FINAL

Instructions

Open book, open notes. You may appeal to any result stated in Do Carmo. References to theorems from other sources will not be accepted.

There are **five** questions. The time limit is **four hours**. No credit will be given for work done after four hours.

Hand in the midterm to the usual box before 4pm on Wednesday, March 17^{th}.

Turn over when beginning exam.

1. Fix a real number $c > 0$, and define a function $F_c : \mathbb{R}^3 \to \mathbb{R}^3$ by $F_c(p) = cp$. (Such a map is called a *similarity*.) Let $S \in \mathbb{R}^3$ be a regular surface and set $\bar{S} = F_c(S)$. Prove that $\bar{S}$ is a regular surface. If S is orientable, find and justify formulae relating the Gaussian and mean curvatures of S with the Gaussian and mean curvatures of $\bar{S}$.

2. Let S be a regular surface. The *gradient* of a differentiable function $f : S \to \mathbb{R}$ is a differentiable map grad $f : S \to \mathbb{R}^3$ which assigns to each point $p \in S$ a vector grad $f(p) \in T_p(S) \subset \mathbb{R}^3$ so that for all $v \in T_p(S)$

$$\langle \text{grad } f(p), v \rangle = df_p(v) \ .$$

   (a) Prove that if E, F, G are the coefficients of the first fundmental form in a parametrisation $\mathbf{x} : U \to S$ then grad f on $\mathbf{x}(U)$ is given by

$$\text{grad } f = \frac{f_u G - f_v F}{EG - F^2}\mathbf{x}_u + \frac{f_v E - f_u F}{EG - F^2}\mathbf{x}_v \ .$$

   In particular, if $S = \mathbb{R}^2$ with coordinates x, y, then

$$\text{grad } f = f_x e_1 + f_y e_2 \ ,$$

   where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$. (Thus the definition agrees with the usual definition of gradient in the plane.)

   (b) Let $p \in S$ be fixed, and suppose that $df_p \not\equiv 0$. Prove that as v varies in the unit circle $|v| = 1$ in $T_p(S)$, the value of $df_p(v)$ is maximum exactly when $v = \frac{\text{grad } f(p)}{||\text{grad } f(p)||}$. (Thus grad $f(p)$ gives the direction of maximum variation of f at p.)

   (c) Show that if grad $f \neq 0$ at all points of the level curve $C = \{q \in S \mid f(q) = const.\}$, then C is a regular curve on S and grad f is normal to C at all points of C.

3. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi}\int_0^\pi k_n(\theta) d\theta$$

   where $k_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

4. Show that the sphere is locally conformal to the plane.

5. Let S be an oriented regular surface and let $\alpha : I \to S$ be a curve parametrized by arc length. At the point $p = \alpha(s)$ consider the three unit vectors (the

Darboux trihedron) $T(s) = \alpha'(s)$, $N(s) =$ the normal vector to S at p, $V(s) = N(s) \wedge T(S)$. Show that

$$\begin{aligned}
\frac{dT}{ds} &= 0 + aV + bN \\
\frac{dV}{ds} &= -aT + 0 + cN \\
\frac{dN}{ds} &= -bT - cV + 0
\end{aligned}$$

where $a = a(s)$, $b = b(s)$, $c = c(s)$, $s \in I$. (These formulas are the analogues of Frenet's formulas for the trihedron T, V, N.) Prove that:

(a) $c = -\langle dN/ds, V \rangle$;

(b) b is the normal curvature of α at p;

(c) a is the geodesic curvature of α at p.