L$  to an orthonormal basis and inner products are preserved so  $L$  is just the restriction of an orthogonal linear map.

Conversely, if  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is an orthogonal map, then  $dL = L$  so it is an isometry when restricted to the tangent space.

4. The geodesic curvature along the boundary is zero and each region is homeomorphic to a disc, so by the Gauss-Bonnet theorem we have

$$\int \int_A K d\sigma = \int \int_B K d\sigma = 2\pi.$$

Using the change of variables formula we find that the area of  $N(A)$  is

$$\int \int_R |N_u \wedge N_v| dudv = \int \int_R K |x_u \wedge x_v| dudv = \int \int_A K d\sigma$$

where  $x : R \rightarrow A$  is a parametrization of  $A$  and similarly for  $B$