# HOMEWORK 8

1. Let $\alpha : I \to S$ be a curve parametrized by arc length $s$, with non-zero curvature, where $I$ is an open interval. Consider the parametrized surface

$$\mathbf{x}(s, v) = \alpha(s) + vb(s), \quad s \in I, \quad -\epsilon < v < \epsilon, \ \epsilon > 0$$

where $b$ is the binormal vector of $\alpha$. Prove that if $\epsilon$ is small, $S = \mathbf{x}(I \times (-\epsilon, \epsilon))$ is a regular surface over which $\alpha(I)$ is a geodesic.

2. Let

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

and let $p \in S^2$. For each piecewise regular parametrized curve $\alpha : [0, l] \to S^2$ with $\alpha(0) = \alpha(l) = p$, let $P_\alpha : T_pS^2 \to T_pS^2$ be the map which assigns to each $v \in T_pS^2$ its parallel transport along $\alpha$ back to $p$.

   (a) Show that $P_\alpha$ is an isometry.

   (b) Prove that for every rotation $R$ of $T_pS$ there exists $\alpha$ such that $R = P_\alpha$.

3. Show that the isometries of the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

are the restrictions to $S^2$ of the linear orthogonal transformations of $\mathbb{R}^3$. (The linear orthogonal transformations of $\mathbb{R}^3$ can be characterized as those linear maps $L : \mathbb{R}^3 \to \mathbb{R}^3$ that preserve the inner product; that is, $\langle Lw, Lv\rangle = \langle w, v\rangle$ for all $w, v \in \mathbb{R}^3$.)

4. Let $S \subseteq \mathbb{R}^3$ be a regular surface homeomorphic to a sphere. Let $\Gamma \subseteq S$ be a simple closed geodesic in $S$, and let $A$ and $B$ be the regions of $S$ which have $\Gamma$ as a common boundary. Let $N : S \to S^2$ be the Gauss map of $S$. Prove that $N(A)$ and $N(B)$ have the same area.